8.5 Achievability of $R_I(D)$

How to Prove Achievability

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- This implies that $R_I(D) \geq R(D)$.

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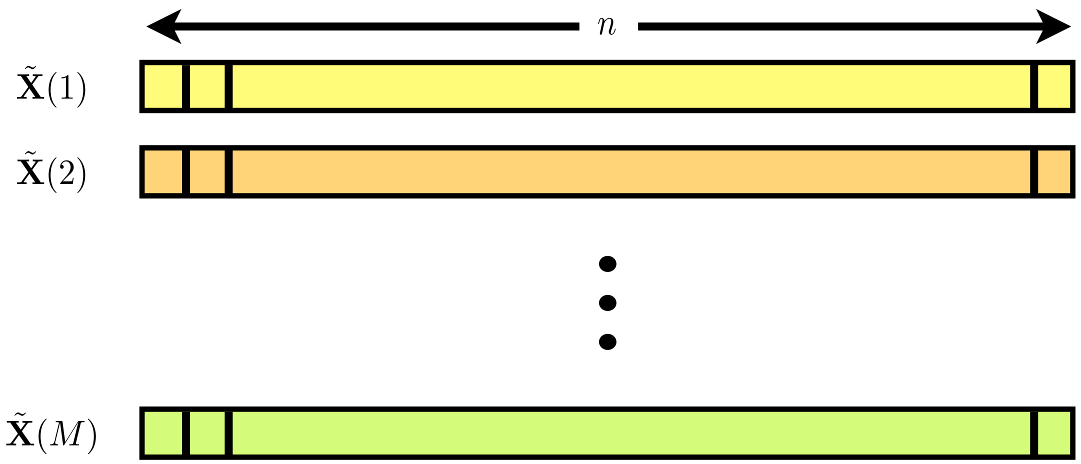

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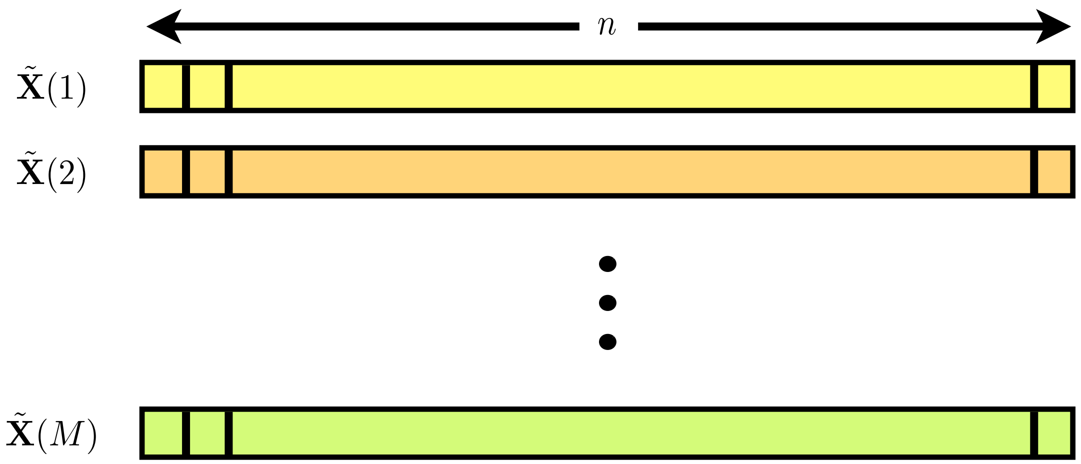

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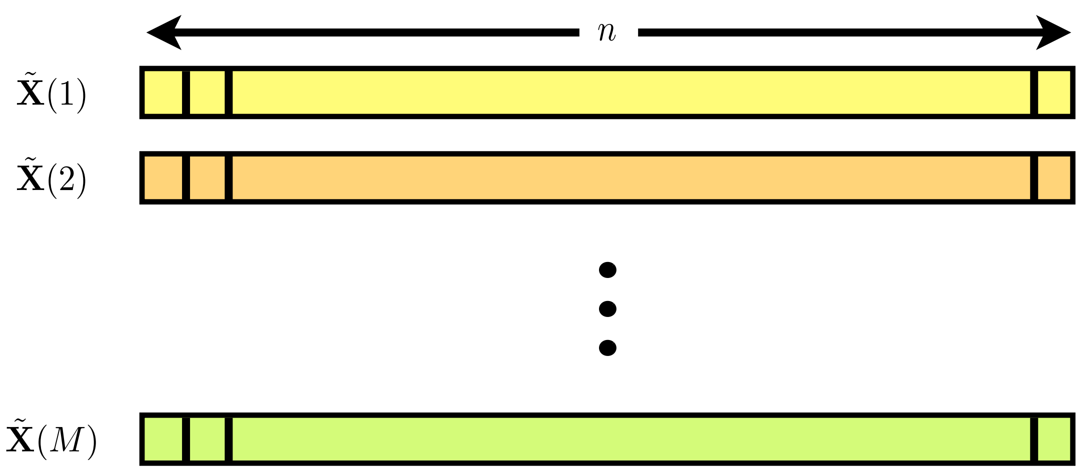

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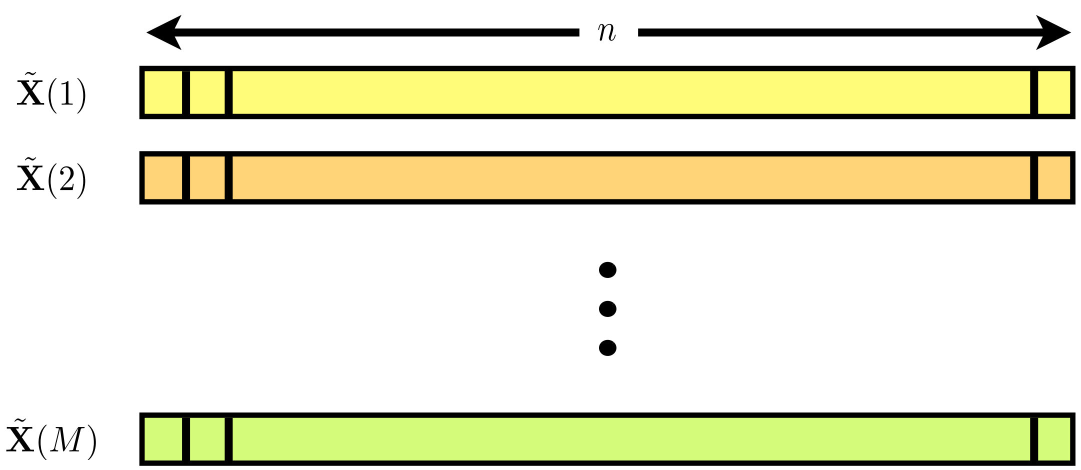

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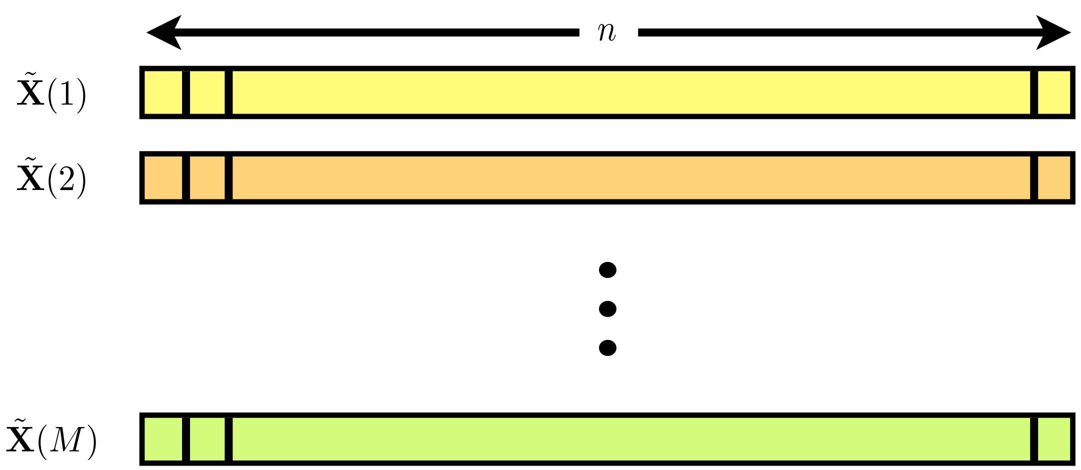

