香港中文大學
The Chinese University of Hong Kong

9.2 The Algorithms

9.2.1 Channel Capacity

Lemma 9.1 Let $r(x)p(y|x)$ be a given joint distribution on $\mathcal{X} \times \mathcal{Y}$ such that $\mathbf{r} > 0$, and let $\mathbf{q}$ be a transition matrix from $\mathcal{Y}$ to $\mathcal{X}$. Then

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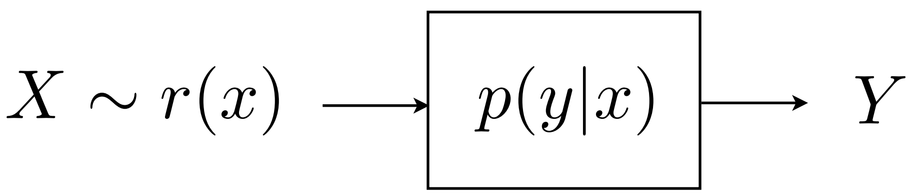

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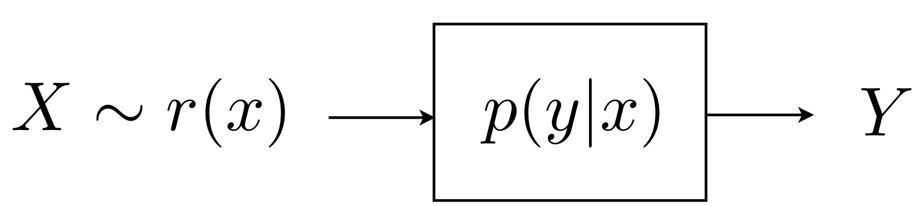

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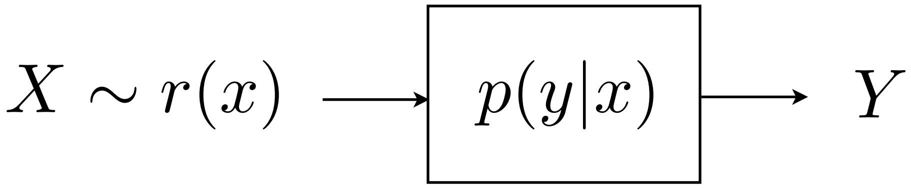

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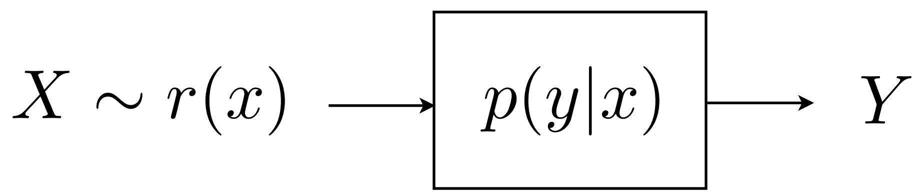

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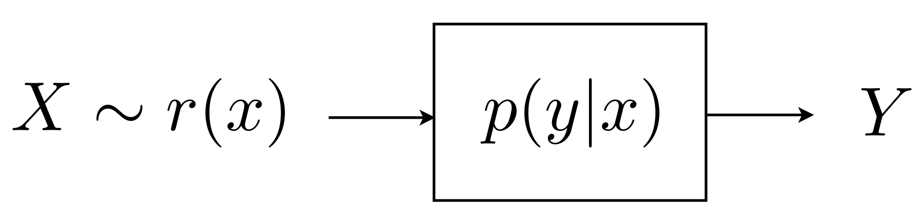

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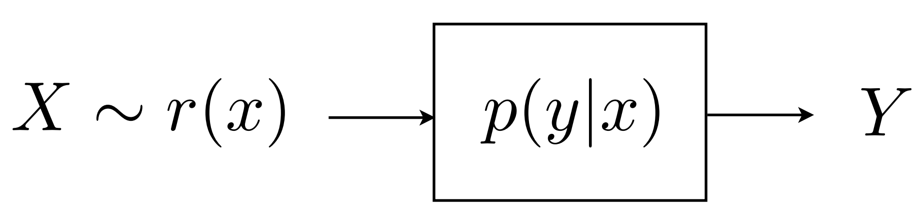

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$$q^*(x|y) = \frac{r(x)p(y|x)}{\sum_{x'} r(x')p(y|x')}. \qquad (2)$$

Proof

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&= \sum_y w(y) \underline{\sum_x q^*(x|y) \log \frac{q^*(x|y)}{q(x|y)}} \\
&= \sum_y w(y) \underline{D(q^*(x|y) \| q(x|y))}
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6. The proof is completed by noting in (2) that $\mathbf{q}^*$ satisfies (1) because $\mathbf{r} > 0$.

Lemma 9.1 Let $r(x)p(y|x)$ be a given joint distribution on $\mathcal{X} \times \mathcal{Y}$ such that $\mathbf{r} > 0$, and let $\mathbf{q}$ be a transition matrix from $\mathcal{Y}$ to $\mathcal{X}$. Then

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$w(y) > 0$

Proof

1. In (2), let

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$$
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$$

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$$
q(x|y) = 0 \quad \text{if and only if} \quad p(y|x) = 0, \tag{1}
$$

and

> 0

$$
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$$

$w(y) > 0$

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w(y) = \sum_{x'} r(x')p(y|x').
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$$

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$$
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&= \quad \sum_y \sum_x w(y)q^*(x|y) \log \frac{q^*(x|y)}{q(x|y)} \\
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\end{aligned}$$

where the maximization is taken over all $\mathbf{q}$ such that

$$q(x|y) = 0 \quad \text{if and only if} \quad p(y|x) = 0, \tag{1}$$

> 0

and

$$\underline{q^*(x|y)} = \frac{r(x)p(y|x)}{\cancel{\sum_{x'} r(x')p(y|x')}}. \tag{2}$$

$w(y) > 0$

Proof

1. In (2), let

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&= \sum_y w(y) D(\underline{q^*(x|y)} \| \underline{q(x|y)}) \quad \forall y \Rightarrow \mathbf{q} = \mathbf{q}^* \\
&\geq 0.
\end{aligned}$$

6. The proof is completed by noting in (2) that $\mathbf{q}^*$ satisfies (1) because $\mathbf{r} > 0$.

Remark The maximizing $\mathbf{q}$ in Lemma 9.1 is unique.

Theorem 9.2 For a discrete memoryless channel $p(y|x)$,

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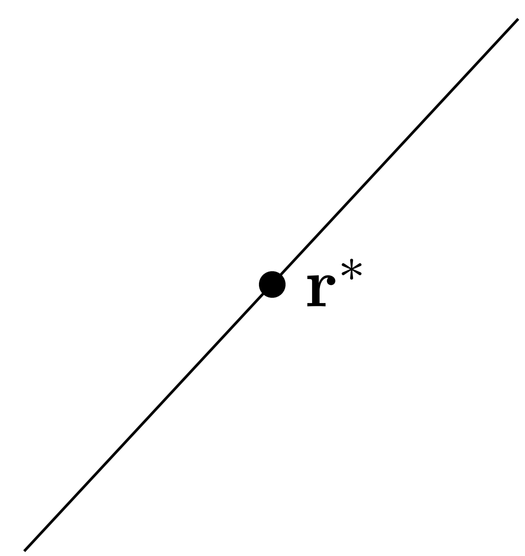

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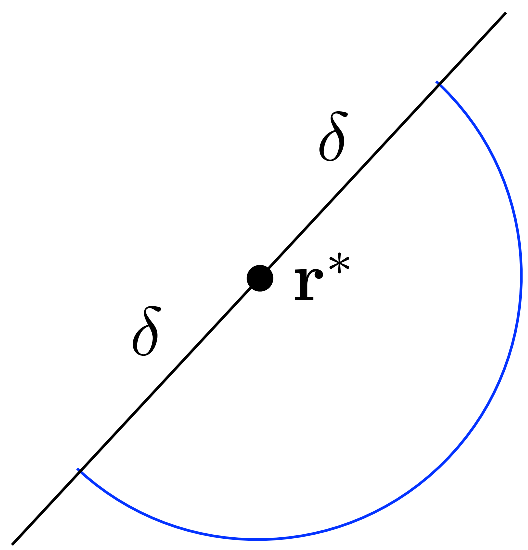

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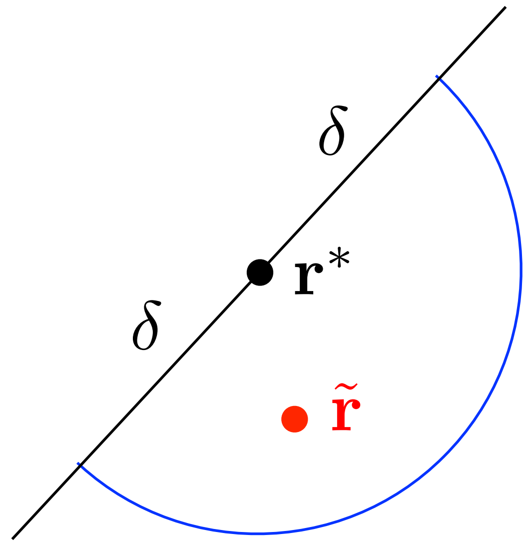

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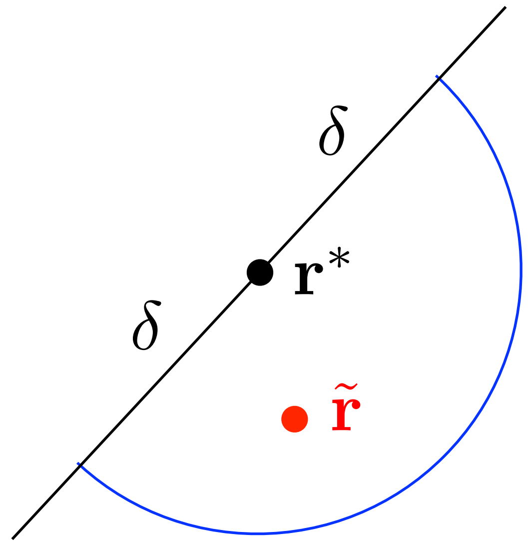

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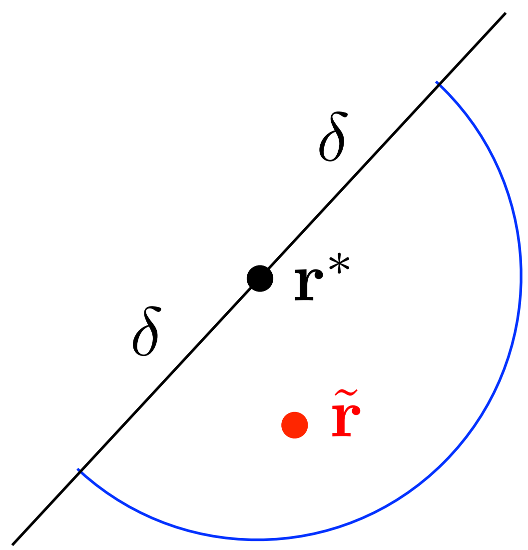

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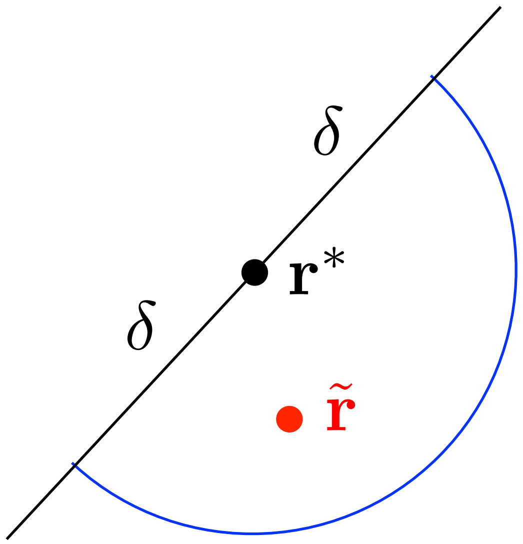

