香港中文大學
The Chinese University of Hong Kong

Chapter 9
The Blahut-Arimoto Algorithms

The Chinese University of Hong Kong

Single-Letter Characterization

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- For a DMC $p(y|x)$, the capacity

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- This characterization of C, in the form of an optimization problem, is called a single-letter characterization because it involves only $p(y|x)$ but not n.

- Similarly, the rate-distortion function

$$R(D) = \min_{Q(\hat{x}|x): Ed(X,\hat{X}) \le D} I(X;\hat{X})$$

for an i.i.d. information source $\{X_k\}$ is a single-letter characterization.

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- So we have to resort to numerical methods.
- The Blahut-Arimoto (BA) algorithms are iterative algorithms devised for this purpose.

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9.1 Alternating Optimization

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    and for all $\mathbf{u}_1 \in A_1$, there exists a unique $c_2(\mathbf{u}_1) \in A_2$ such that

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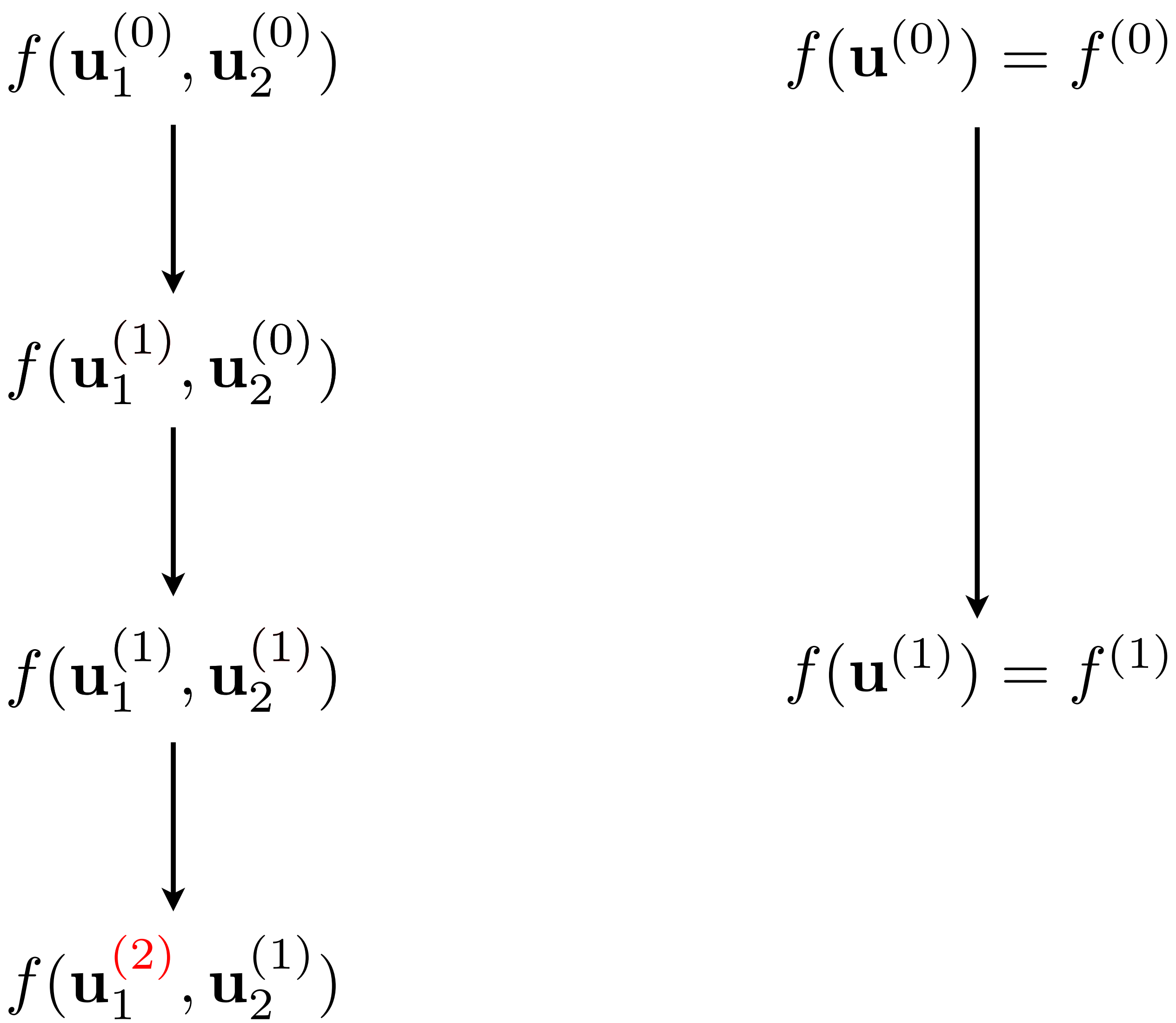