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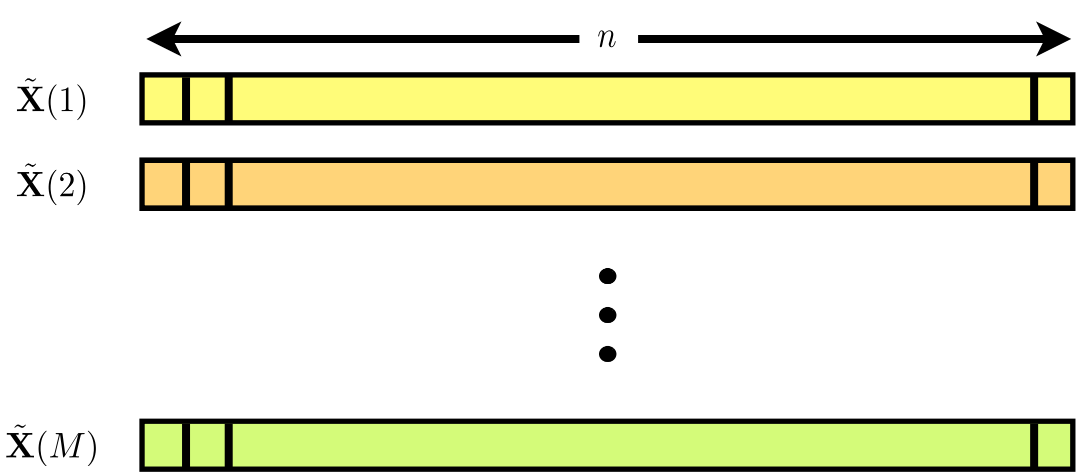

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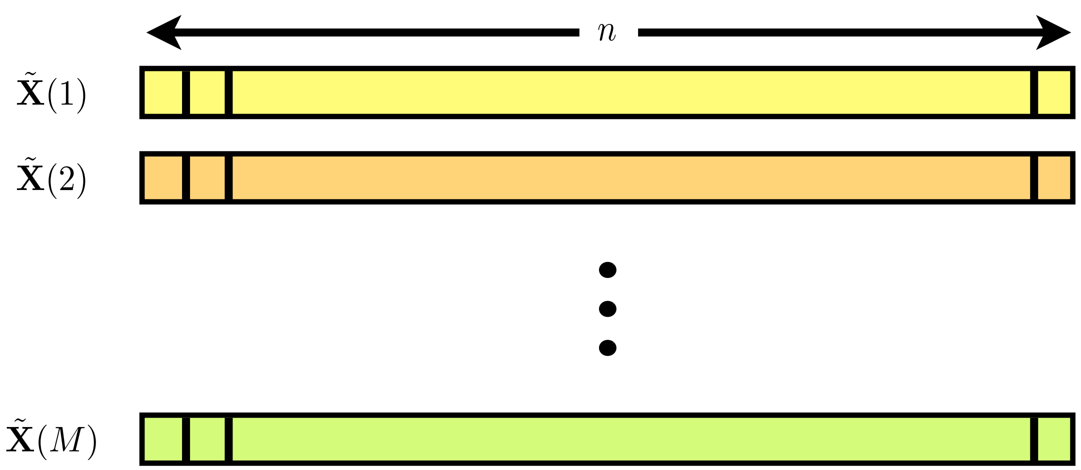

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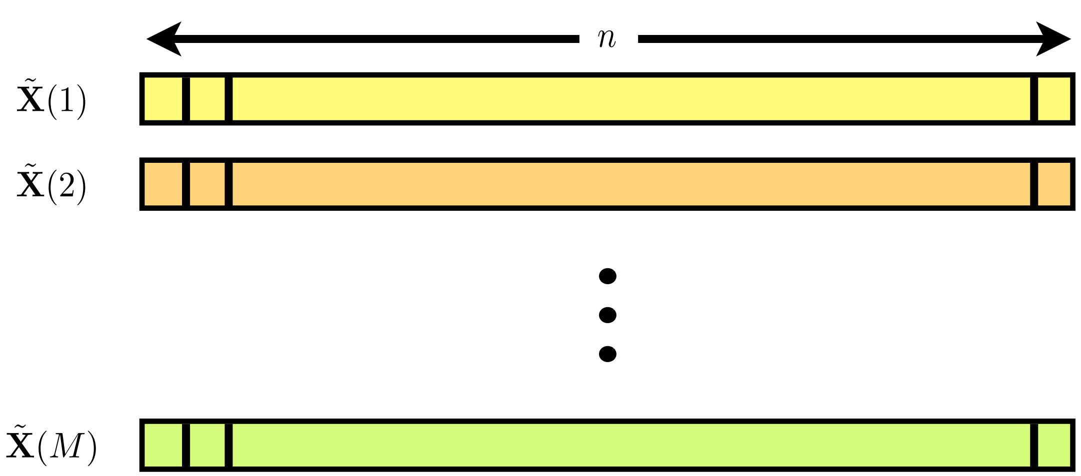

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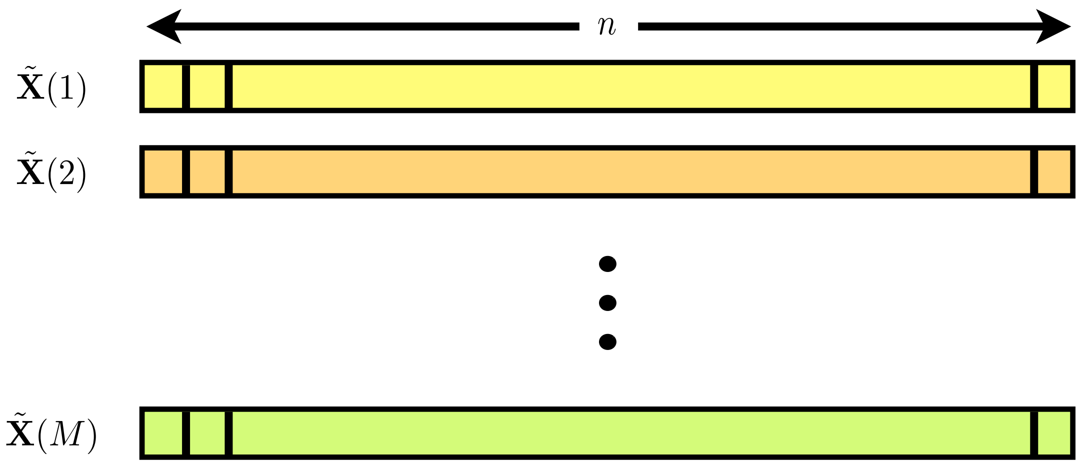

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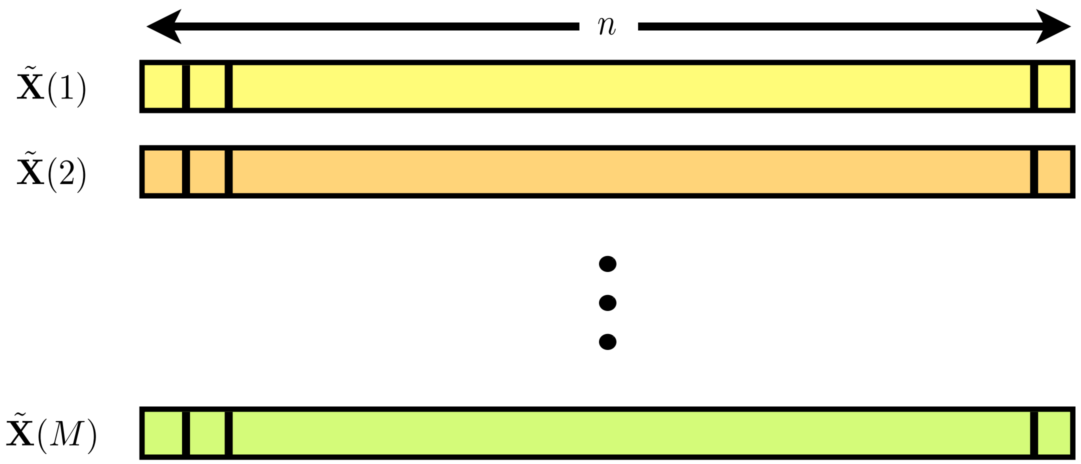