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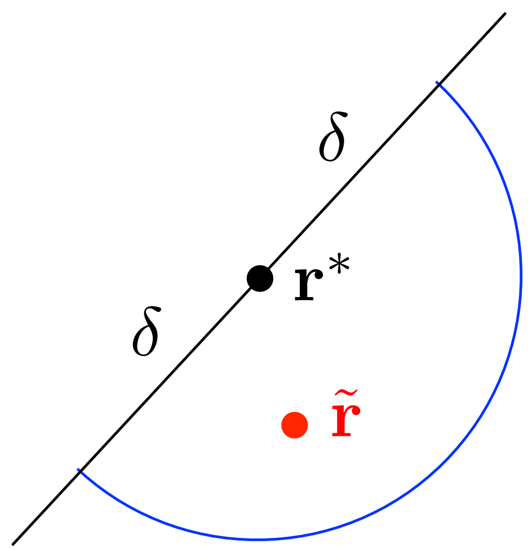

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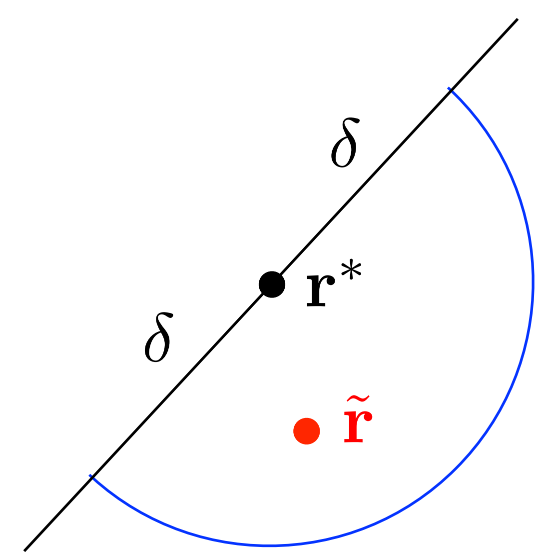

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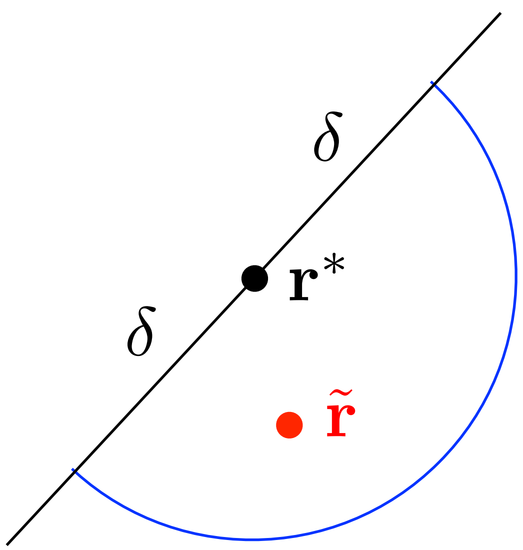

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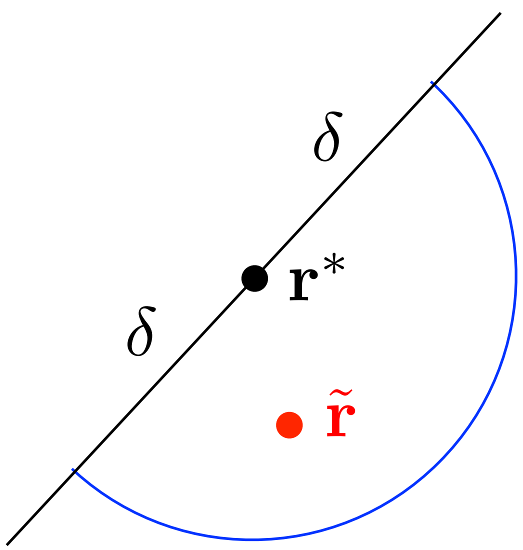

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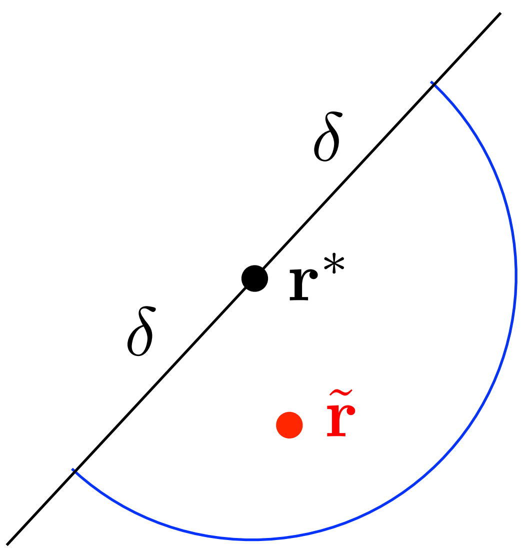

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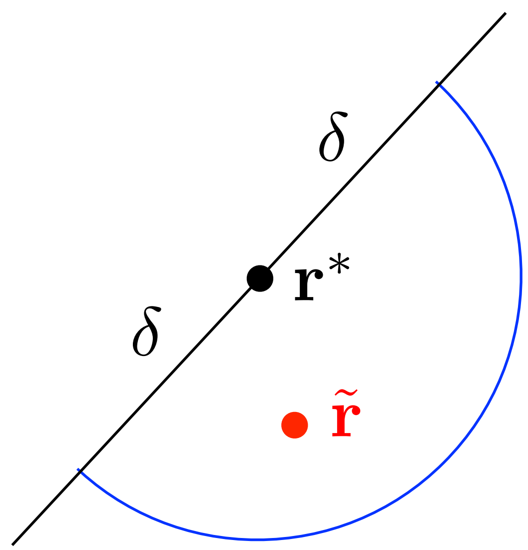

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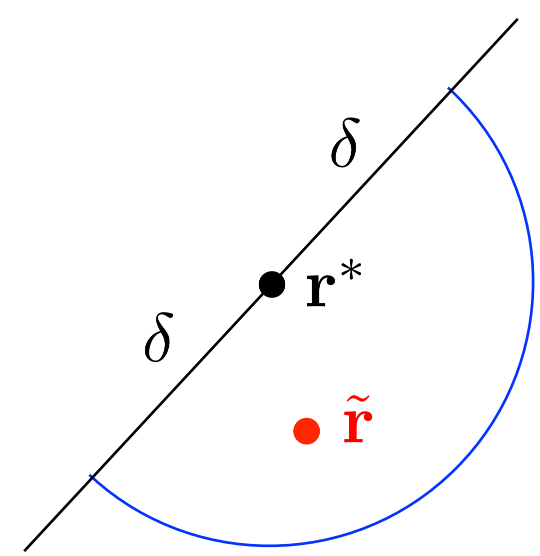

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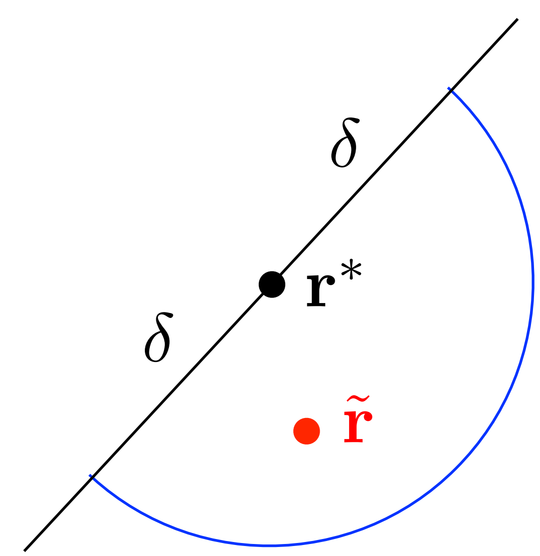
$$\|\tilde{\mathbf{r}} - \mathbf{r}^*\| < \delta.$$

7. Then

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}) \geq \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}) \geq I(\tilde{\mathbf{r}},\mathbf{p}) > C - \cancel{\epsilon}.$$

8. Let $\epsilon \to 0$ to conclude that



Theorem 9.2 For a discrete memoryless channel $p(y|x)$,

$$C = \sup_{\mathbf{r}>0} \max_{\mathbf{q}} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)},$$

where the maximization is taken over all $\mathbf{q}$ that satisfies (1) in Lemma 9.1.

Proof

1. Write $I(X;Y)$ as $I(\mathbf{r},\mathbf{p})$ where $\mathbf{r}$ is the input distribution and $\mathbf{p}$ denotes the transition matrix of the generic channel $p(y|x)$. Then

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}).$$

2. By Lemma 9.1, we need to prove that

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}) = \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}).$$

3. Let $\mathbf{r}^*$ achieves C.

4. If $\mathbf{r}^* > 0$, then

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}) = \max_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}) = \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}).$$

5. Next, consider $\mathbf{r}^* \geq 0$. Since $I(\mathbf{r},\mathbf{p})$ is continuous in $\mathbf{r}$, for any $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\|\mathbf{r} - \mathbf{r}^*\| < \delta,$$

then

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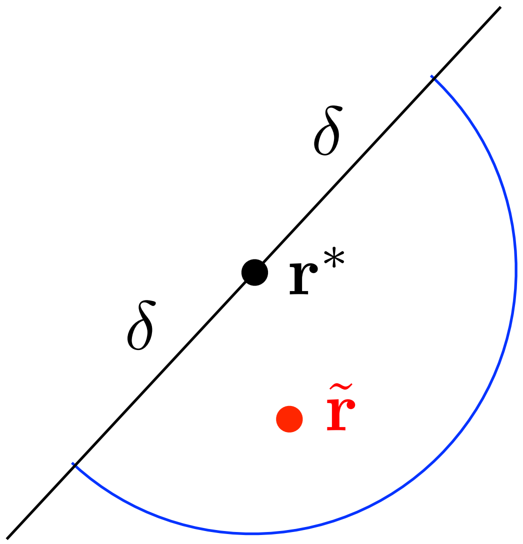

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$\leq I(X;Y) \leq H(X) \leq \log|\mathcal{X}|$

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where $\mathbf{u}_1 \leftarrow \mathbf{r}$ and $\mathbf{u}_2 \leftarrow \mathbf{q}$.

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$$A_1 = \left\{(r(x), x \in \mathcal{X}) : r(x) > 0 \text{ and } \textstyle\sum_x r(x) = 1\right\} \subset \Re^{|\mathcal{X}|}$$

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3. Let $\mathbf{r}^{(0)}$ be an arbitrarily chosen strictly positive input distribution in A_1. Then $\mathbf{q}^{(0)} \in A_2$ can be computed according to (1). This forms $(\mathbf{r}^{(0)}, \mathbf{q}^{(0)})$.

4. Compute $\mathbf{r}^{(1)}$, $\mathbf{q}^{(1)}$, $\mathbf{r}^{(2)}$, $\mathbf{q}^{(2)}$, $\cdots$ iteratively by applying (2) and (1) alternately.

5. It can be verified from (1) that if $\mathbf{r}^{(k)} \in A_1$, i.e., $\mathbf{r}^{(k)} > 0$, then $q^{(k)}(x|y) > 0$ iff $p(y|x) > 0$, i.e., $\mathbf{q}^{(k)} \in A_2$.

6. Likewise, it can be verified from (2) that if $\mathbf{q}^{(k)} \in A_2$, then $\mathbf{r}^{(k+1)} > 0$, i.e., $\mathbf{r}^{(k+1)} \in A_1$.