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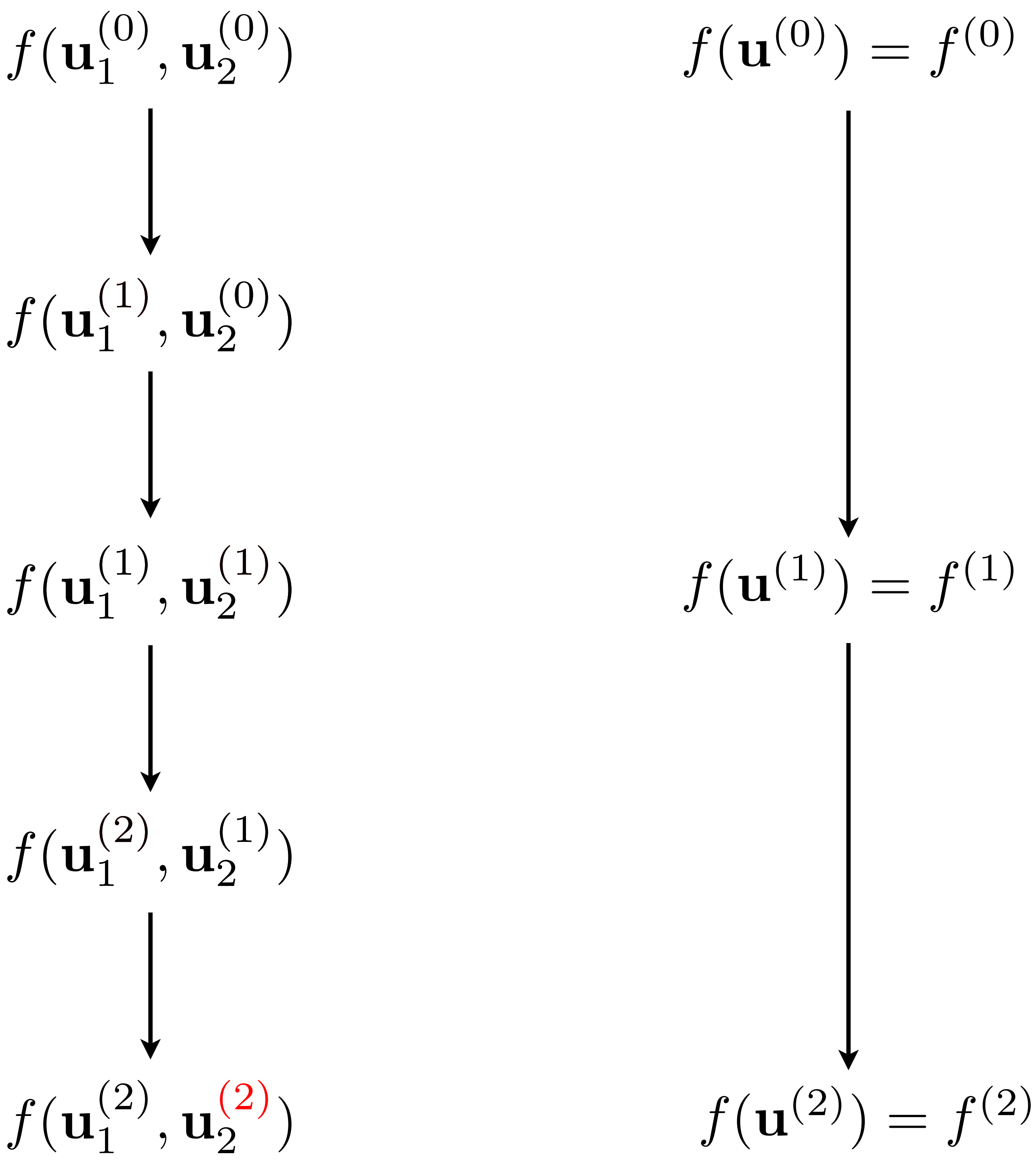