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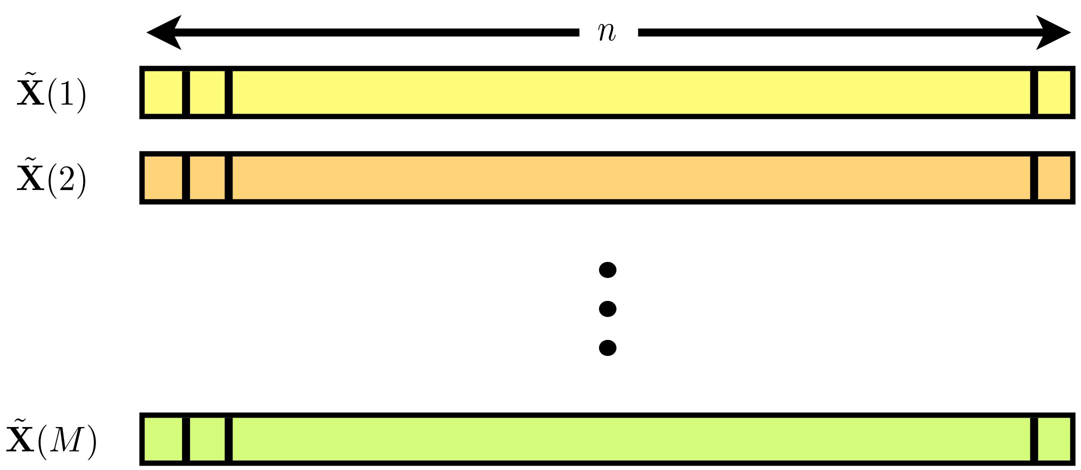

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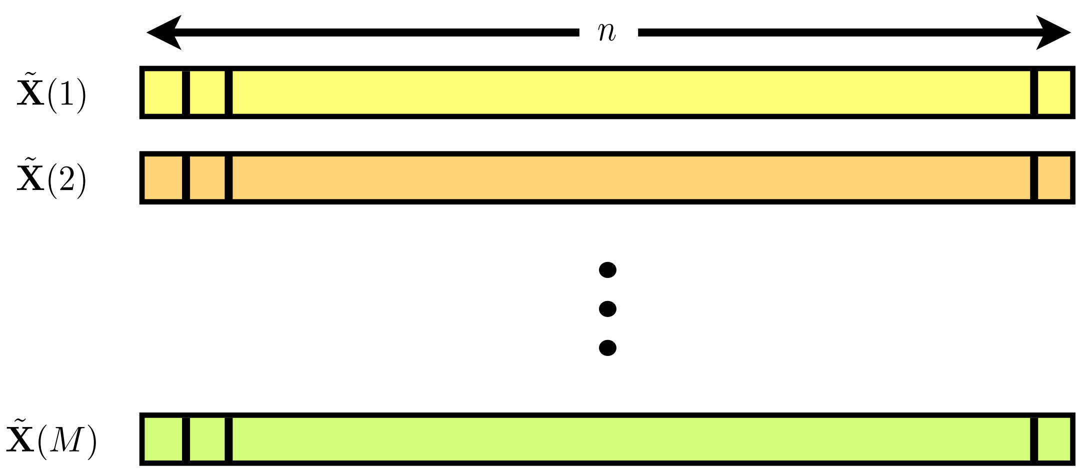

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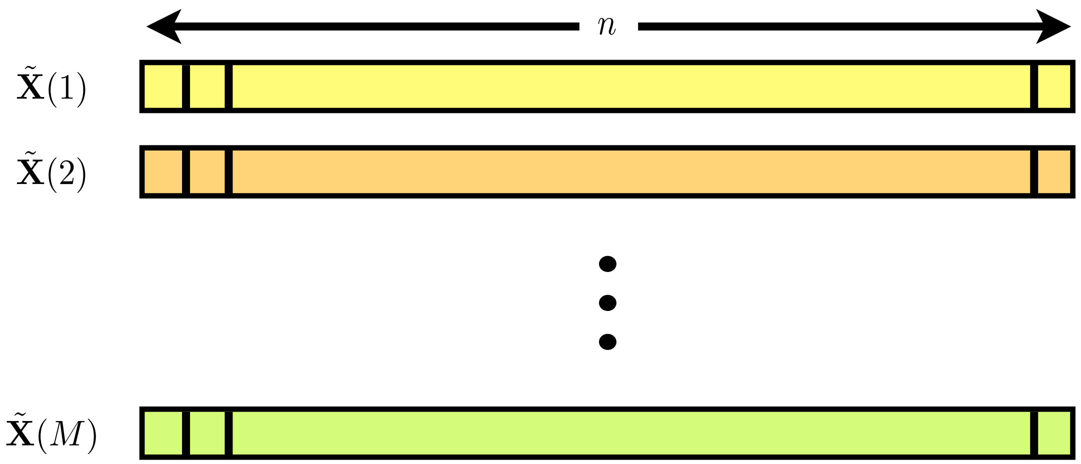

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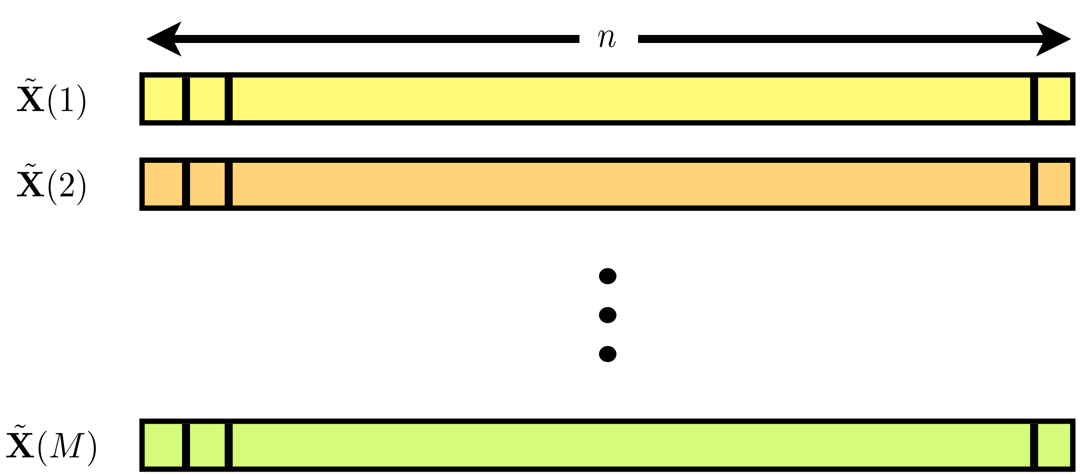

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5. The index K is delivered to the decoder.

6. The decoder outputs $\hat{\mathbf{X}}(K)$ as the reproduction sequence $\hat{\mathbf{X}}$.

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Random Coding Scheme

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1. Fix $\epsilon > 0$ and $\hat{X}$ with $Ed(X, \hat{X}) \le D$, where $0 \le D \le D_{max}$. Let δ be specified later.

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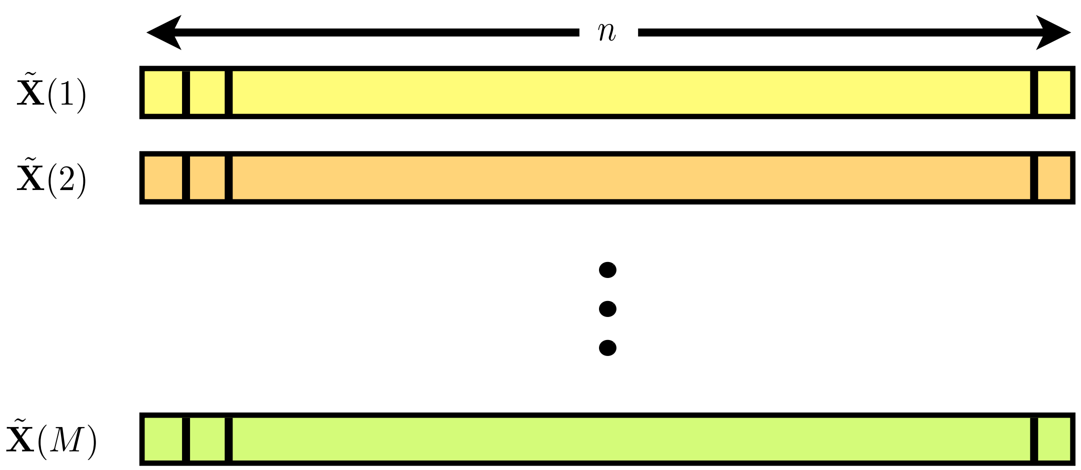