7. Therefore, $\mathbf{r}^{(k)} \in A_1$ and $\mathbf{q}^{(k)} \in A_2$ for all $k \geq 0$.

8. It will be shown in Section 9.3 that $f^{(k)} = f(\mathbf{r}^{(k)}, \mathbf{q}^{(k)}) \to f^* = C$.

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

Maximizing *f*(r,q) for a Fixed q

$$\max_{\mathbf{r} \in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

Maximizing *f*(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r} \in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

and

$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

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2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

Maximizing f(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}}$$

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2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \underline{\sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}} - \lambda \sum_x r(x).$$

Maximizing f(r,q) for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

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$$J = \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)} - \lambda \underline{\sum_x r(x)}.$$

Maximizing f(r,q) for a Fixed q

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For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

Maximizing f(r,q) for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

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For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x)\log q(x|y) - \log r(x) - 1 - \lambda.$$

Maximizing *f*(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

and

$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x) \log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

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$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

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$$J = \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\cancel{\frac{\partial J}{\partial r(x)}}^{\,0} = \sum_y p(y|x) \log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

Maximizing *f*(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}}$$

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For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\cancelto{0}{\frac{\partial J}{\partial r(x)}} = \sum_y p(y|x) \log q(x|y) - \underline{\log r(x)} - 1 - \lambda.$$

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Maximizing *f*(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}}$$

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$$\cancelto{0}{\frac{\partial J}{\partial r(x)}} = \sum_y p(y|x) \log q(x|y) - \underline{\log r(x)} - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\underline{\log r(x)} = \sum_y p(y|x) \log q(x|y) - 1 - \lambda,$$

Maximizing f(r,q) for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}$$

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Maximizing *f*(r,q) for a Fixed q

$$\max_{\mathbf{r} \in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}$$

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Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x) \log q(x|y) - 1 - \lambda,$$

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$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

Maximizing $f(r,q)$ for a Fixed q

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3. By considering the normalization constraint in (1), we can eliminate λ and obtain

Maximizing *f*(r,q) for a Fixed q

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$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

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3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'}\prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed q

$$\boxed{\max_{\mathbf{r} \in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}}$$

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Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

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$$r(x) = \underline{e^{-(\lambda+1)}} \prod_y q(x|y)^{p(y|x)}.$$

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Maximizing f(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}}$$

1. The constraints on $\mathbf{r}$ are

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Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

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or

$$r(x) = \underline{e^{-(\lambda+1)}} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

Maximizing f(r,q) for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}}$$

1. The constraints on $\mathbf{r}$ are

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For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x) \log q(x|y) - \log r(x) - 1 - \lambda.$$

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$$\log r(x) = \sum_y p(y|x) \log q(x|y) - 1 - \lambda,$$

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$$r(x) = \underline{e^{-(\lambda+1)}} \prod_y q(x|y)^{p(y|x)}.$$

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$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed $\mathbf{q}$

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

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2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x)\log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

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or

$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

4. The above product is over all y such that $p(y|x) > 0$, and $q(x|y) > 0$ for all such y. This implies that both the numerator and the denominator on the right hand side above are positive, and therefore $r(x) > 0$.

Maximizing f(r,q) for a Fixed q

$$\max_{\mathbf{r} \in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

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$$J = \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x) \log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x) \log q(x|y) - 1 - \lambda,$$

or

$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y q(x|y)^{\underline{p(y|x)}}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

4. The above product is over all y such that $p(y|x) > 0$, and $q(x|y) > 0$ for all such y. This implies that both the numerator and the denominator on the right hand side above are positive, and therefore $r(x) > 0$.

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log \frac{q(x|y)}{r(x)}}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

and

$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x)p(y|x)\log \frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x)\log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x)\log q(x|y) - 1 - \lambda,$$

or

$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y \underline{q(x|y)^{p(y|x)}}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

4. The above product is over all y such that $p(y|x) > 0$, and $q(x|y) > 0$ for all such y. This implies that both the numerator and the denominator on the right hand side above are positive, and therefore $r(x) > 0$.

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed $\mathbf{q}$

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

and

$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x) \log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x) \log q(x|y) - 1 - \lambda,$$

or

$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y \underline{q(x|y)}^{\underline{p(y|x)}}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

4. The above product is over all y such that $p(y|x) > 0$, and $q(x|y) > 0$ for all such y. This implies that both the numerator and the denominator on the right hand side above are positive, and therefore $\underline{r(x) > 0}$.

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed $\mathbf{q}$

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

and

$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x) \log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x) \log q(x|y) - 1 - \lambda,$$

or

$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

4. The above product is over all y such that $p(y|x) > 0$, and $q(x|y) > 0$ for all such y. This implies that both the numerator and the denominator on the right hand side above are positive, and therefore $r(x) > 0$.

5. In other words, the $\mathbf{r}$ thus obtained happen to satisfy the positivity constraints in (2) although these constraints were ignored when we set up the Lagrange multipliers.

Maximizing $f(\mathbf{r},\mathbf{q})$ for a Fixed q

$$\boxed{\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

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Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x)\log q(x|y) - 1 - \lambda,$$

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$$r(x) = e^{-(\lambda+1)}\prod_y q(x|y)^{p(y|x)}.$$

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5. In other words, the $\mathbf{r}$ thus obtained happen to satisfy the positivity constraints in (2) although these constraints were ignored when we set up the Lagrange multipliers.

Maximizing *f*(r,q) for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

and

$$r(x) > 0 \quad \text{for all } x \in \mathcal{X}. \tag{2}$$

2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

For convenience sake, we assume that the logarithm is the natural logarithm. Differentiating with respect to $r(x)$ gives

$$\frac{\partial J}{\partial r(x)} = \sum_y p(y|x)\log q(x|y) - \log r(x) - 1 - \lambda.$$

Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x)\log q(x|y) - 1 - \lambda,$$

or

$$r(x) = e^{-(\lambda+1)} \prod_y q(x|y)^{p(y|x)}.$$

3. By considering the normalization constraint in (1), we can eliminate λ and obtain

$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

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5. In other words, the $\mathbf{r}$ thus obtained happen to satisfy the positivity constraints in (2) although these constraints were ignored when we set up the Lagrange multipliers.

6. We will show in Section 9.3.2 that f is concave. Then $\mathbf{r}$ as given in (3), which is unique, indeed achieves the maximum of f for a given $\mathbf{q} \in A_2$ because $\mathbf{r}$ is in the interior of A_1.

Maximizing *f*(r,q) for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

$$\sum_x r(x) = 1 \tag{1}$$

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2. Use the method of Lagrange multipliers to find the best $\mathbf{r}$ by ignoring temporarily the positivity constraints on $\mathbf{r}$ in (2). Let

$$J = \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)} - \lambda \sum_x r(x).$$

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Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

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Maximizing *f*(r,q) for a Fixed q

$$\max_{\mathbf{r}\in A_1} \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

1. The constraints on $\mathbf{r}$ are

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Upon setting $\frac{\partial J}{\partial r(x)} = 0$, we have

$$\log r(x) = \sum_y p(y|x)\log q(x|y) - 1 - \lambda,$$

or

$$r(x) = e^{-(\lambda+1)}\prod_y q(x|y)^{p(y|x)}.$$

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$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'}\prod_y q(x'|y)^{p(y|x')}}. \tag{3}$$

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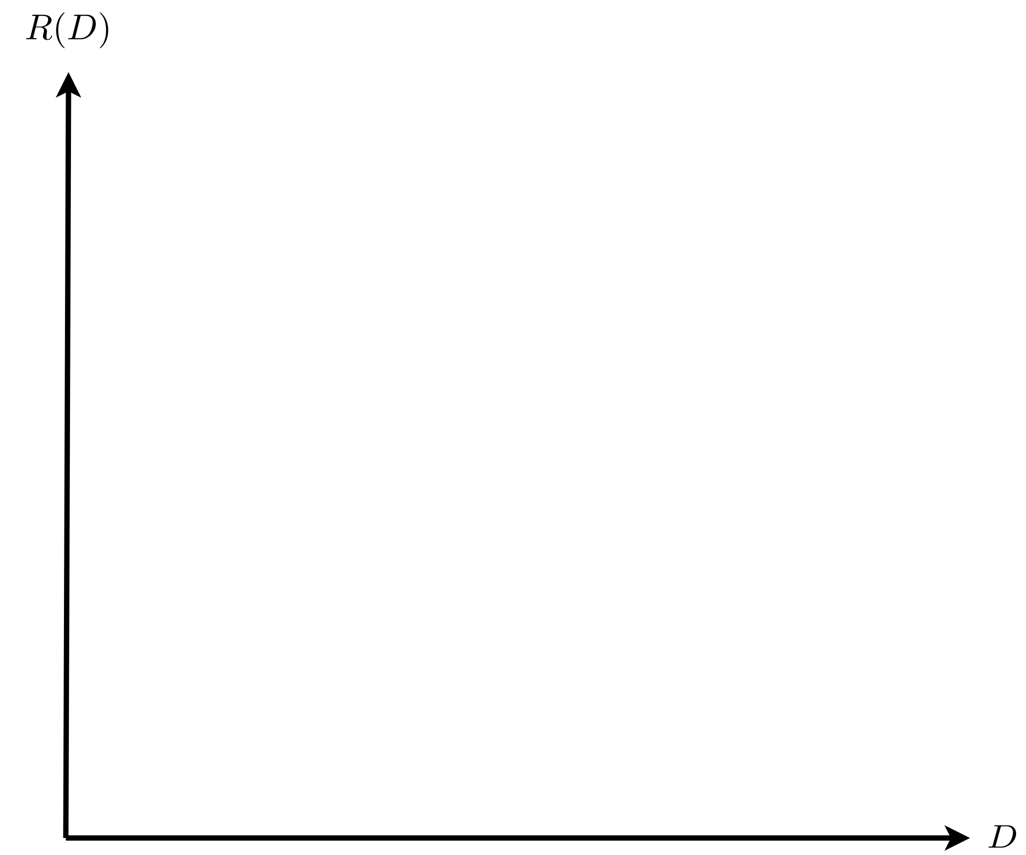
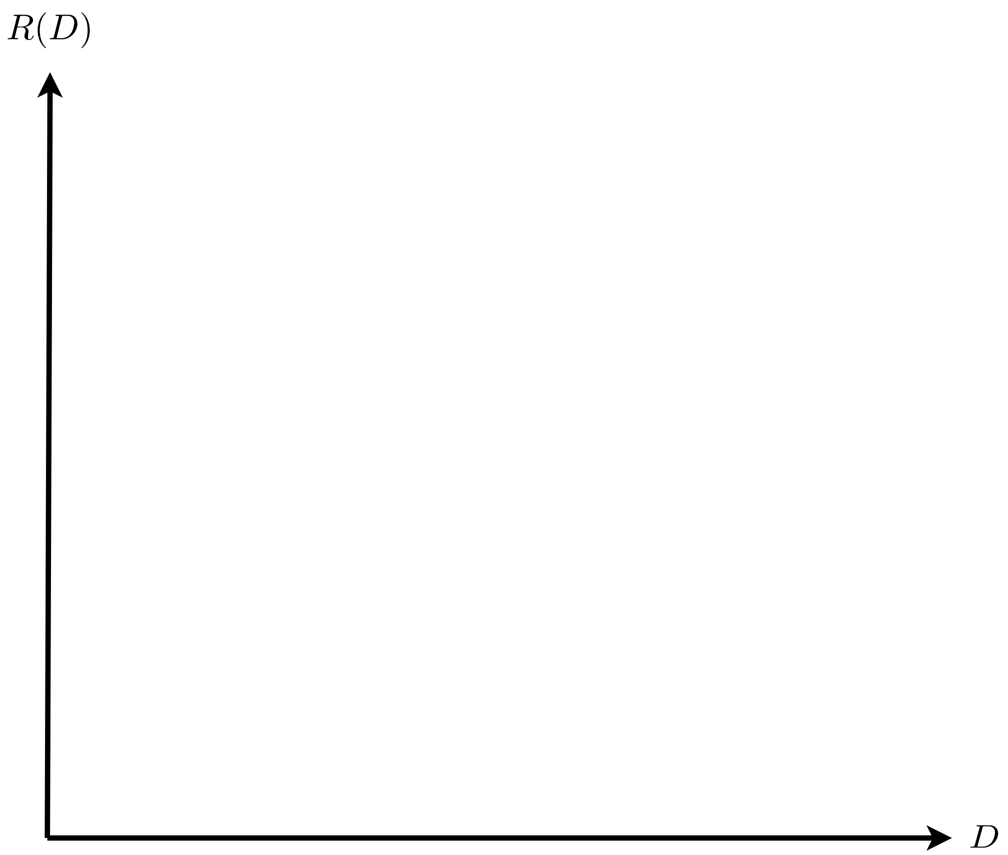
9.2.2 The Rate-Distortion Function

The $R(D)$ Curve



The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \leq D \leq D_{max}$.