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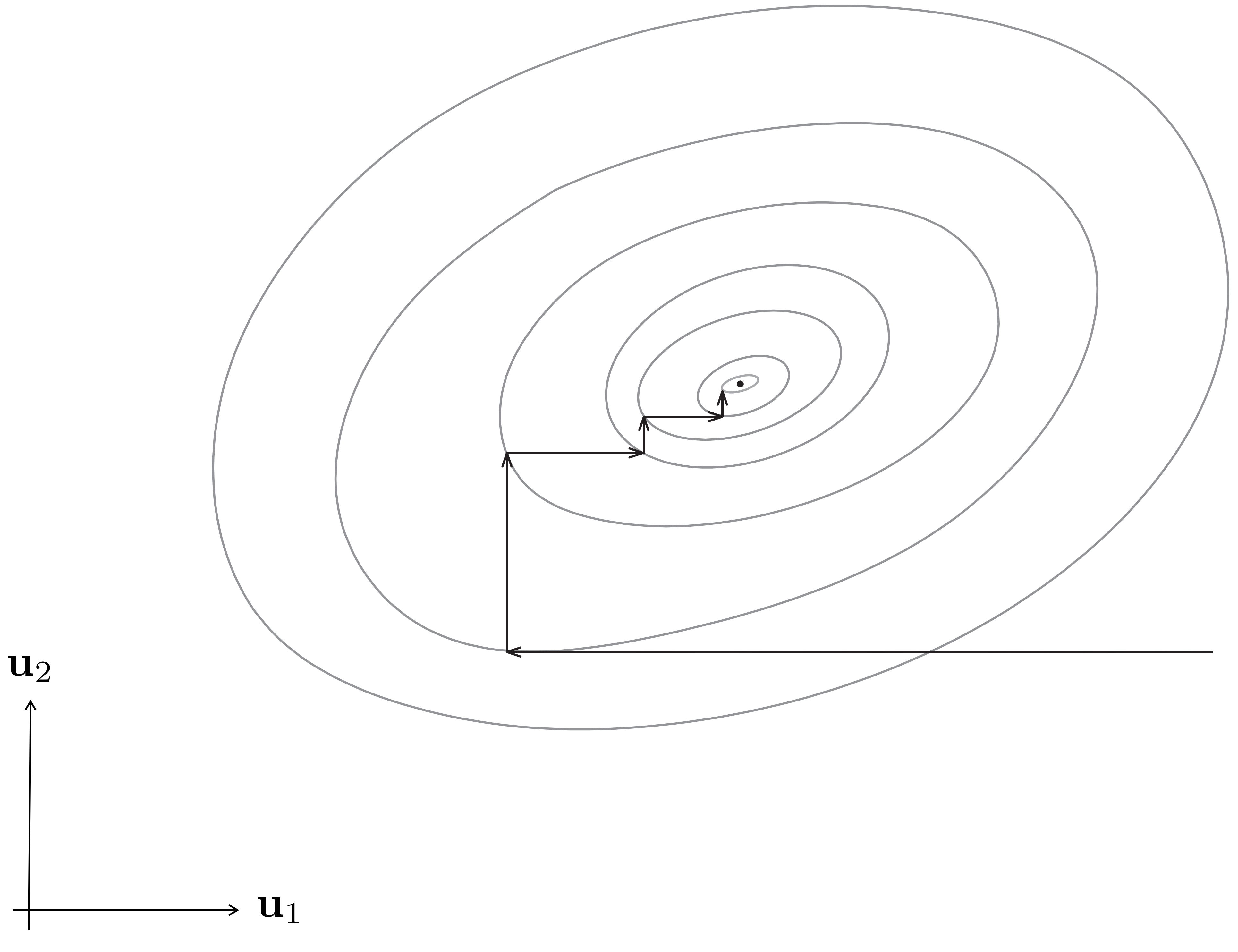
- The alternating optimization algorithm will be specialized for computing C and $R(D)$.