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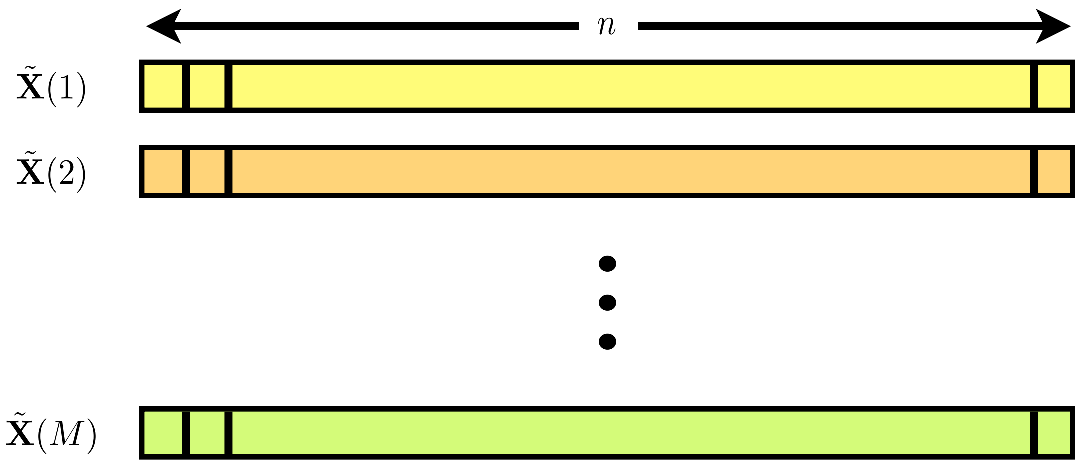

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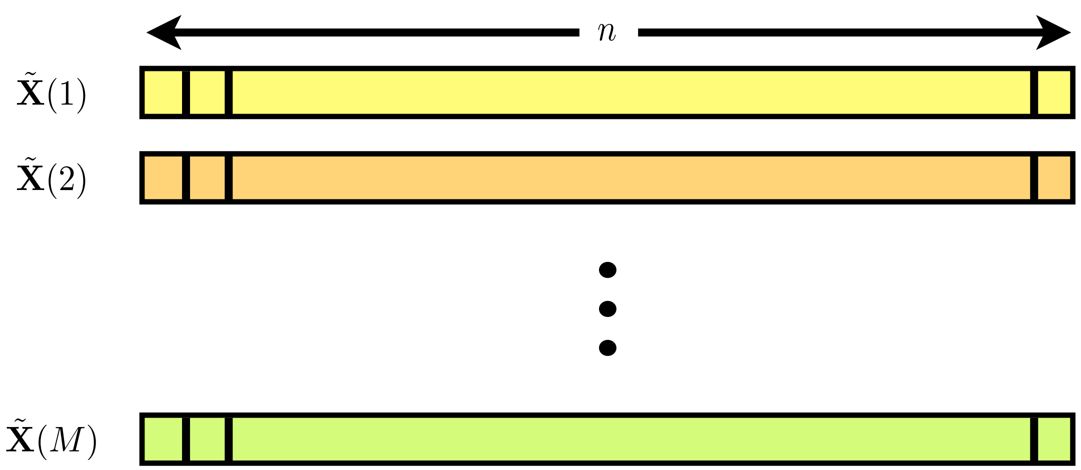

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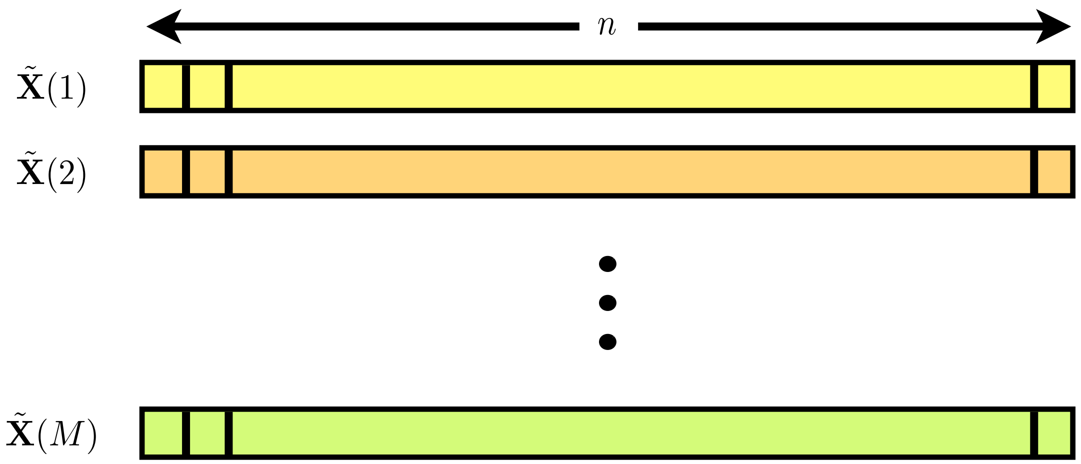

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5. Then for any $\mathbf{x} \in \mathcal{X}^n$,

$$\begin{aligned}
&\Pr\{K = 1|\mathbf{X} = \mathbf{x}\} \\
&\quad\leq \Pr\{E_2^c \cap E_3^c \cap \cdots \cap E_M^c|\mathbf{X} = \mathbf{x}\} \\
&\quad= \prod_{i=2}^{M} \Pr\{E_i^c|\mathbf{X} = \mathbf{x}\} \\
&\quad= (\Pr\{E_1^c|\mathbf{X} = \mathbf{x}\})^{M-1} \\
&\quad= (1 - \Pr\{E_1|\mathbf{X} = \mathbf{x}\})^{M-1}.
\end{aligned}$$

6. We will focus on $\mathbf{x} \in S^n_{[X]\delta}$ where

$$S^n_{[X]\delta} = \{\mathbf{x} \in T^n_{[X]\delta} : |T^n_{[\hat{X}|X]\delta}(\mathbf{x})| \geq 1\},$$

Performance Analysis

1. The event $\{K = 1\}$ occurs in one of the following two scenarios:

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2. Define the event

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because $\Pr\{\mathbf{X} \in S^n_{[X]\delta}\} \approx 1$ for large n.

Proposition 6.13 With respect to a joint distribution $p(x, y)$ on $\mathcal{X} \times \mathcal{Y}$, for any $\delta > 0$,

$$\Pr\{\mathbf{X} \in S^n_{[X]\delta}\} > 1 - \delta$$

for n sufficiently large.

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$$\Pr\{E_1|\mathbf{X} = \mathbf{x}\} \quad = \quad \Pr\left\{(\mathbf{x}, \hat{\mathbf{X}}(1)) \in T^n_{[X\hat{X}]\delta}\right\}$$

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\Pr\{E_1|\mathbf{X} = \mathbf{x}\} &= \Pr\left\{(\mathbf{x}, \hat{\mathbf{X}}(1)) \in T^n_{[X\hat{X}]\delta}\right\} \\
&= \sum_{\hat{\mathbf{x}} \in T^n_{[\hat{X}|X]\delta}(\mathbf{x})} p(\hat{\mathbf{x}})
\end{aligned}$$

> **Theorem 6.7 (Consistency)** If $(\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}$, then $\mathbf{x} \in T^n_{[X]\delta}$ and $\mathbf{y} \in T^n_{[Y]\delta}$.

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&\geq \sum_{\hat{\mathbf{x}} \in T^n_{[\hat{X}|X]\delta}(\mathbf{x})} 2^{-n(H(\hat{X})+\eta)}
\end{aligned}$$

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Theorem 6.10 (Conditional Strong AEP) If $|T^n_{[Y|X]\delta}(\mathbf{x})| \ge 1$, then

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3. Then

$$\{K = 1\} \subset E_2^c \cap E_3^c \cap \cdots \cap E_M^c.$$

4. Since the codewords are generated i.i.d., conditioning on $\{\mathbf{X} = \mathbf{x}\}$ for any $\mathbf{x} \in \mathcal{X}^n$, the events E_i are mutually independent and have the same probability.