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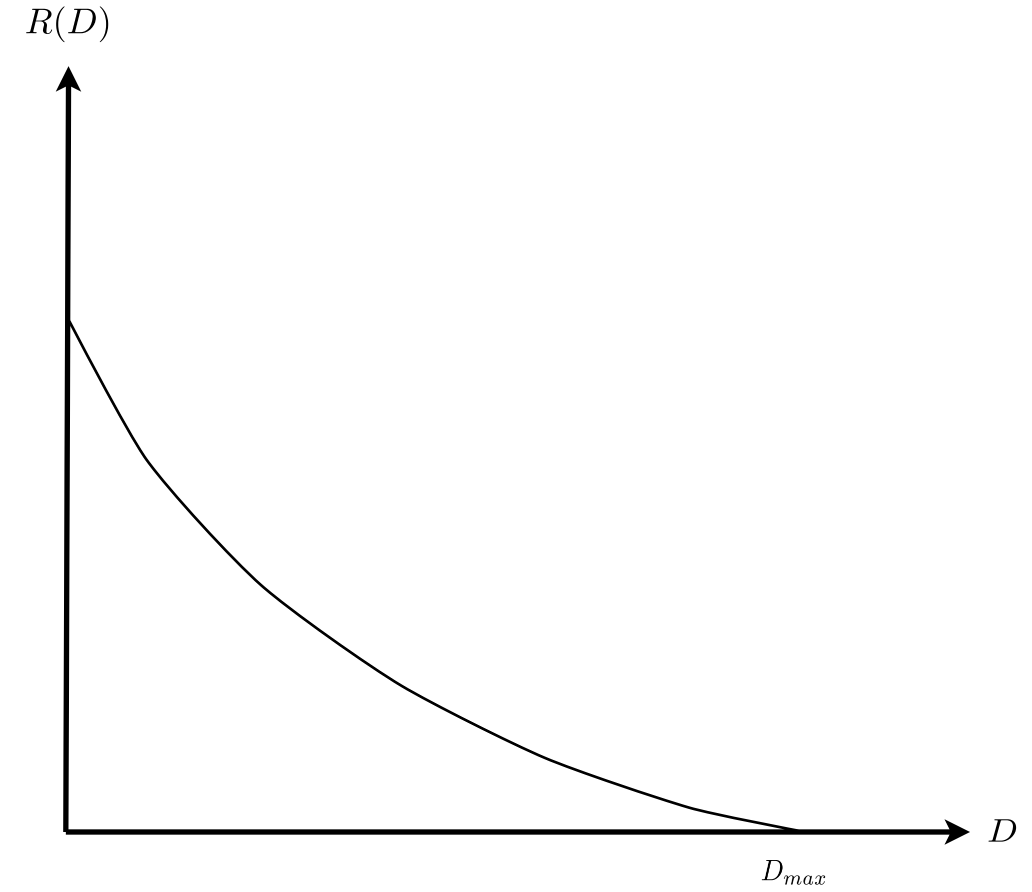
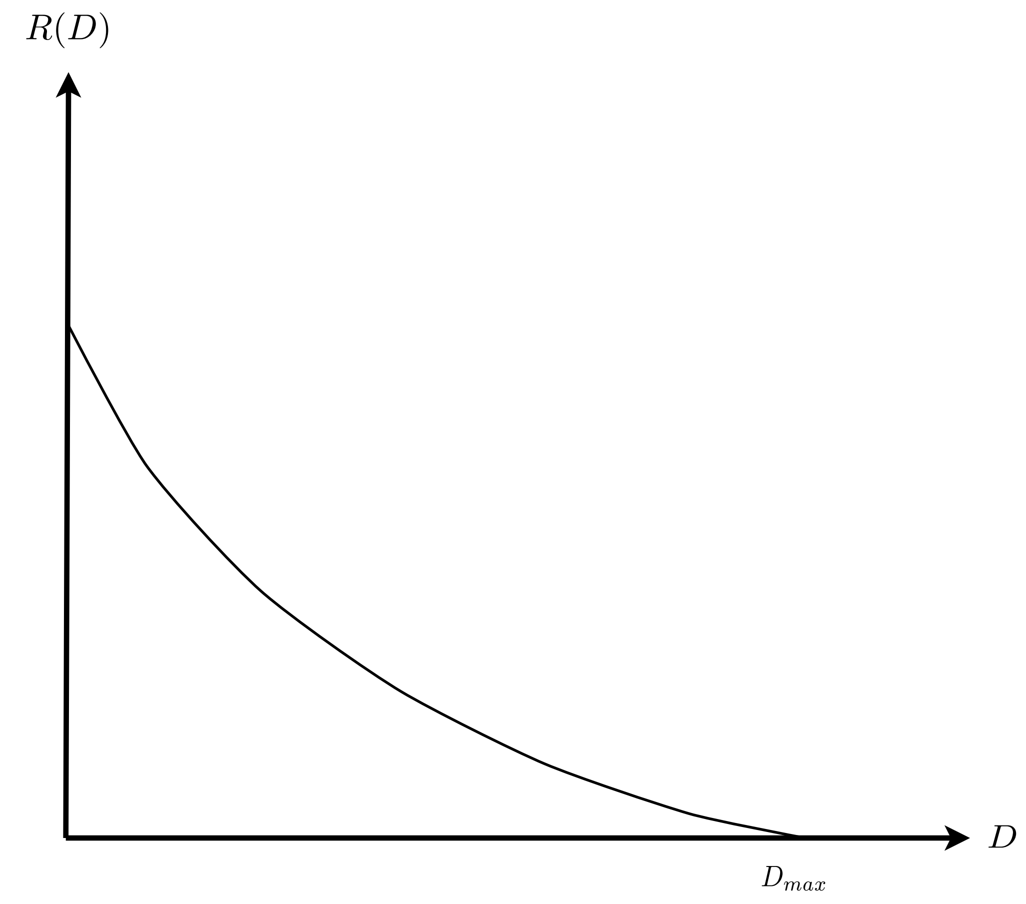


The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \leq D \leq D_{max}$.

2. Since $R(D)$ is convex, for any $s \leq 0$, there exists a point on the $R(D)$ curve for $0 \leq D \leq D_{max}$ such that the slope of a tangent to the $R(D)$ curve at that point is equal to s.



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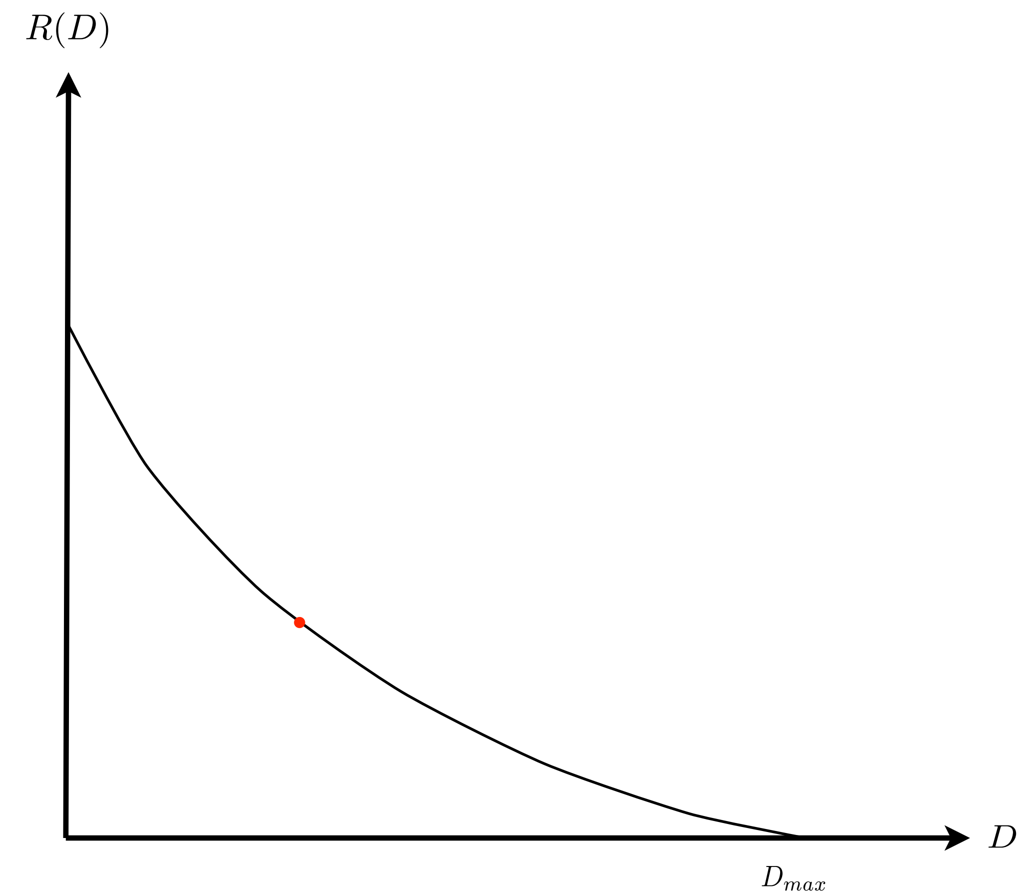

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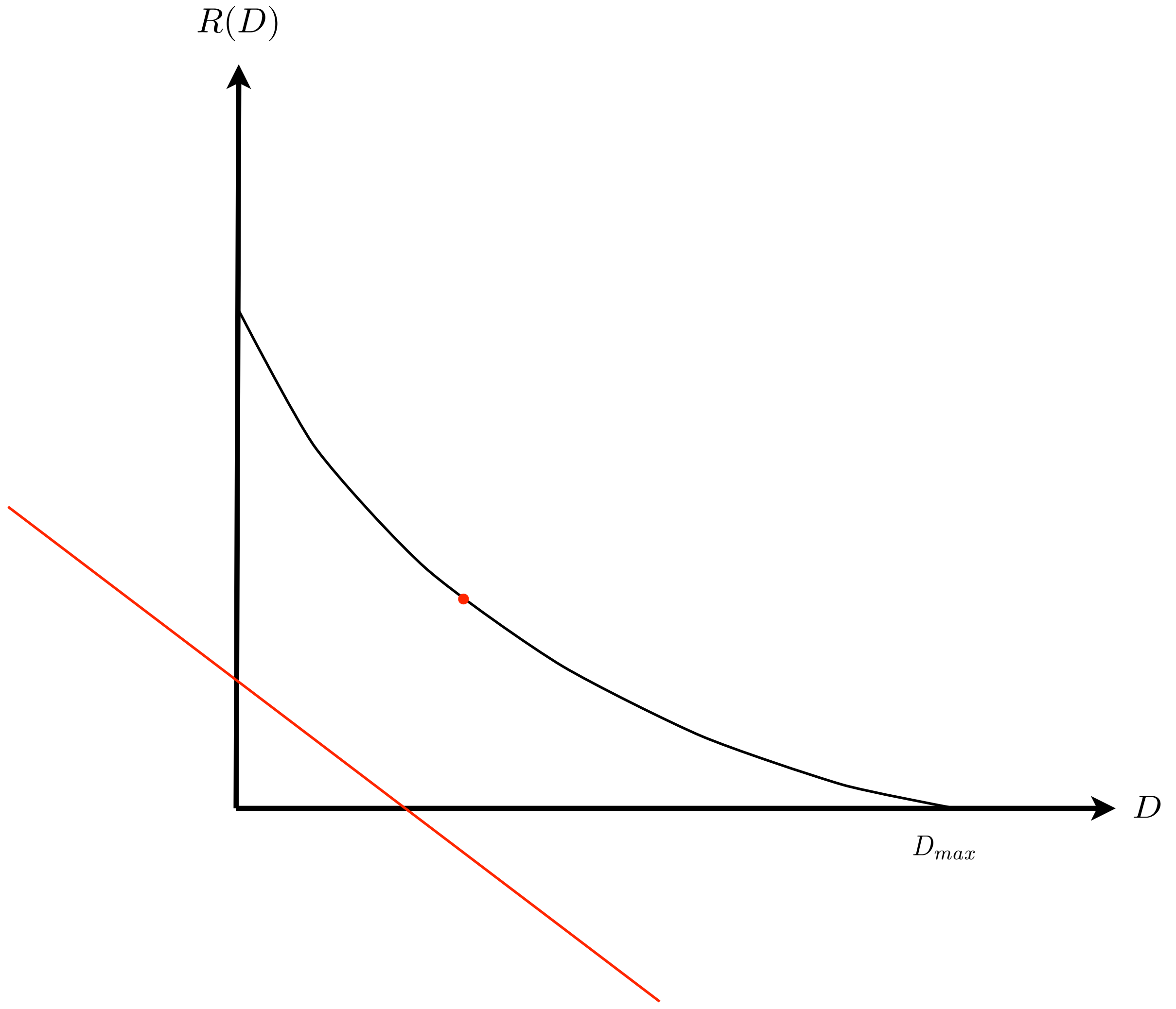