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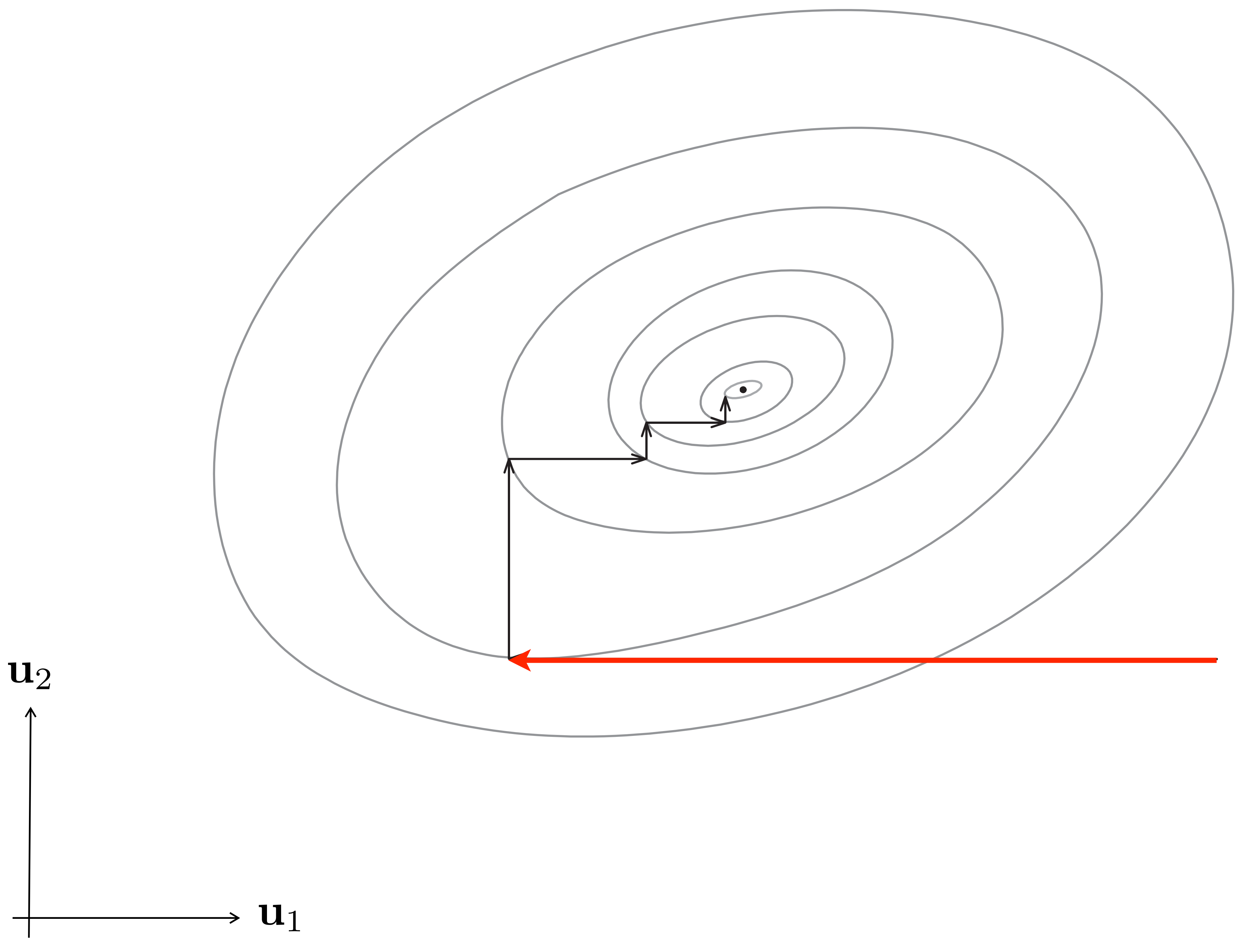
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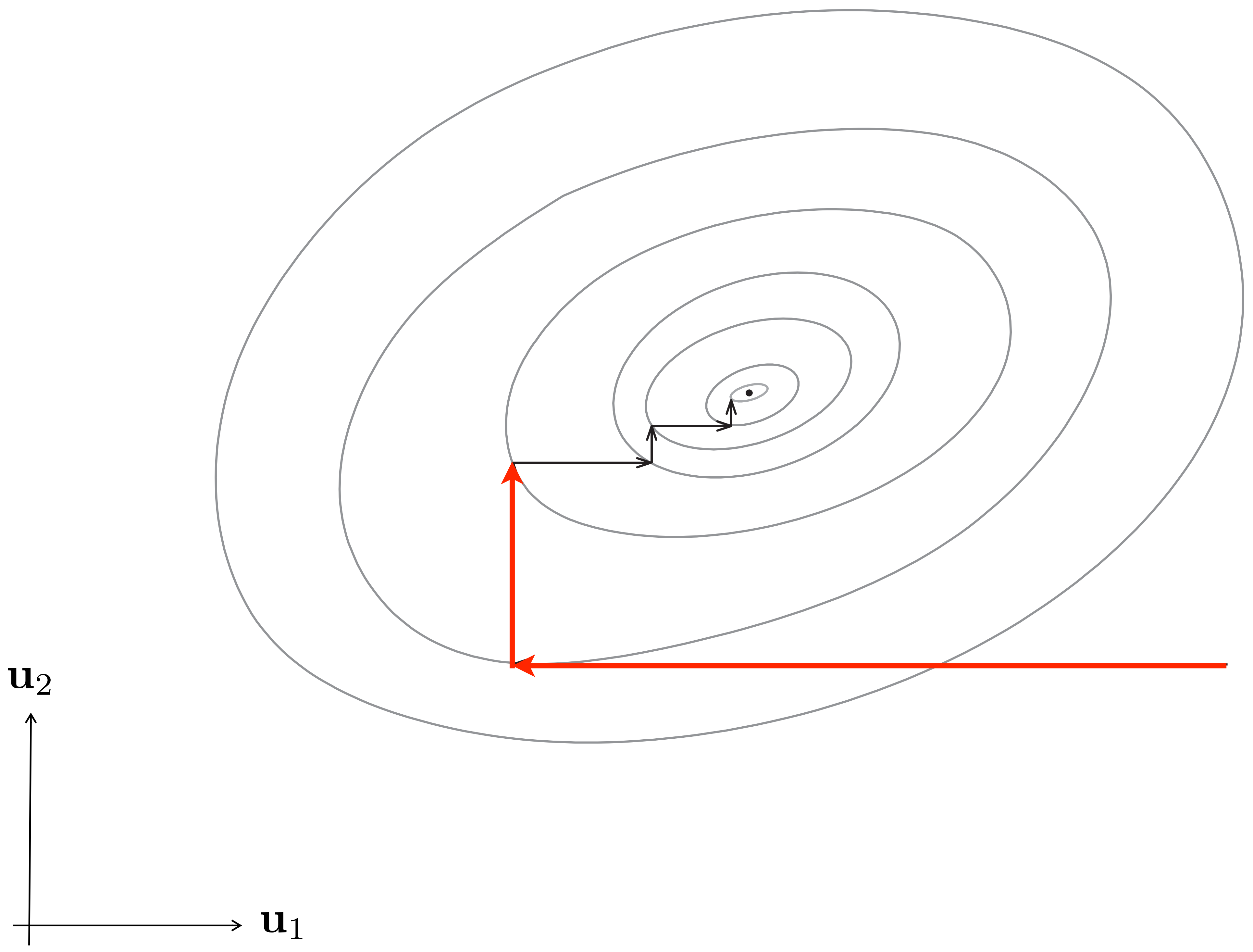