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1. The event $\{K = 1\}$ occurs in one of the following two scenarios:

- $\hat{X}(1)$ is the only codeword in $\mathcal{C}$ which is jointly typical with $\mathbf{X}$.
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In other words, if $K = 1$, then $\mathbf{X}$ is jointly typical with none of the codewords $\hat{X}(2), \hat{X}(3), \cdots, \hat{X}(M)$. We will show that $\Pr\{K = 1\}$ can be made arbitrarily small.

2. Define the event

$$E_i = \left\{(\mathbf{X}, \hat{\mathbf{X}}(i)) \in T^n_{[X\hat{X}]\delta}\right\}.$$

3. Then

$$\{K = 1\} \subset E_2^c \cap E_3^c \cap \cdots \cap E_M^c.$$

4. Since the codewords are generated i.i.d., conditioning on $\{\mathbf{X} = \mathbf{x}\}$ for any $\mathbf{x} \in \mathcal{X}^n$, the events E_i are mutually independent and have the same probability.

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&\quad \leq \quad (1 - \Pr\{E_1|\mathbf{X} = \mathbf{x}\})^{M-1} \\
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where $\zeta = \xi + \eta \to 0$ as $n \to \infty$ and $\delta \to 0$.

8. Therefore,

$$\begin{aligned}
&\Pr\{K = 1|\mathbf{X} = \mathbf{x}\} \\
&\quad\leq (1 - \Pr\{E_1|\mathbf{X} = \mathbf{x}\})^{M-1} \\
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&\leq -\left(2^{n(I(X;\hat{X})+\frac{\epsilon}{2})} - 1\right)2^{-n(I(X;\hat{X})+\zeta)} \\
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10. Let n be sufficiently large and δ be sufficiently small so that

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Then the upper bound on $\ln \Pr\{K = 1|\mathbf{X} = \mathbf{x}\}$ tends to $-\infty$ as $n \to \infty$, i.e., $\Pr\{K = 1|\mathbf{X} = \mathbf{x}\} \to 0$ as $n \to \infty$.

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6. We will focus on $\mathbf{x} \in S^n_{[X]\delta}$ where

$$S^n_{[X]\delta} = \{\mathbf{x} \in T^n_{[X]\delta} : |T^n_{[\hat{X}|X]\delta}(\mathbf{x})| \geq 1\},$$

because $\Pr\{\mathbf{X} \in S^n_{[X]\delta}\} \approx 1$ for large n.

7. For $\mathbf{x} \in S^n_{[X]\delta}$, obtain the following lower bound on $\Pr\{E_1|\mathbf{X} = \mathbf{x}\}$:

$$\begin{aligned}
\Pr\{E_1|\mathbf{X} = \mathbf{x}\} &= \Pr\left\{(\mathbf{x}, \hat{\mathbf{X}}(1)) \in T^n_{[X\hat{X}]\delta}\right\} \\
&= \sum_{\hat{\mathbf{x}} \in T^n_{[\hat{X}|X]\delta}(\mathbf{x})} p(\hat{\mathbf{x}}) \\
&\geq \sum_{\hat{\mathbf{x}} \in T^n_{[\hat{X}|X]\delta}(\mathbf{x})} 2^{-n(H(\hat{X})+\eta)} \\
&\geq 2^{n(H(\hat{X}|X)-\xi)} 2^{-n(H(\hat{X})+\eta)} \\
&= 2^{-n(H(\hat{X})-H(\hat{X}|X)+\xi+\eta)} \\
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&\leq \sum_{\mathbf{x}\in S^n_{[X]\delta}} \frac{\epsilon}{2}\cdot\Pr\{\mathbf{X}=\mathbf{x}\} + \sum_{\mathbf{x}\notin S^n_{[X]\delta}} \underline{1}\cdot\Pr\{\mathbf{X}=\mathbf{x}\}
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Proposition 6.13 With respect to a joint distribution $p(x,y)$ on $\mathcal{X}\times\mathcal{Y}$, for any $\delta>0$,

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for n sufficiently large.

9. Now

$$\frac{1}{n}\log M \geq I(X;\hat{X}) + \frac{\epsilon}{2} \iff M \geq 2^{n(I(X;\hat{X})+\frac{\epsilon}{2})}.$$

Then

$$\begin{aligned}
&\ln \Pr\{K=1|\mathbf{X}=\mathbf{x}\} \\
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- If M grows with n at a rate higher than $I(X;\hat{X})$, then the probability that there exists at least one $\hat{\mathbf{X}}(i)$ which is jointly typical with the source sequence $\mathbf{X}$ with respect to $p(x, \hat{x})$ is high.

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- Such an $\hat{\mathbf{X}}(i)$, if exists, would have $d(\mathbf{X}, \hat{\mathbf{X}}) \approx Ed(X, \hat{X}) \leq D$, because the joint relative frequency of $(\mathbf{x}, \hat{\mathbf{X}}(i)) \approx p(x, \hat{x})$. See the next proposition.

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- Use this $\hat{\mathbf{X}}(i)$ to represent $\mathbf{X}$ to satisfy the distortion constraint.

Proposition For $\hat{X}$ such that $Ed(X, \hat{X}) \leq D$, if $(\mathbf{x}, \hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

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$$\begin{aligned} d(\mathbf{x}, \hat{\mathbf{x}}) &= \frac{1}{n}\sum_{k=1}^{n} d(x_k, \hat{x}_k) \\ &= \frac{1}{n}\sum_{x,\hat{x}} d(x, \hat{x}) N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) \\ &= \frac{1}{n}\sum_{x,\hat{x}} \underline{d(x, \hat{x})}\,(\underline{np(x, \hat{x})} + N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - np(x, \hat{x})) \end{aligned}$$

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

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&= \underline{\sum_{x,\hat{x}} p(x,\hat{x})d(x,\hat{x})} + \sum_{x,\hat{x}} d(x,\hat{x})\left(\frac{1}{n}N(x,\hat{x}|\mathbf{x},\hat{\mathbf{x}}) - p(x,\hat{x})\right)
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&= Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x}) \right)
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&= Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n}N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right) \\
&\leq Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left|\frac{1}{n}N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right|
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&\leq Ed(X, \hat{X}) + d_{max} \sum_{x,\hat{x}} \underline{\left|\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right|}
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&= Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right) \\
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Proof For $(\mathbf{x}, \hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, consider

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d(\mathbf{x}, \hat{\mathbf{x}}) &= \frac{1}{n}\sum_{k=1}^{n} d(x_k, \hat{x}_k) \\
&= \frac{1}{n}\sum_{x,\hat{x}} d(x, \hat{x}) N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) \\
&= \frac{1}{n}\sum_{x,\hat{x}} d(x, \hat{x})(np(x, \hat{x}) + N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - np(x, \hat{x})) \\
&= \sum_{x,\hat{x}} p(x, \hat{x}) d(x, \hat{x}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right) \\
&= Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right) \\
&\leq Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left|\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right| \\
&\leq Ed(X, \hat{X}) + d_{max} \sum_{x,\hat{x}} \left|\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right| \\
&\leq \underline{Ed(X, \hat{X})} + d_{max}\delta
\end{aligned}$$

Proposition For $\hat{X}$ such that $Ed(X, \hat{X}) \leq D$, if $(\mathbf{x}, \hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

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&= \sum_{x,\hat{x}} p(x, \hat{x}) d(x, \hat{x}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right) \\
&= Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left(\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right) \\
&\leq Ed(X, \hat{X}) + \sum_{x,\hat{x}} d(x, \hat{x}) \left|\frac{1}{n} N(x, \hat{x}|\mathbf{x}, \hat{\mathbf{x}}) - p(x, \hat{x})\right| \\
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&\leq D + d_{max}\delta.
\end{aligned}$$

The Remaining Details

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4. Therefore, $\Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon | K \neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon\} \leq \epsilon$.

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&\quad = \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\
&\qquad + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\
&\quad \leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\
&\quad = \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}. \quad 0
\end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K = 1\}\Pr\{K = 1\} \\
&\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K \neq 1\}\Pr\{K \neq 1\} \\
&\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K \neq 1\}\cdot 1 \\
&= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K \neq 1\}.
\end{aligned}$$

0

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K \neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} &= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K=1\}\Pr\{K=1\} \\ &\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\}\Pr\{K\neq 1\} \\ &\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\}\cdot 1 \\ &= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\}. \quad 0\end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} &= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\ &\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\ &\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\ &= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}.\end{aligned}$$

0

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

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$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq I(X;\hat{X}) + \epsilon$$

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$
\begin{aligned}
\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} &= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K=1\}\Pr\{K=1\} \\
&\quad + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\}\Pr\{K\neq 1\} \\
&\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\}\cdot 1 \\
&= \epsilon + \underbrace{\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\}}_{0}.
\end{aligned}
$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon \mid K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq I(X;\hat{X}) + \epsilon$$

and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon.$$

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\
&\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\
&\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\
&= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}.
\end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq I(X;\hat{X}) + \epsilon$$

and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, $(I(X;\hat{X}), D)$ is achievable.