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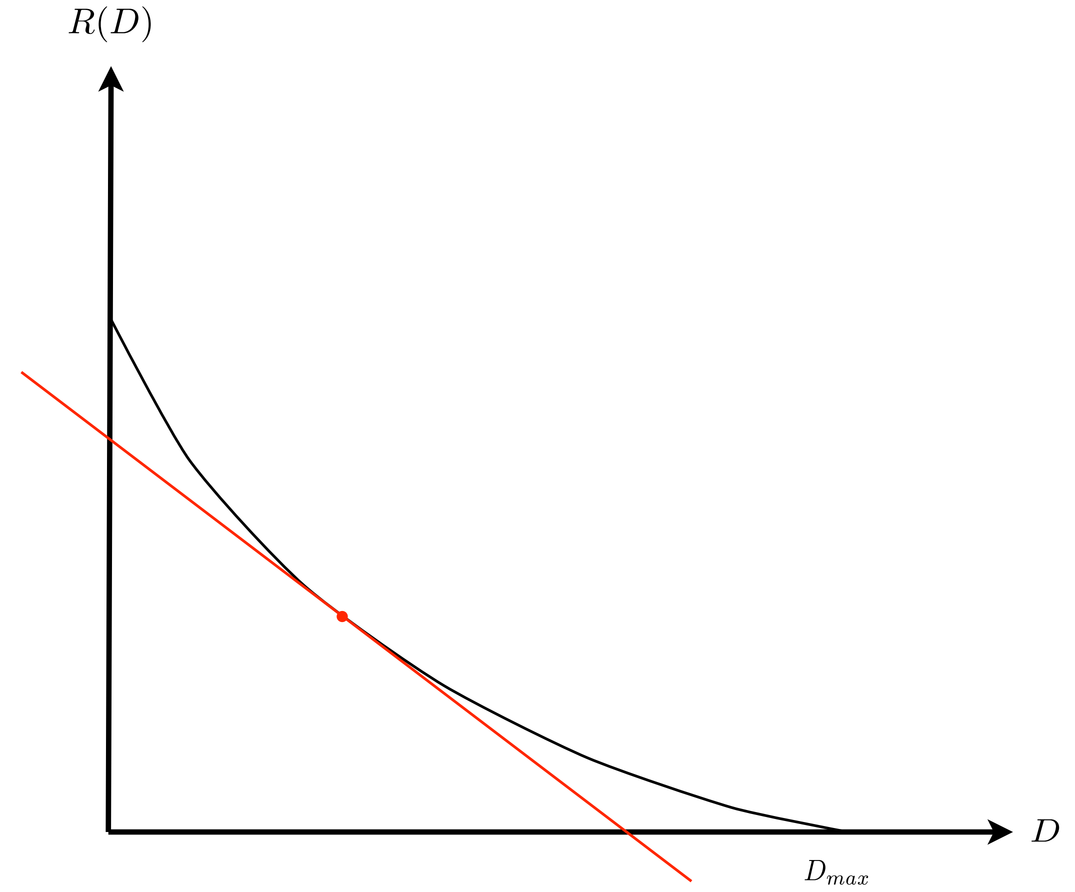
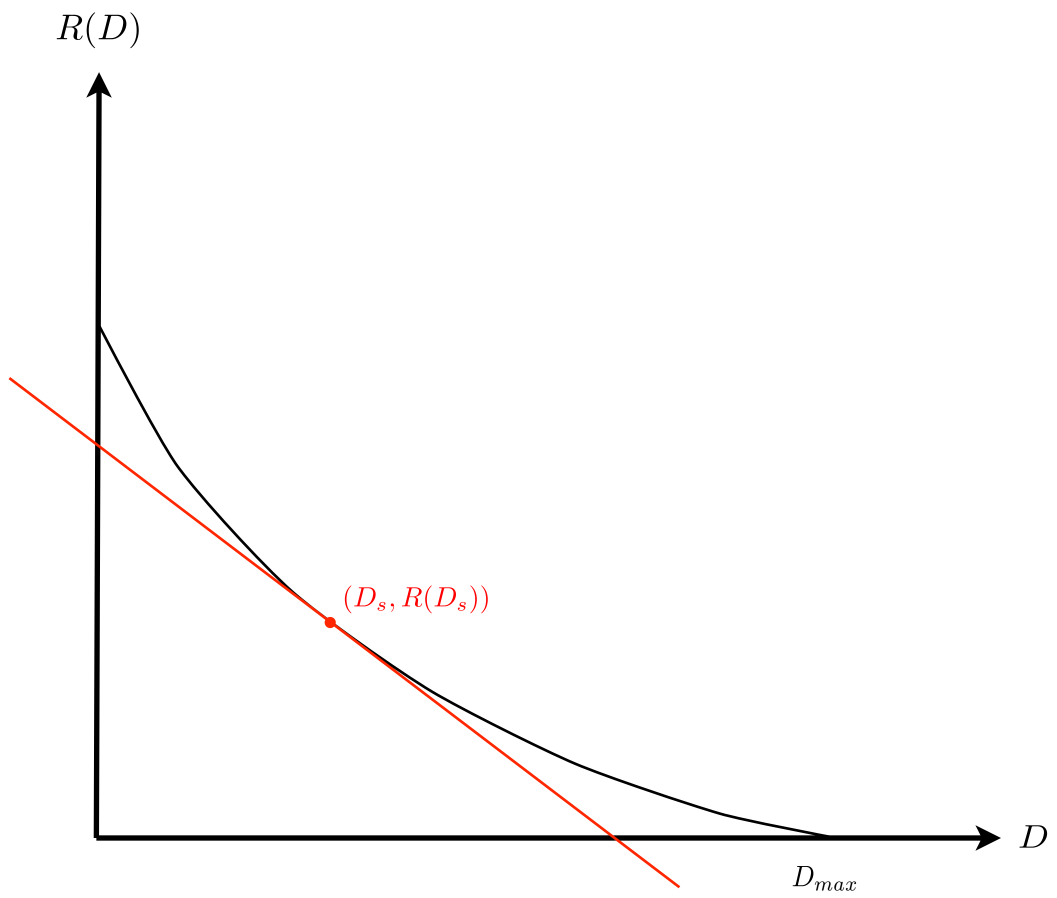


The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \leq D \leq D_{max}$.

2. Since $R(D)$ is convex, for any $s \leq 0$, there exists a point on the $R(D)$ curve for $0 \leq D \leq D_{max}$ such that the slope of a tangent to the $R(D)$ curve at that point is equal to s.

3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.



The $R(D)$ Curve

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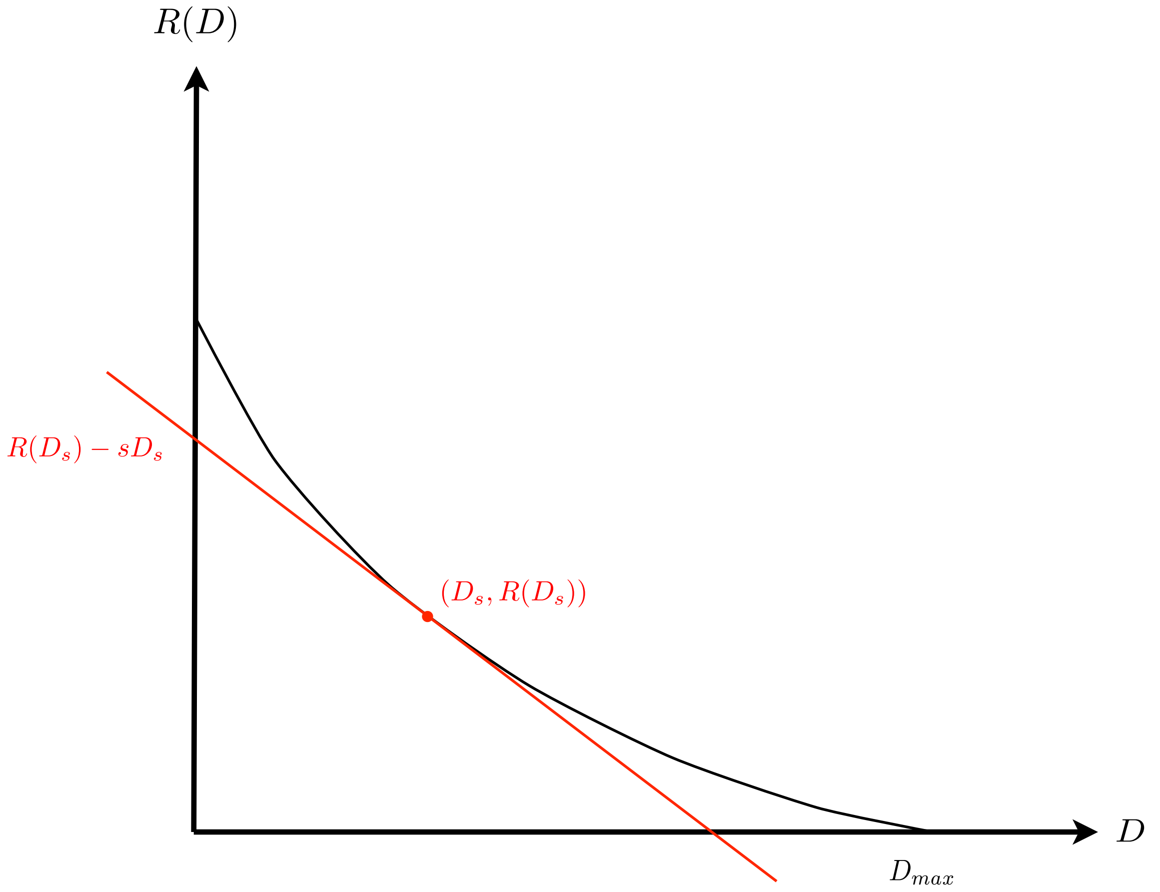
3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.