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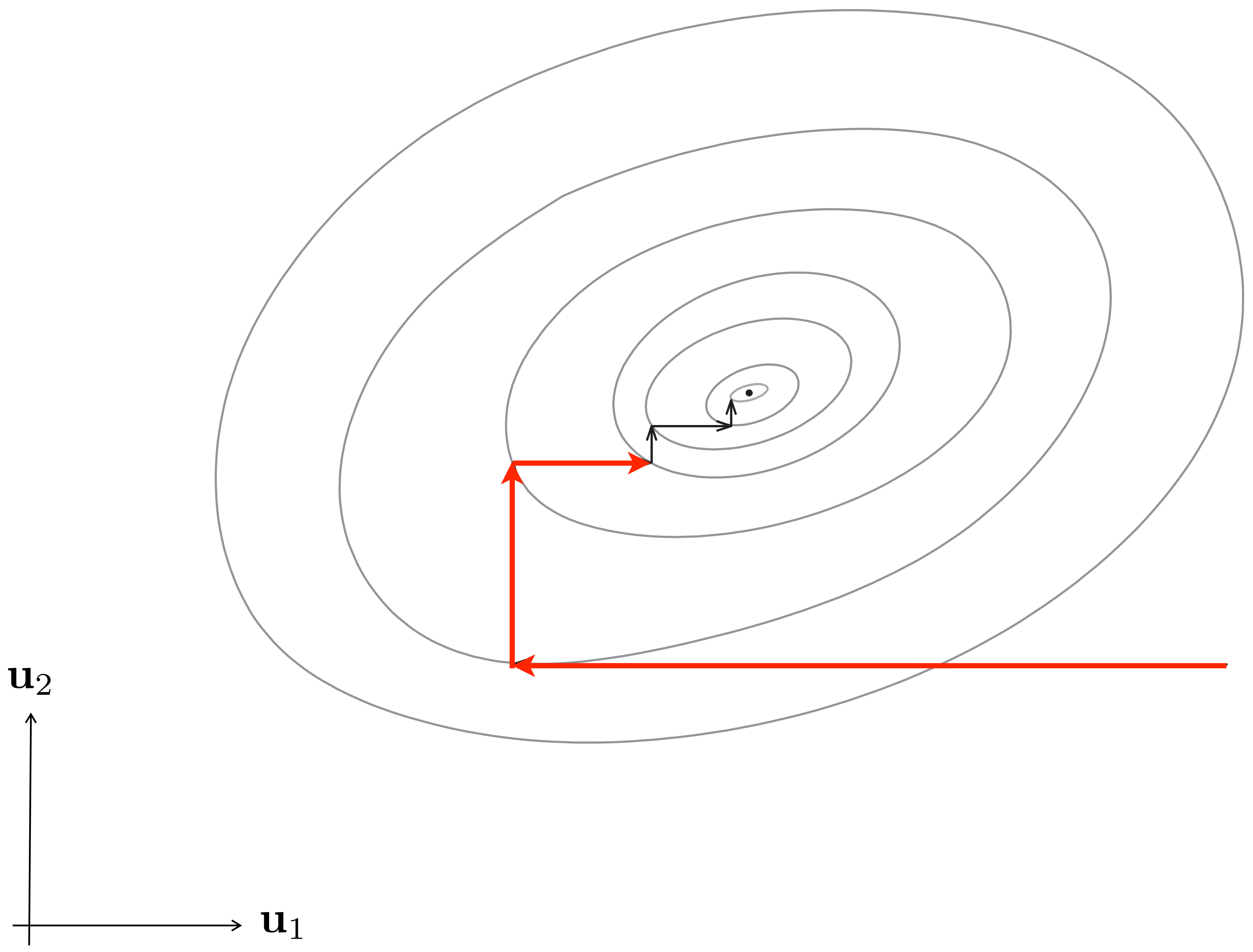