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$
\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\
&\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\
&\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\
&= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}.
\end{aligned}
$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

> **Proposition** For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then
>
> $$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq I(X;\hat{X}) + \epsilon$$

and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, $(I(X;\hat{X}), D)$ is achievable.

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&\quad = \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\
&\qquad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\
&\quad \leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\
&\quad = \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}.
\end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

> **Proposition** For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then
>
> $$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq \underline{I(X;\hat{X})} + \epsilon$$

and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > \underline{D} + \epsilon\} \leq \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X,\hat{X}) \leq \underline{D}$, $(\underline{I(X;\hat{X})}, \underline{D})$ is achievable.

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon\} \\
&= \Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon | K = 1\}\Pr\{K = 1\} \\
&\quad + \Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon | K \neq 1\}\Pr\{K \neq 1\} \\
&\le 1 \cdot \epsilon + \Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon | K \neq 1\} \cdot 1 \\
&= \epsilon + \Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon | K \neq 1\}.
\end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X}, \hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X}, \hat{\mathbf{X}}) \le D + d_{max}\delta.$$

3. By taking $\delta \le \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X}, \hat{\mathbf{X}}) \le D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D + \epsilon.$$

Proposition For $\hat{X}$ such that $Ed(X, \hat{X}) \le D$, if $(\mathbf{x}, \hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x}, \hat{\mathbf{x}}) \le D + d_{max}\delta.$$

4. Therefore, $\Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon | K \neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon\} \le \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \le I(X; \hat{X}) + \epsilon$$

and

$$\Pr\{d(\mathbf{X}, \hat{\mathbf{X}}) > D + \epsilon\} \le \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X, \hat{X}) \le D$, $(I(X; \hat{X}), D)$ is achievable.

6. Finally, minimize $I(X; \hat{X})$ over all such $\hat{X}$ to conclude that $(R_I(D), D)$ is achievable, i.e.,

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\
&\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\
&\le 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\
&= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}.
\end{aligned}$$

2. Recall that conditioning on $\{K\neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \le D + d_{max}\delta.$$

3. By taking $\delta \le \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \le D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \le \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \le I(X;\hat{X}) + \epsilon$$

and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \le \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X,\hat{X}) \le D$, $(I(X;\hat{X}), D)$ is achievable.

6. Finally, minimize $I(X;\hat{X})$ over all such $\hat{X}$ to conclude that $(R_I(D), D)$ is achievable, i.e.,

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \le D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \le D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned} &\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\ &= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\ &\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\ &\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\ &= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}. \end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq I(X;\hat{X}) + \epsilon$$

and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, $(I(X;\hat{X}), D)$ is achievable.

6. Finally, minimize $I(X;\hat{X})$ over all such $\hat{X}$ to conclude that $(R_I(D), D)$ is achievable, i.e.,

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

$$d(\mathbf{x},\hat{\mathbf{x}}) \leq D + d_{max}\delta.$$

The Remaining Details

1. For sufficiently large n, consider

$$\begin{aligned}
&\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \\
&= \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K=1\}\Pr\{K=1\} \\
&\quad +\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\Pr\{K\neq 1\} \\
&\leq 1\cdot\epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}\cdot 1 \\
&= \epsilon + \Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\}.
\end{aligned}$$

2. Recall that conditioning on $\{K \neq 1\}$, $(\mathbf{X},\hat{\mathbf{X}}) \in T^n_{[X\hat{X}]\delta}$. Then

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\delta.$$

3. By taking $\delta \leq \frac{\epsilon}{d_{max}}$, we obtain

$$d(\mathbf{X},\hat{\mathbf{X}}) \leq D + d_{max}\left(\frac{\epsilon}{d_{max}}\right) = D+\epsilon.$$

Proposition For $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, if $(\mathbf{x},\hat{\mathbf{x}}) \in T^n_{[X\hat{X}]\delta}$, then

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4. Therefore, $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon | K\neq 1\} = 0$, which implies $\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon$.

5. Thus we have shown the existence of a code such that

$$\frac{1}{n}\log M \leq I(X;\hat{X}) + \epsilon$$

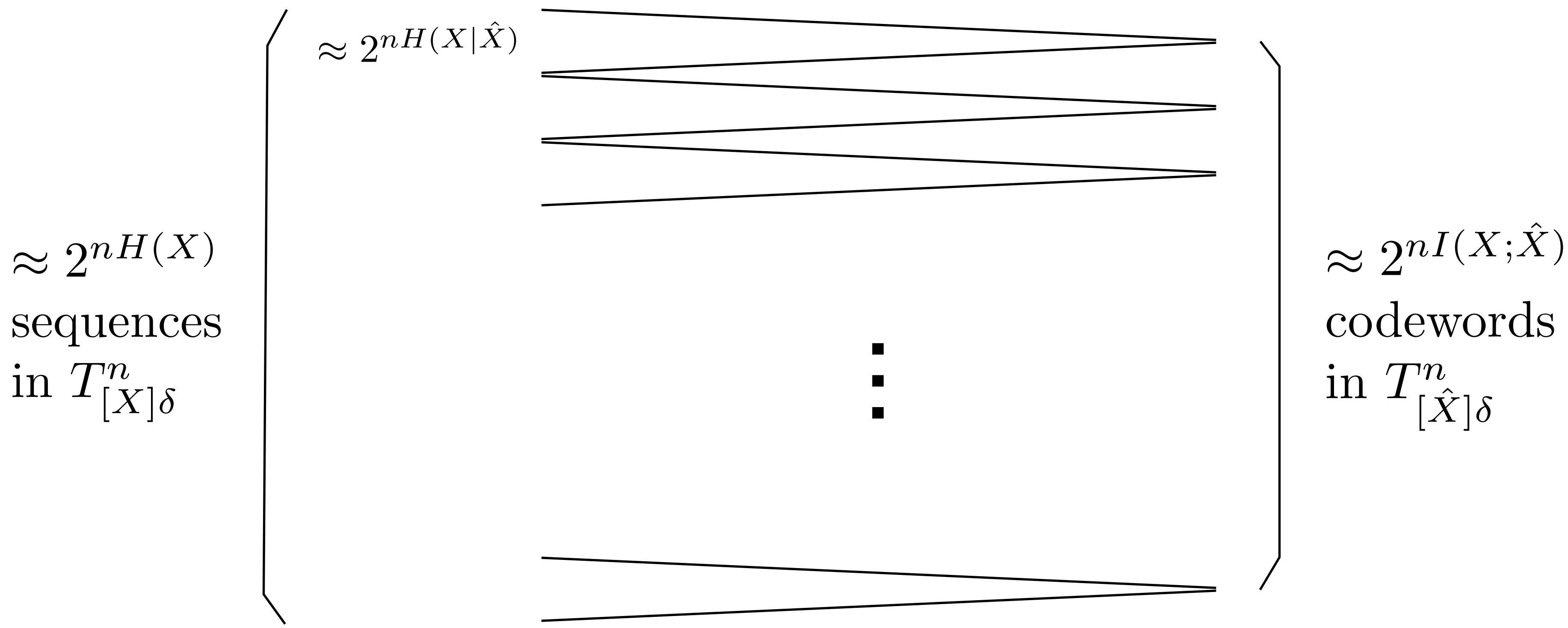
and

$$\Pr\{d(\mathbf{X},\hat{\mathbf{X}}) > D+\epsilon\} \leq \epsilon.$$

Hence, for any $\hat{X}$ such that $Ed(X,\hat{X}) \leq D$, $(I(X;\hat{X}), D)$ is achievable.