The $R(D)$ Curve

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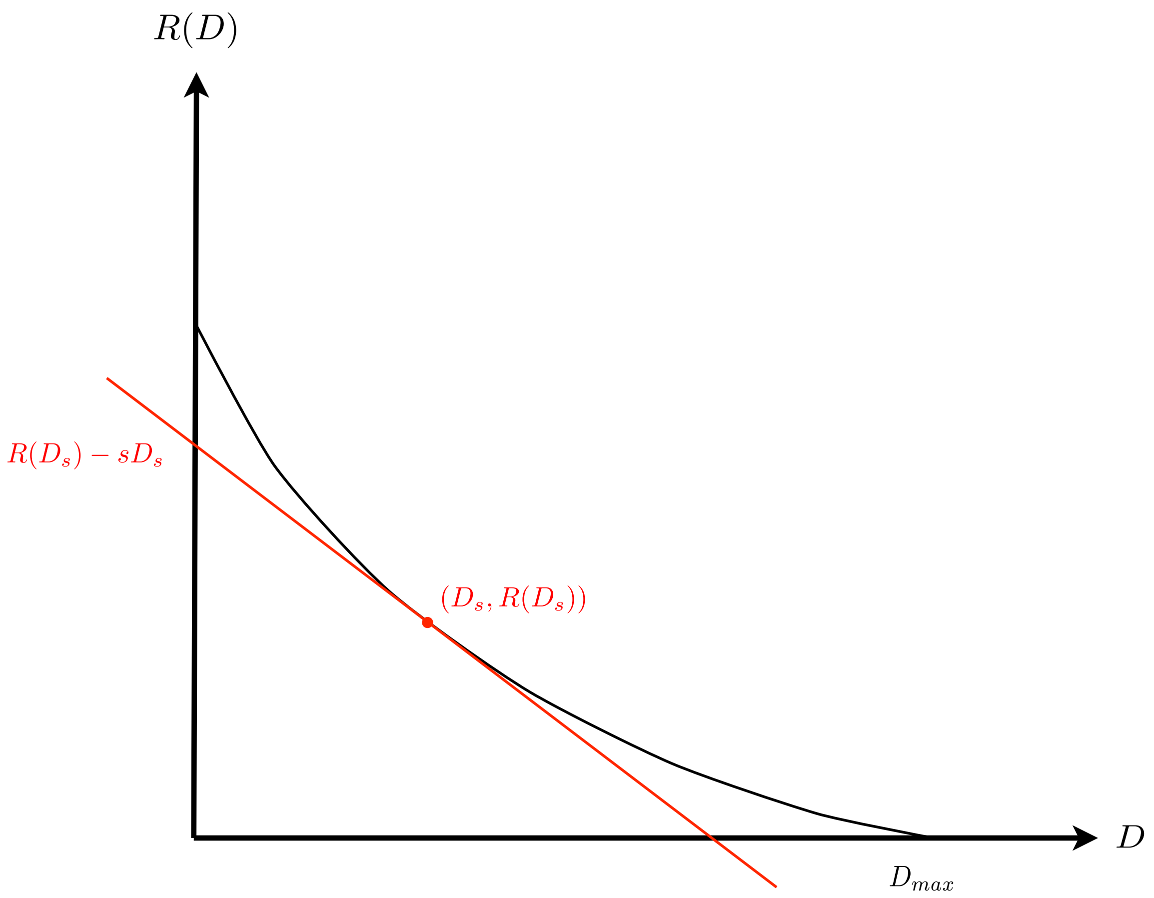
3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

5. Write $I(X;\hat{X})$ and $Ed(X,\hat{X})$ as $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$, respectively, where $\mathbf{p}$ is the distribution for X and $\mathbf{Q}$ is the transition matrix from $\mathcal{X}$ to $\hat{\mathcal{X}}$ defining $\hat{X}$.



The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \le D \le D_{max}$.

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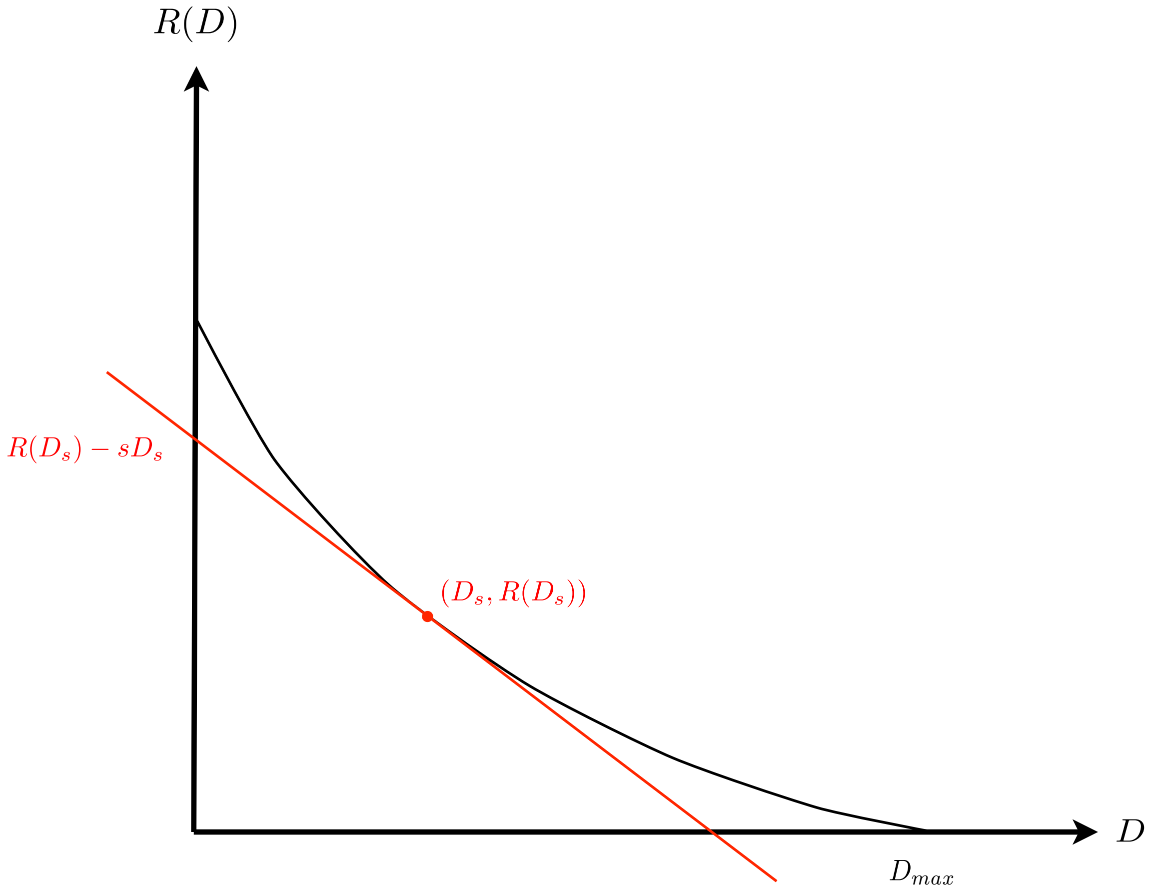
3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

5. Write $I(X;\hat{X})$ and $Ed(X,\hat{X})$ as $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$, respectively, where $\mathbf{p}$ is the distribution for X and $\mathbf{Q}$ is the transition matrix from $\mathcal{X}$ to $\hat{\mathcal{X}}$ defining $\hat{X}$.

6. For any $\mathbf{Q}$, $(D(\mathbf{p},\mathbf{Q}), I(\mathbf{p},\mathbf{Q}))$ is a point in the rate-distortion region, and the line with slope s passing through $(D(\mathbf{p},\mathbf{Q}), I(\mathbf{p},\mathbf{Q}))$ intersects the ordinate at $I(\mathbf{p},\mathbf{Q}) - sD(\mathbf{p},\mathbf{Q})$.



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3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

5. Write $I(X;\hat{X})$ and $Ed(X,\hat{X})$ as $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$, respectively, where $\mathbf{p}$ is the distribution for X and $\mathbf{Q}$ is the transition matrix from $\mathcal{X}$ to $\hat{\mathcal{X}}$ defining $\hat{X}$.

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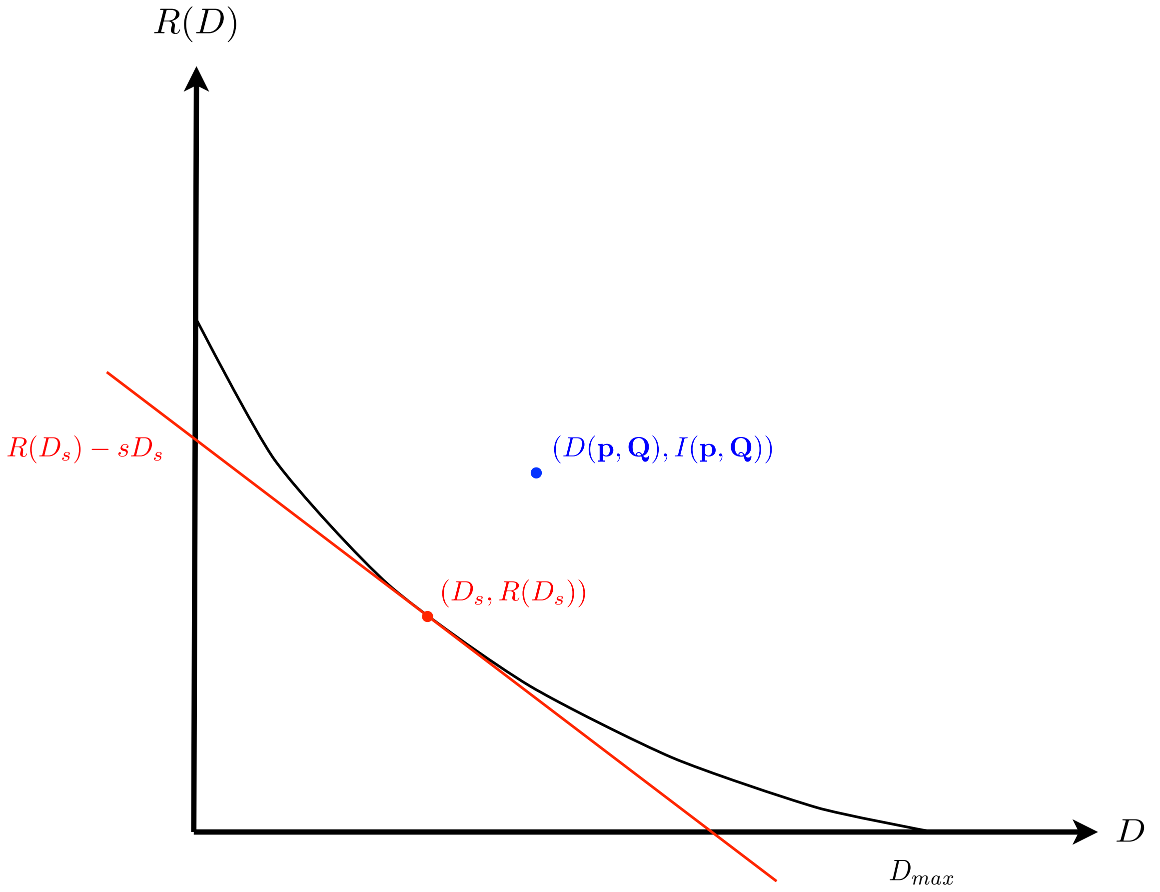

The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \le D \le D_{max}$.

2. Since $R(D)$ is convex, for any $s \le 0$, there exists a point on the $R(D)$ curve for $0 \le D \le D_{max}$ such that the slope of a tangent to the $R(D)$ curve at that point is equal to s.

3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

5. Write $I(X;\hat{X})$ and $Ed(X,\hat{X})$ as $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$, respectively, where $\mathbf{p}$ is the distribution for X and $\mathbf{Q}$ is the transition matrix from $\mathcal{X}$ to $\hat{\mathcal{X}}$ defining $\hat{X}$.