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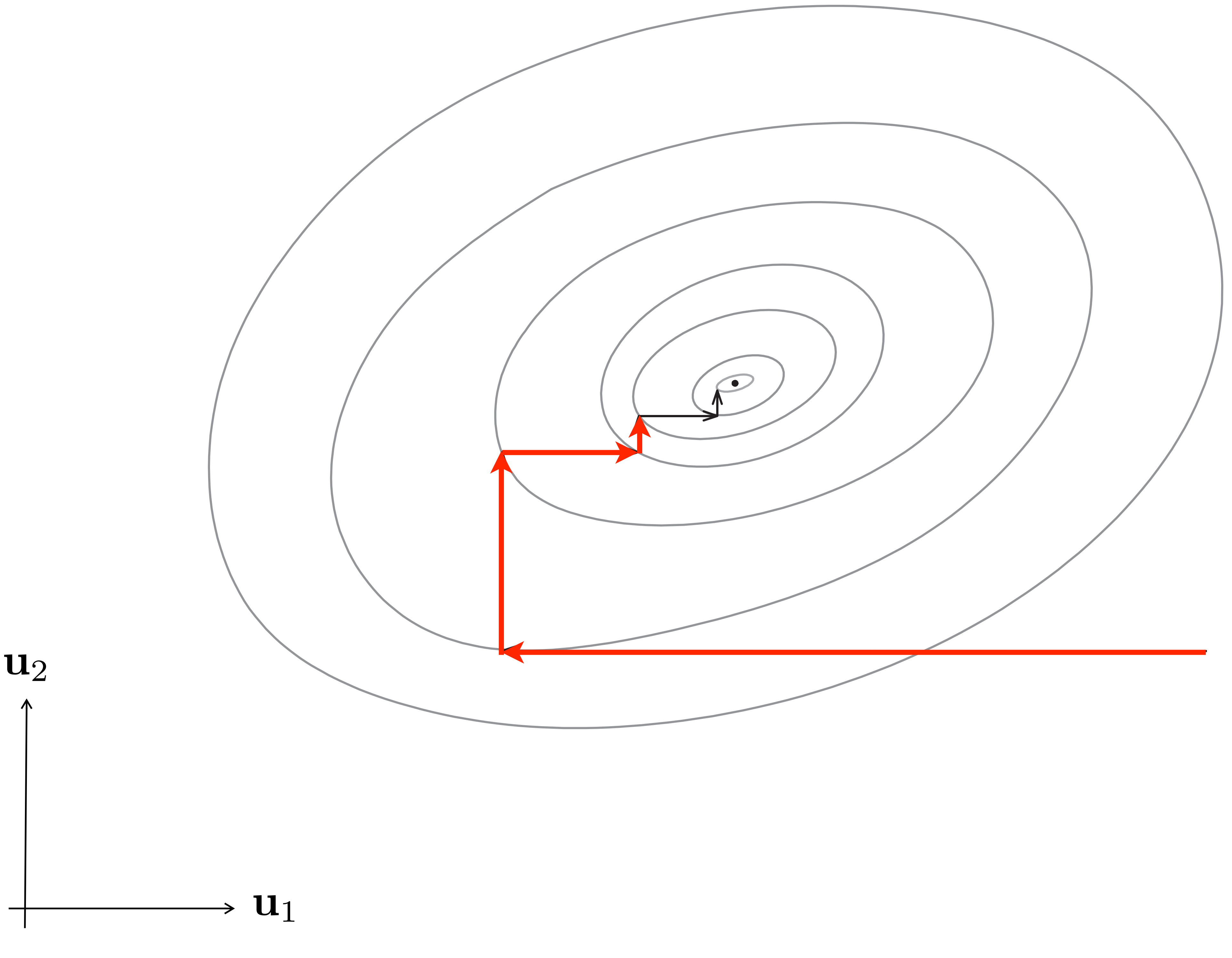