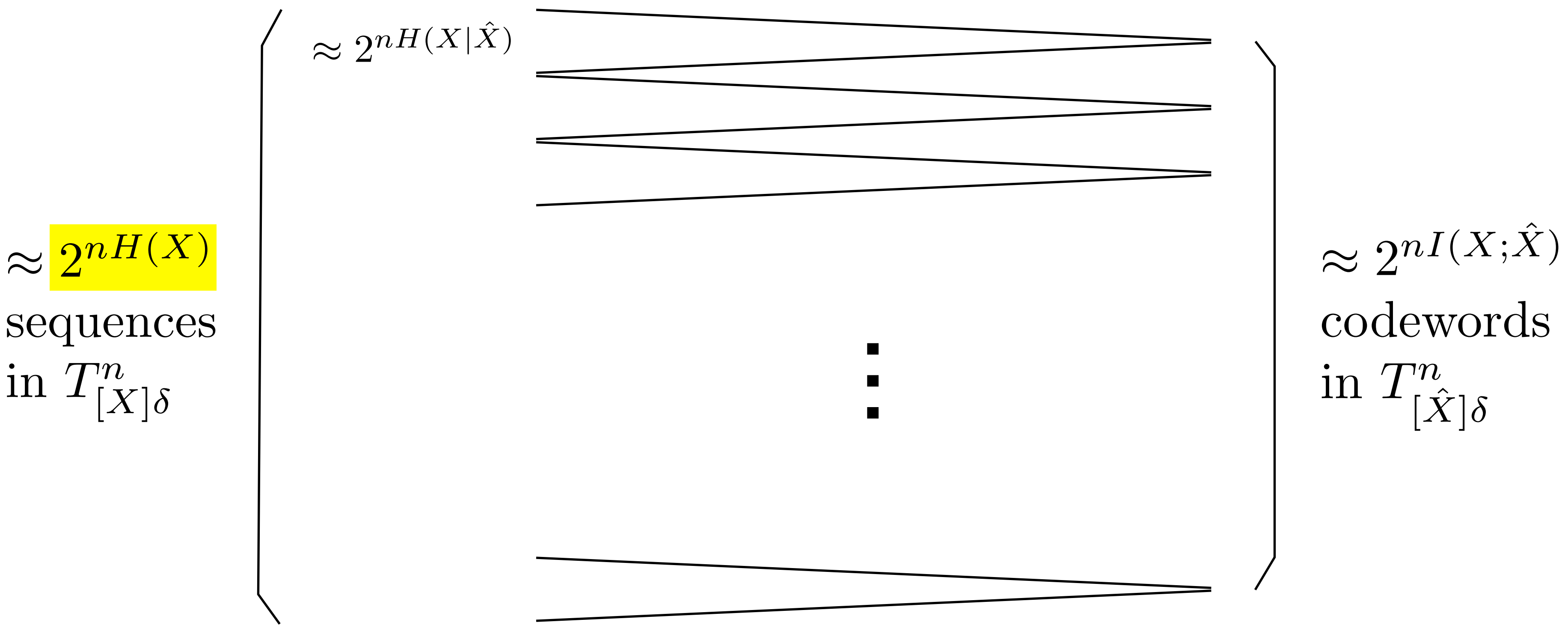
6. Finally, minimize $I(X;\hat{X})$ over all such $\hat{X}$ to conclude that $(R_I(D), D)$ is achievable, i.e.,

$$R_I(D) \geq R(D).$$



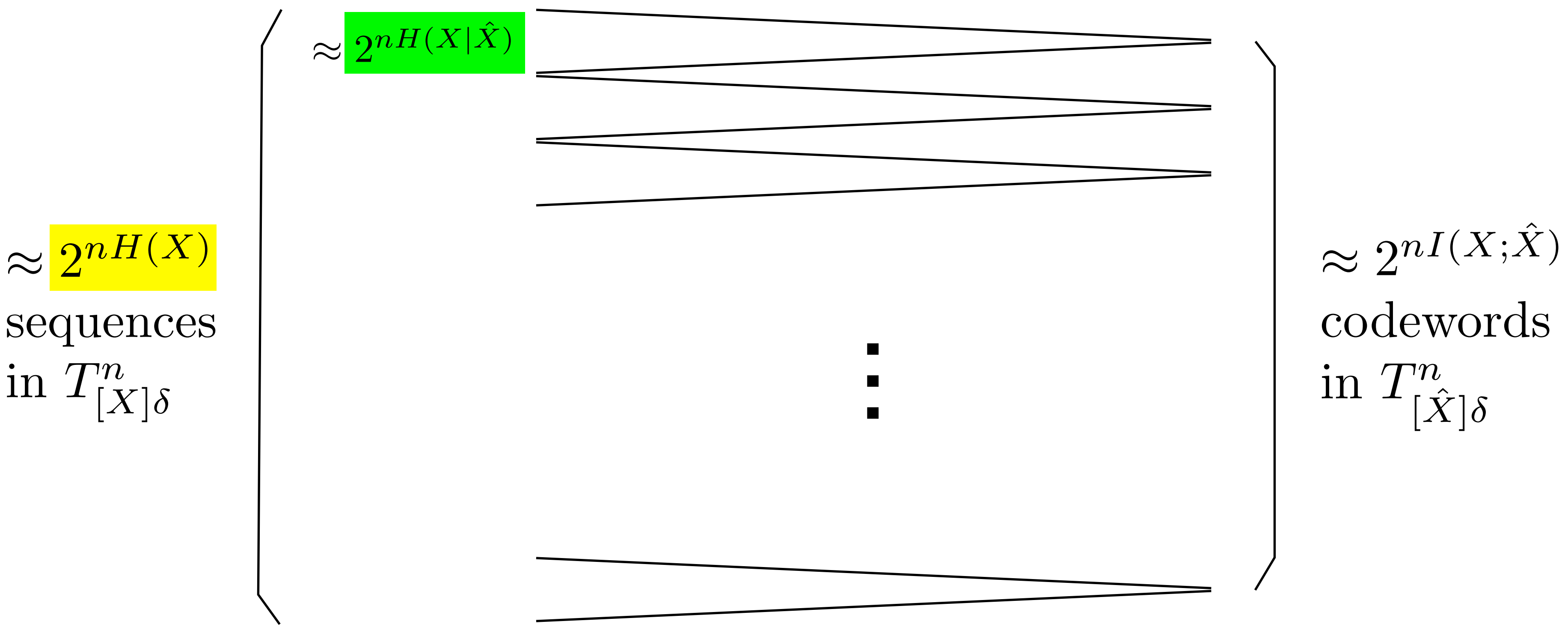
The number of codewords must be at least

$$\frac{2^{nH(X)}}{2^{nH(X|\hat{X})}} \approx 2^{nI(X;\hat{X})} \geq 2^{nR_I(D)}.$$



The number of codewords must be at least

$$\frac{2^{nH(X)}}{2^{nH(X|\hat{X})}} \approx 2^{nI(X;\hat{X})} \geq 2^{nR_I(D)}.$$



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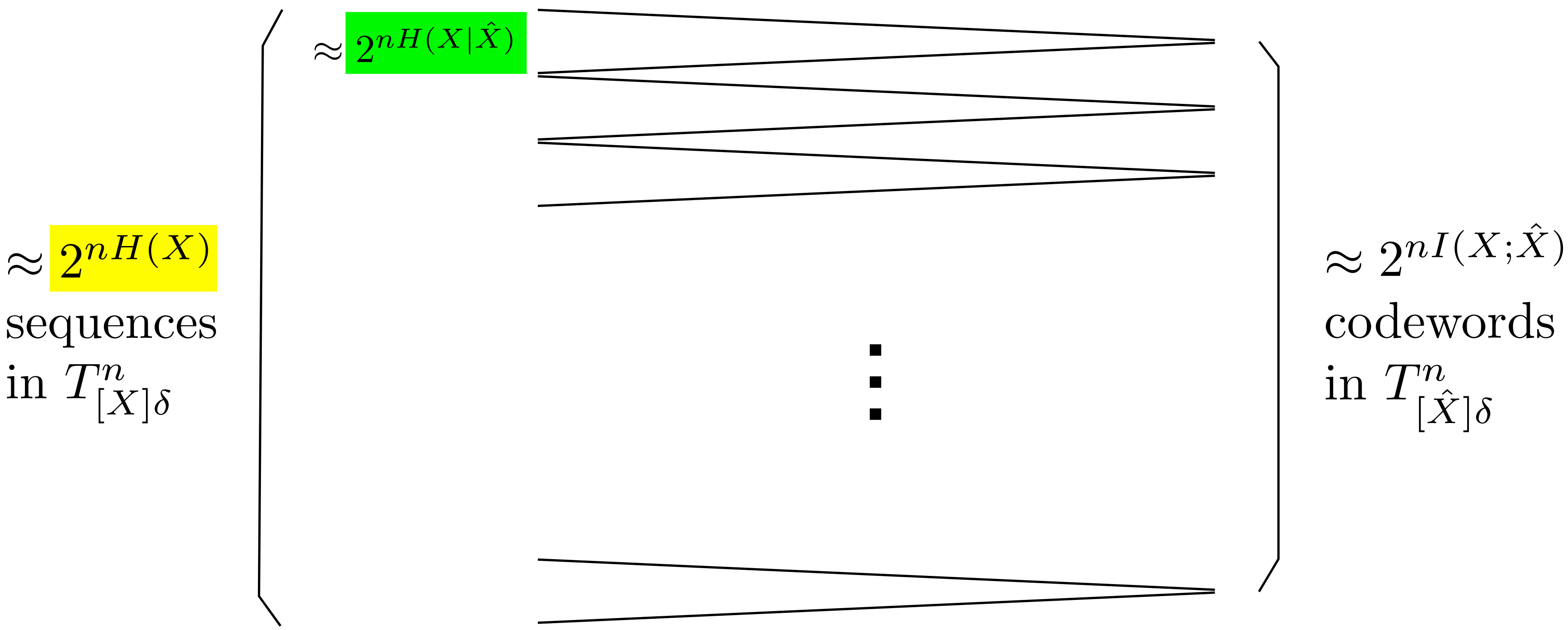