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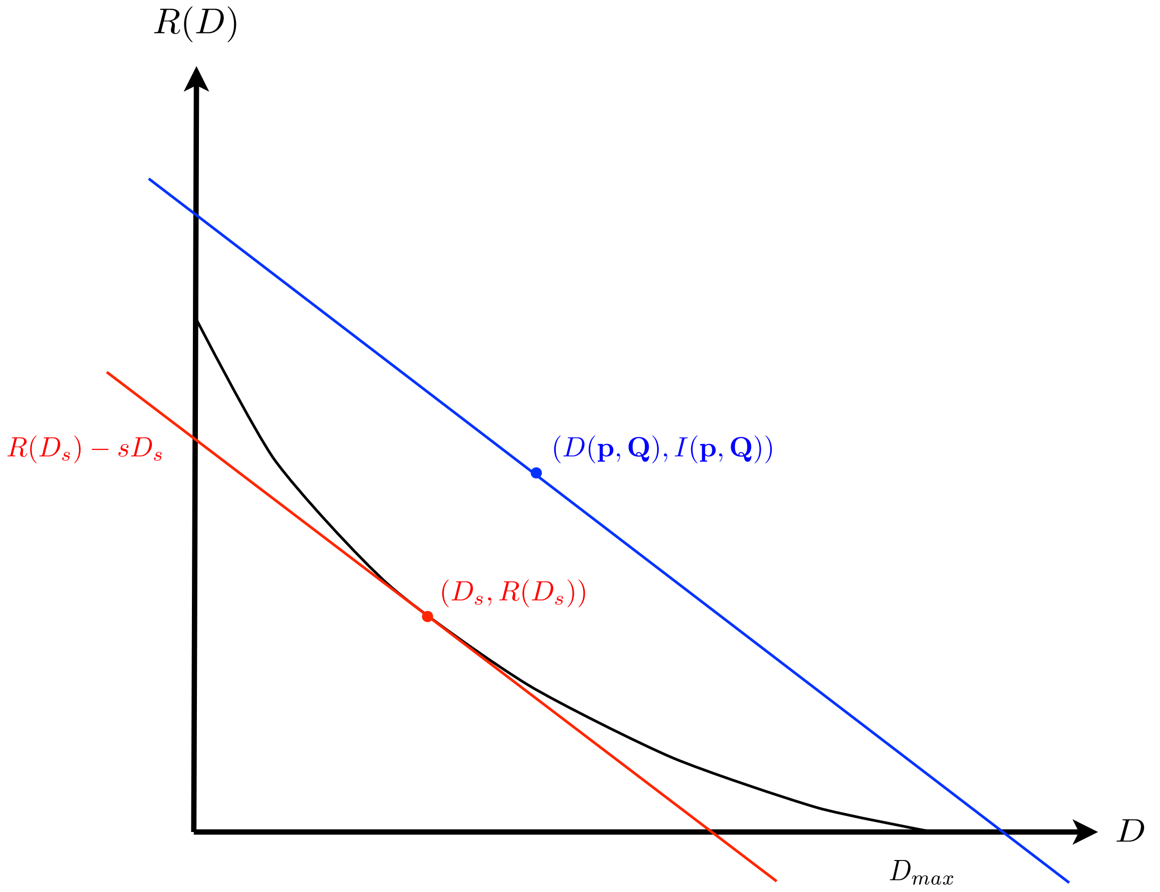


The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \leq D \leq D_{max}$.

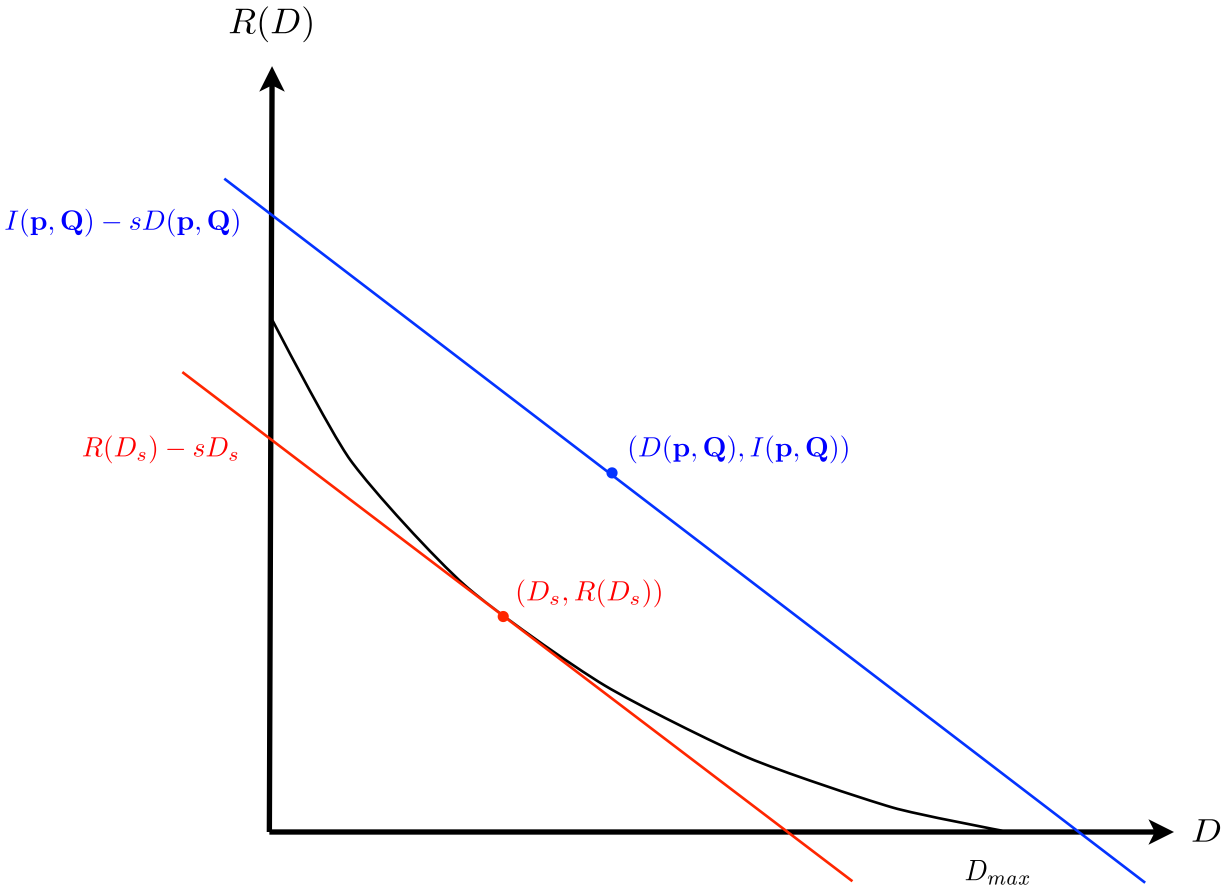
2. Since $R(D)$ is convex, for any $s \leq 0$, there exists a point on the $R(D)$ curve for $0 \leq D \leq D_{max}$ such that the slope of a tangent to the $R(D)$ curve at that point is equal to s.

3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

5. Write $I(X;\hat{X})$ and $Ed(X,\hat{X})$ as $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$, respectively, where $\mathbf{p}$ is the distribution for X and $\mathbf{Q}$ is the transition matrix from $\mathcal{X}$ to $\hat{\mathcal{X}}$ defining $\hat{X}$.

6. For any $\mathbf{Q}$, $(D(\mathbf{p},\mathbf{Q}), I(\mathbf{p},\mathbf{Q}))$ is a point in the rate-distortion region, and the line with slope s passing through $(D(\mathbf{p},\mathbf{Q}), I(\mathbf{p},\mathbf{Q}))$ intersects the ordinate at $I(\mathbf{p},\mathbf{Q}) - sD(\mathbf{p},\mathbf{Q})$.



The $R(D)$ Curve

1. Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \leq D \leq D_{max}$.

2. Since $R(D)$ is convex, for any $s \leq 0$, there exists a point on the $R(D)$ curve for $0 \leq D \leq D_{max}$ such that the slope of a tangent to the $R(D)$ curve at that point is equal to s.

3. Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.

4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

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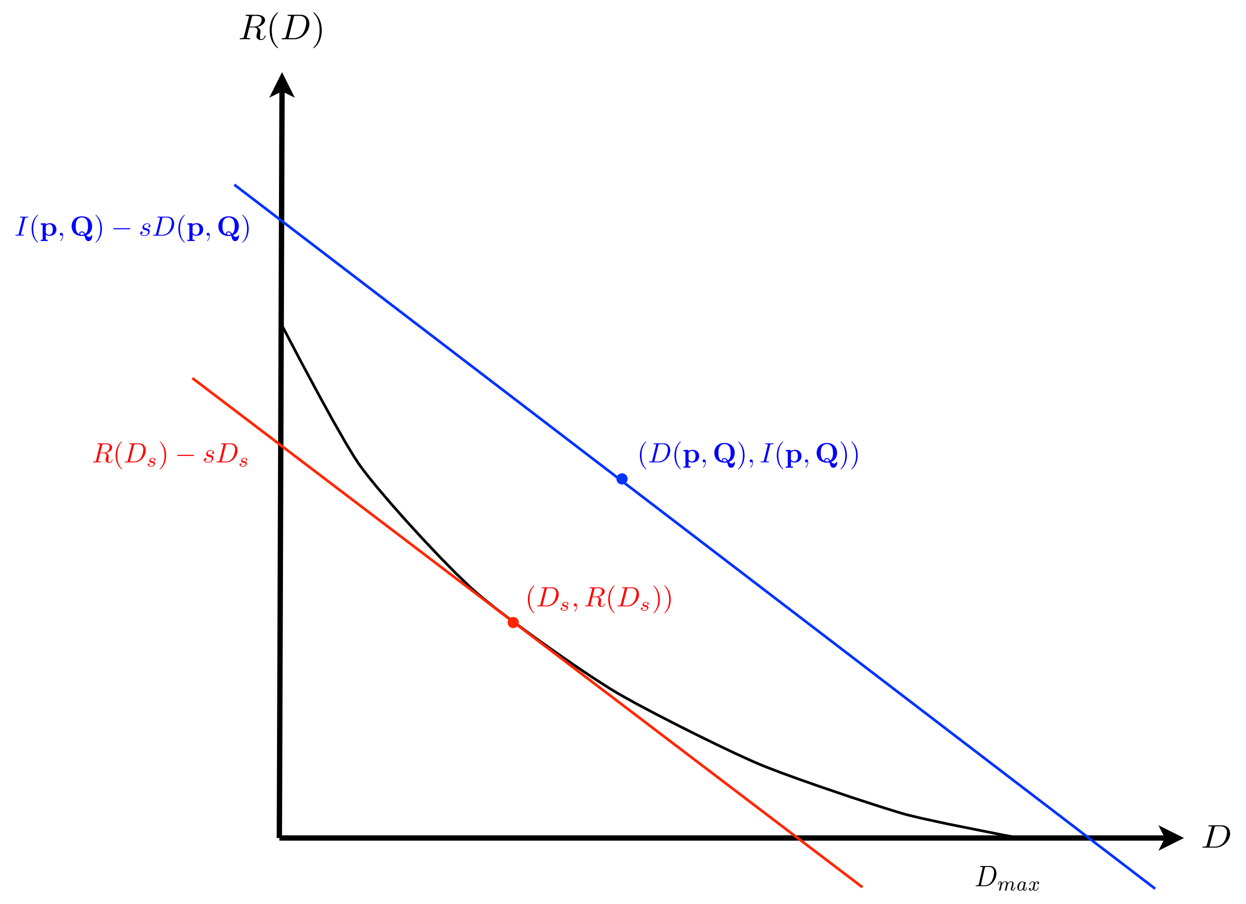
7. Then

$$R(D_s) - sD_s = \min_{\mathbf{Q}} [I(\mathbf{p},\mathbf{Q}) - sD(\mathbf{p},\mathbf{Q})].$$



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4. Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.