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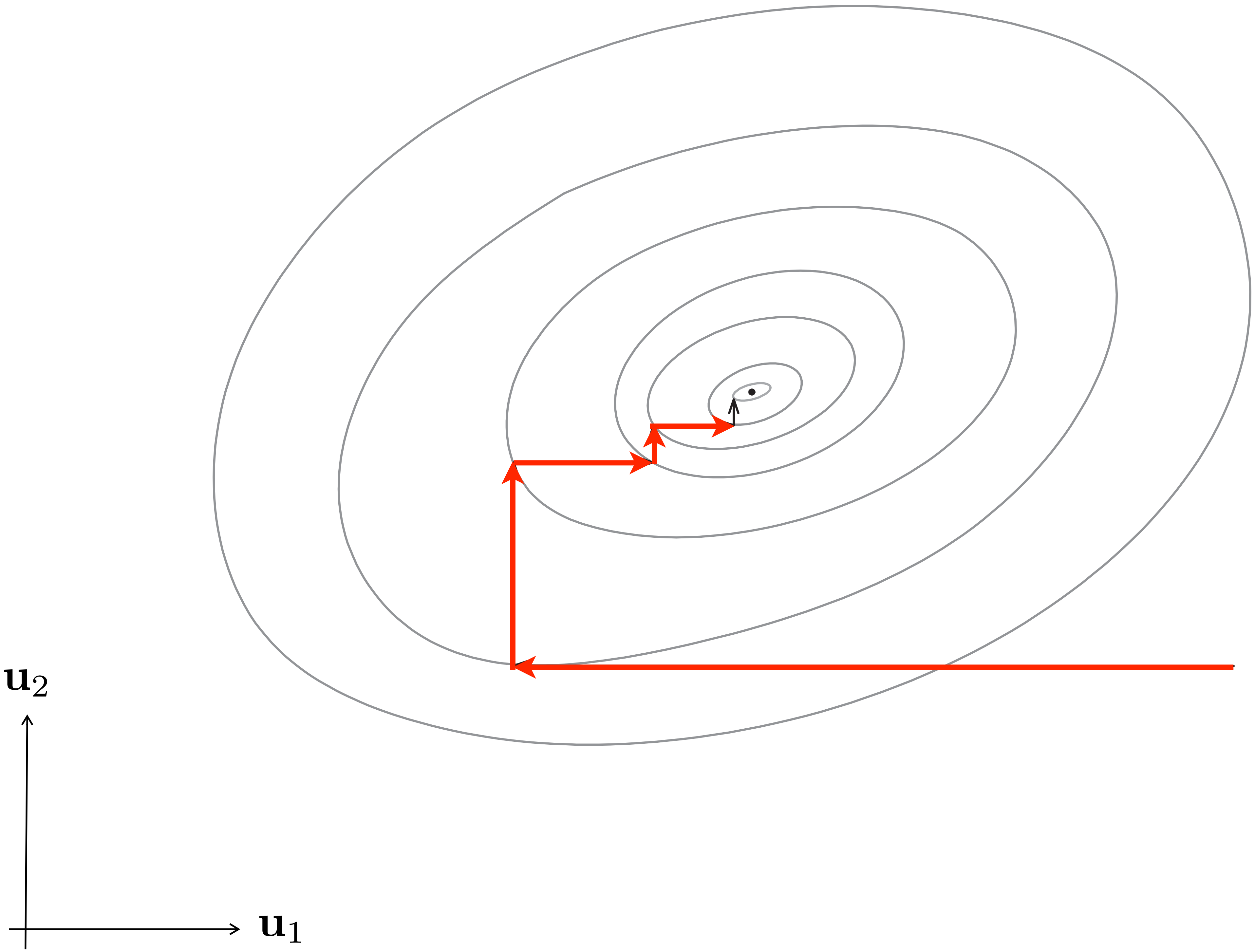