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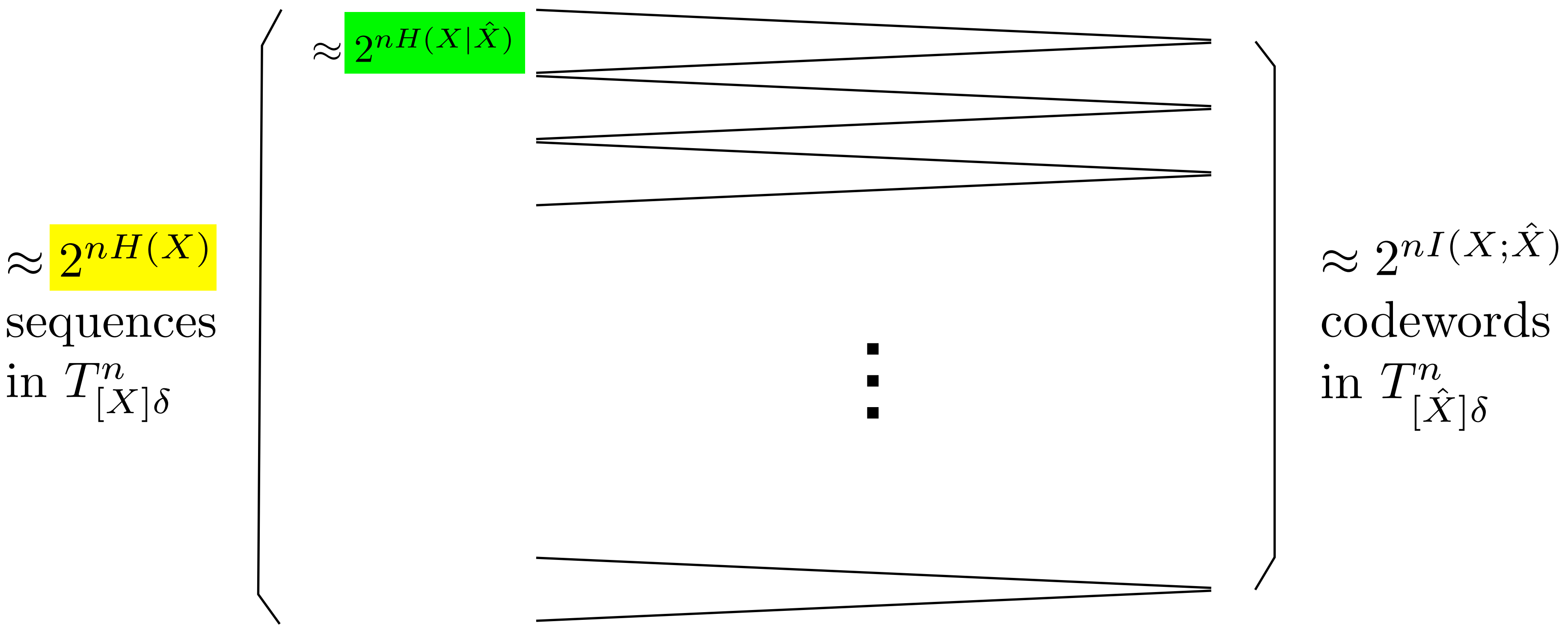
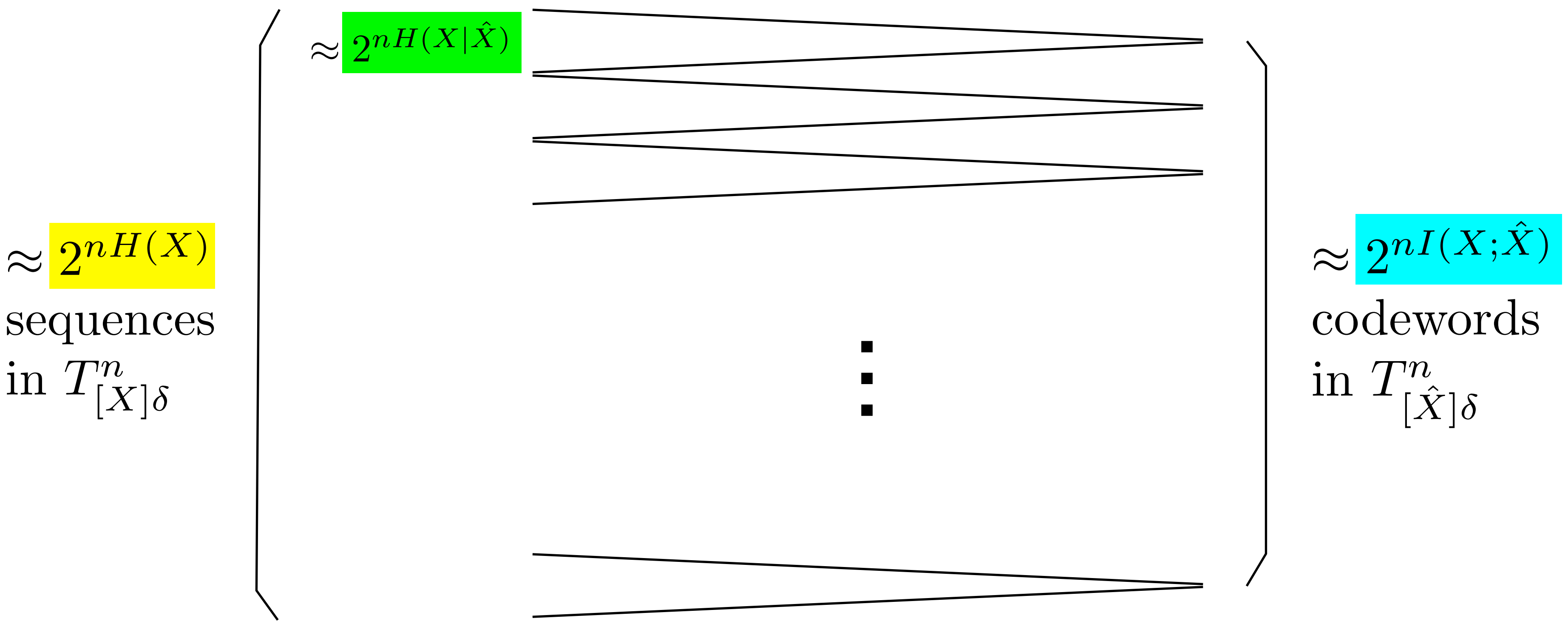
$$\frac{2^{nH(X)}}{2^{nH(X|\hat{X})}} \approx 2^{nI(X;\hat{X})} \geq 2^{nR_I(D)}.$$



The number of codewords must be at least

$$\frac{2^{nH(X)}}{2^{nH(X|\hat{X})}} \approx 2^{nI(X;\hat{X})} \geq 2^{nR_I(D)}.$$



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$$\frac{2^{nH(X)}}{2^{nH(X|\hat{X})}} \approx 2^{nI(X;\hat{X})} \geq 2^{nR_I(D)}.$$