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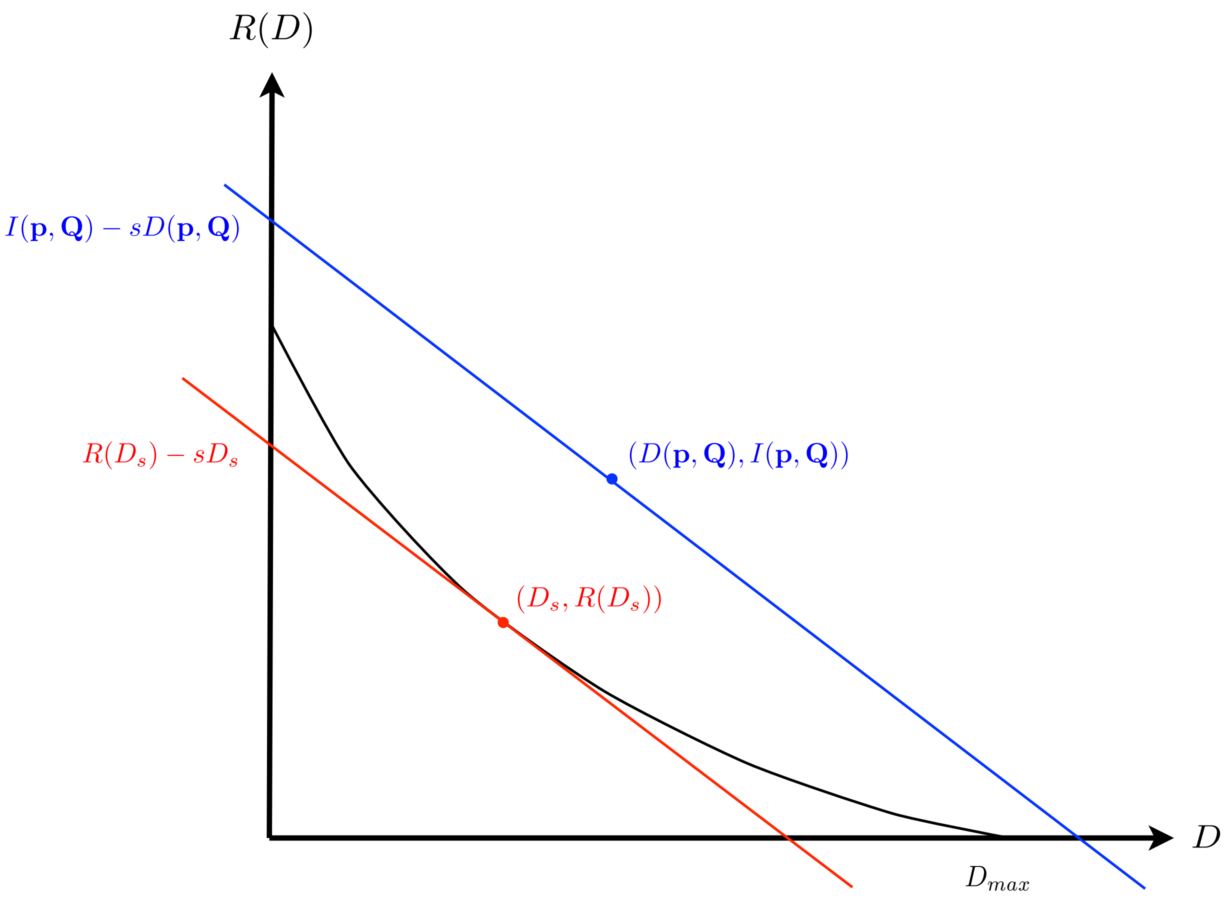
7. Then

$$R(D_s) - sD_s = \min_{\mathbf{Q}} [I(\mathbf{p}, \mathbf{Q}) - sD(\mathbf{p}, \mathbf{Q})].$$

8. By varying over all $s \le 0$, we can then trace out the whole $R(D)$ curve.



Lemma 9.3 Let $p(x)Q(\hat{x}|x)$ be a given joint distribution on $\mathcal{X} \times \hat{\mathcal{X}}$ such that $\mathbf{Q} > 0$, and let $\mathbf{t}$ be any distribution on $\hat{\mathcal{X}}$ such that $\mathbf{t} > 0$. Then

$$\min_{\mathbf{t}>0} \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} = \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t^*(\hat{x})},$$

where

$$t^*(\hat{x}) = \sum_x p(x)Q(\hat{x}|x),$$

i.e., the minimizing $\mathbf{t}$ is the one which corresponds to the input distribution $\mathbf{p}$ and the transition matrix $\mathbf{Q}$.

Proof

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Proof

- Similar to Lemma 9.1.

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Proof

- Similar to Lemma 9.1.
- Note that $\mathbf{t}^* > 0$ because $\mathbf{Q} > 0$, so that it suffices to minimize over all $\mathbf{t} > 0$ instead of $\mathbf{t} \geq 0$.

Lemma 9.3 Let $p(x)Q(\hat{x}|x)$ be a given joint distribution on $\mathcal{X} \times \hat{\mathcal{X}}$ such that $\mathbf{Q} > 0$, and let $\mathbf{t}$ be any distribution on $\hat{\mathcal{X}}$ such that $\mathbf{t} > 0$. Then

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$$\boxed{R(D_S) - sD_S = \min_{\mathbf{Q}} \left[I(\mathbf{p}, \mathbf{Q}) - sD(\mathbf{p}, \mathbf{Q}) \right].}$$

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$R(D_s) - sD_s$

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&\quad = \inf_{\mathbf{Q}>0} \min_{\mathbf{t}>0} \left[\sum_{x,\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_{x,\hat{x}} p(x)Q(\hat{x}|x)d(x, \hat{x}) \right].
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The BA Algorithm for Computing $R(D_s) - sD_s$

The BA Algorithm for Computing $R(D_s)$ - sD_s

Recall the double infimum in Section 9.1:

$$\inf_{\mathbf{u}_1 \in A_1} \inf_{\mathbf{u}_2 \in A_2} f(\mathbf{u}_1, \mathbf{u}_2).$$

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Cast the computation of $R(D_s) - sD_s$ into this optimization problem:

1. Let

$$f(\mathbf{Q}, \mathbf{t}) = \underbrace{\sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})}}_{\geq I(\mathbf{p}, \mathbf{Q}) \geq 0} - s \underline{\sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) d(x, \hat{x})},$$

where $\mathbf{u}_1 \leftarrow \mathbf{Q}$ and $\mathbf{u}_2 \leftarrow \mathbf{t}$.

2. Let

$$A_1 = \left\{ (Q(\hat{x}|x), (x, \hat{x}) \in \mathcal{X} \times \hat{\mathcal{X}}) : Q(\hat{x}|x) > 0, \sum_{\hat{x}} Q(\hat{x}|x) = 1 \text{ for all } x \in \mathcal{X} \right\} \subset \Re^{|\mathcal{X}||\hat{\mathcal{X}}|}$$

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Remarks

1. Both A_1 and A_2 are convex.
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where $\mathbf{u}_1 \leftarrow \mathbf{Q}$ and $\mathbf{u}_2 \leftarrow \mathbf{t}$.

2. Let

$$\begin{aligned} A_1 &= \Big\{ (Q(\hat{x}|x), (x, \hat{x}) \in \mathcal{X} \times \hat{\mathcal{X}}) : Q(\hat{x}|x) > 0, \\ &\quad \sum_{\hat{x}} Q(\hat{x}|x) = 1 \text{ for all } x \in \mathcal{X} \Big\} \subset \Re^{|\mathcal{X}||\hat{\mathcal{X}}|} \end{aligned}$$

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$$A_1 = \Big\{ (Q(\hat{x}|x), (x, \hat{x}) \in \mathcal{X} \times \hat{\mathcal{X}}) : Q(\hat{x}|x) > 0, \sum_{\hat{x}} Q(\hat{x}|x) = 1 \text{ for all } x \in \mathcal{X} \Big\} \subset \Re^{|\mathcal{X}||\hat{\mathcal{X}}|}$$

and

$$A_2 = \{ (t(\hat{x}), \hat{x} \in \hat{\mathcal{X}}) : t(\hat{x}) > 0 \}.$$

3. The double infimum now becomes

$$\inf_{\mathbf{Q} \in A_1} \inf_{\mathbf{t} \in A_2} f(\mathbf{Q}, \mathbf{t}) = \inf_{\mathbf{Q} \in A_1} \inf_{\mathbf{t} \in A_2} \Bigg[\sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) d(x, \hat{x}) \Bigg],$$

where the infimum over all $\mathbf{t} \in A_2$ is in fact a minimum, and

$$f^* = \inf_{\mathbf{Q} \in A_1} \inf_{\mathbf{t} \in A_2} f(\mathbf{Q}, \mathbf{t}) = R(D_s) - sD_s.$$

Algorithm Details

$$f^* = \inf_{\mathbf{Q} \in A_1} \inf_{\mathbf{t} \in A_2} f(\mathbf{Q}, \mathbf{t})$$

$$f(\mathbf{Q}, \mathbf{t}) = \sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) d(x, \hat{x}),$$

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1. By Lemma 9.3, for any given $\mathbf{Q} \in A_1$, the unique $\mathbf{t} \in A_2$ that minimizes f is given by

Algorithm Details

$$\boxed{f^* = \inf_{\mathbf{Q} \in A_1} \inf_{\mathbf{t} \in A_2} f(\mathbf{Q}, \mathbf{t})}$$

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1. By Lemma 9.3, for any given $\mathbf{Q} \in A_1$, the unique $\mathbf{t} \in A_2$ that minimizes f is given by

$$t^*(\hat{x}) = \sum_x p(x) Q(\hat{x}|x). \tag{1}$$

Algorithm Details

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2. By Lagrange multipliers, it can be shown that for a given $\mathbf{t} \in A_2$, the transition matrix $\mathbf{Q}$ that minimizes f is given by

Algorithm Details

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$$Q(\hat{x}|x) = \frac{t(\hat{x}) e^{s d(x, \hat{x})}}{\sum_{\hat{x}'} t(\hat{x}') e^{s d(x, \hat{x}')}} > 0, \tag{2}$$

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and so $\mathbf{Q} \in A_1$.

Algorithm Details

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and so $\mathbf{Q} \in A_1$.

3. Let $\mathbf{Q}^{(0)}$ be an arbitrarily chosen strictly positive transition matrix in A_1. Then $\mathbf{t}^{(0)} \in A_2$ can be determined accordingly. This forms $(\mathbf{Q}^{(0)}, \mathbf{t}^{(0)})$.

Algorithm Details

$$f^* = \inf_{\mathbf{Q}\in A_1} \inf_{\mathbf{t}\in A_2} f(\mathbf{Q}, \mathbf{t})$$

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1. By Lemma 9.3, for any given $\mathbf{Q} \in A_1$, the unique $\mathbf{t} \in A_2$ that minimizes f is given by

$$t^*(\hat{x}) = \sum_x p(x)Q(\hat{x}|x). \qquad (1)$$

2. By Lagrange multipliers, it can be shown that for a given $\mathbf{t} \in A_2$, the transition matrix $\mathbf{Q}$ that minimizes f is given by

$$Q(\hat{x}|x) = \frac{t(\hat{x})e^{sd(x,\hat{x})}}{\sum_{\hat{x}'} t(\hat{x}')e^{sd(x,\hat{x}')}} > 0, \qquad (2)$$

and so $\mathbf{Q} \in A_1$.

3. Let $\mathbf{Q}^{(0)}$ be an arbitrarily chosen strictly positive transition matrix in A_1. Then $\mathbf{t}^{(0)} \in A_2$ can be determined accordingly. This forms $(\mathbf{Q}^{(0)}, \mathbf{t}^{(0)})$.

4. Compute $\mathbf{Q}^{(1)}, \mathbf{t}^{(1)}, \mathbf{Q}^{(2)}, \mathbf{t}^{(2)}, \cdots$ iteratively by applying (2) and (1) alternately.

Algorithm Details

$$f^* = \inf_{\mathbf{Q}\in A_1} \inf_{\mathbf{t}\in A_2} f(\mathbf{Q},\mathbf{t})$$

$$f(\mathbf{Q},\mathbf{t}) = \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x)\log\frac{Q(\hat{x}|x)}{t(\hat{x})} - s\sum_x\sum_{\hat{x}} p(x)Q(\hat{x}|x)d(x,\hat{x}),$$

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3. Let $\mathbf{Q}^{(0)}$ be an arbitrarily chosen strictly positive transition matrix in A_1. Then $\mathbf{t}^{(0)} \in A_2$ can be determined accordingly. This forms $(\mathbf{Q}^{(0)}, \mathbf{t}^{(0)})$.

4. Compute $\mathbf{Q}^{(1)}$, $\mathbf{t}^{(1)}$, $\mathbf{Q}^{(2)}$, $\mathbf{t}^{(2)}$, $\cdots$ iteratively by applying (2) and (1) alternately.

5. It will be shown in Section 9.3 that $f^{(k)} = f(\mathbf{Q}^{(k)}, \mathbf{t}^{(k)}) \to f^* = R(D_s) - sD_s$.