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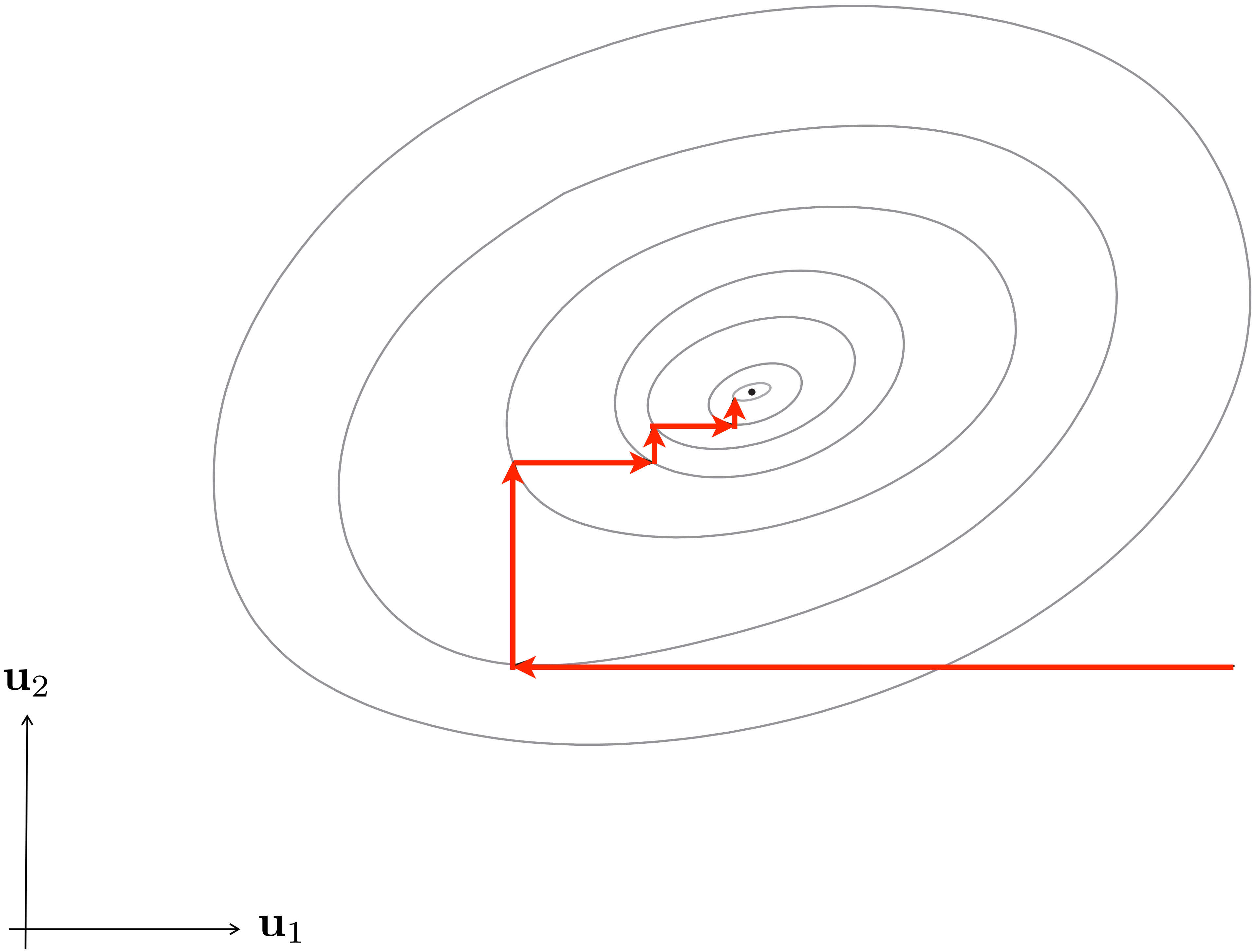

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