8.3 The Rate-Distortion Theorem

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- The minimization is taken over the set of all transition matrices $p(\hat{x}|x)$ such that $Ed(X,\hat{X}) \leq D$, namely the set

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- Since this set is compact (closed and bounded) in $\Re^{|\mathcal{X}||\hat{\mathcal{X}}|}$ and $I(X;\hat{X})$ is a continuous functional of $p(\hat{x}|x)$, the minimum value of $I(X;\hat{X})$ can be attained.

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- Equivalently, the minimization can be taken over the set of all joint distributions $p(x,\hat{x})$ with marginal distribution $p(x)$, the given source distribution, such that $Ed(X,\hat{X}) \leq D$.

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- Since

$$E\tilde{d}(X,\hat{X}) = Ed(X,\hat{X}) - \Delta,$$

where Δ does not depend on $p(\hat{x}|x)$, we can always replace d by $\tilde{d}$ and D by $D - \Delta$ in the definition of $R_I(D)$ without changing the minimization problem.

Definition 8.16 For $D \geq 0$, the information rate-distortion function is defined by

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- Without loss of generality, we can assume d is normal.

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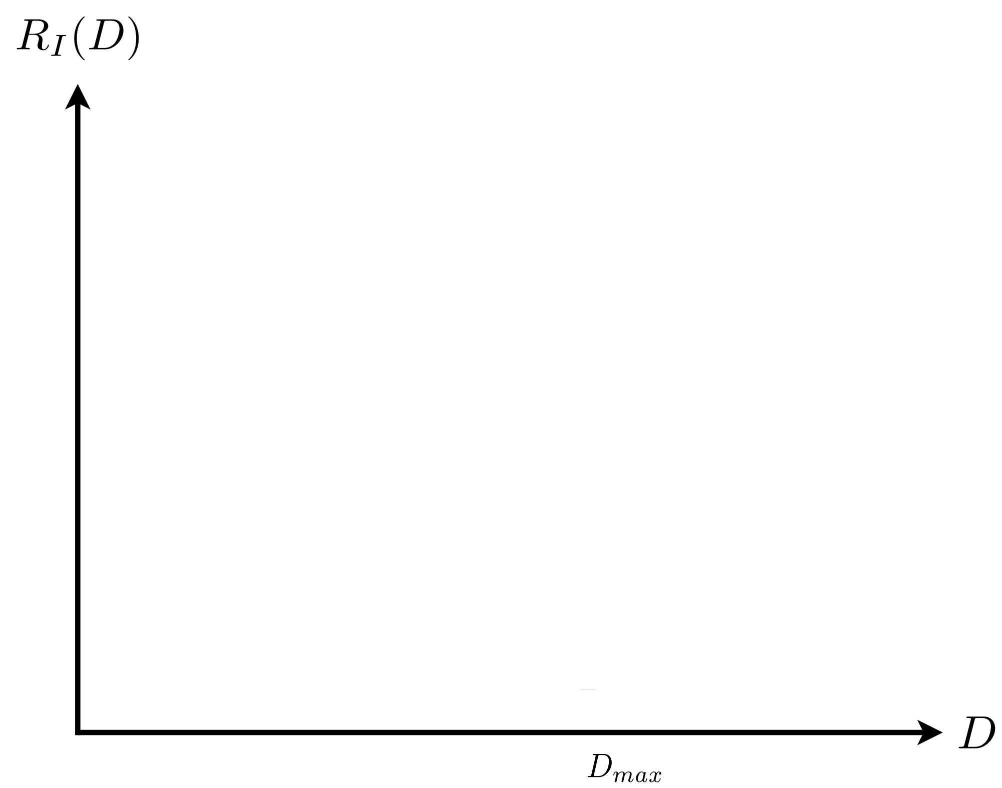

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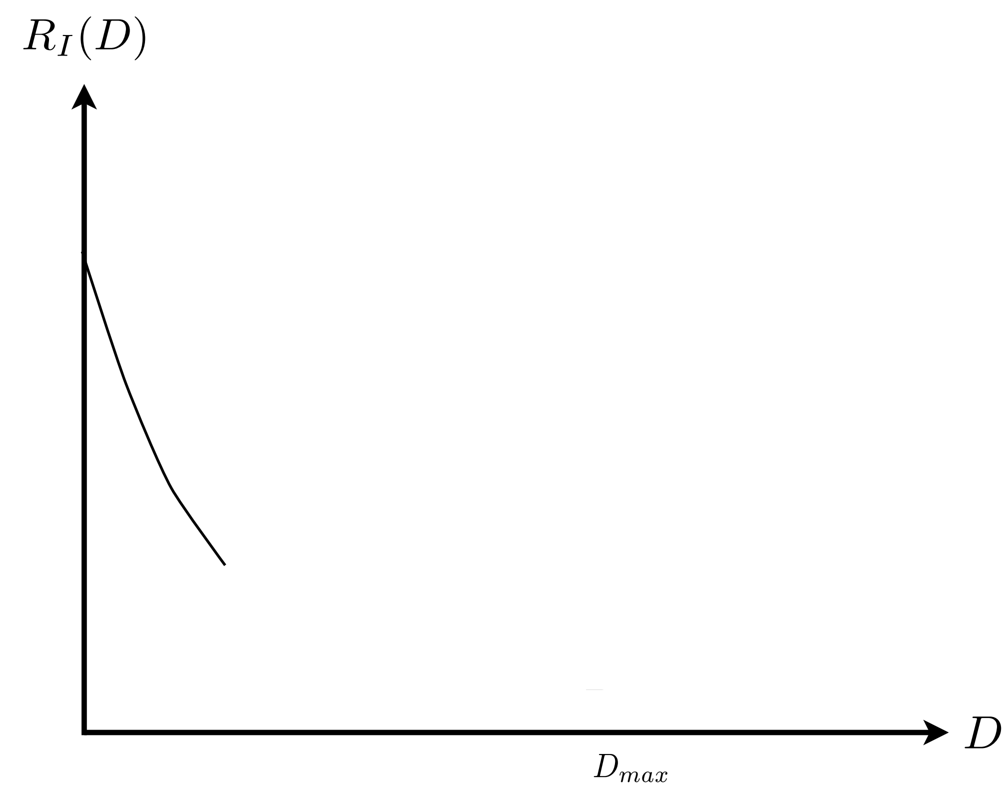

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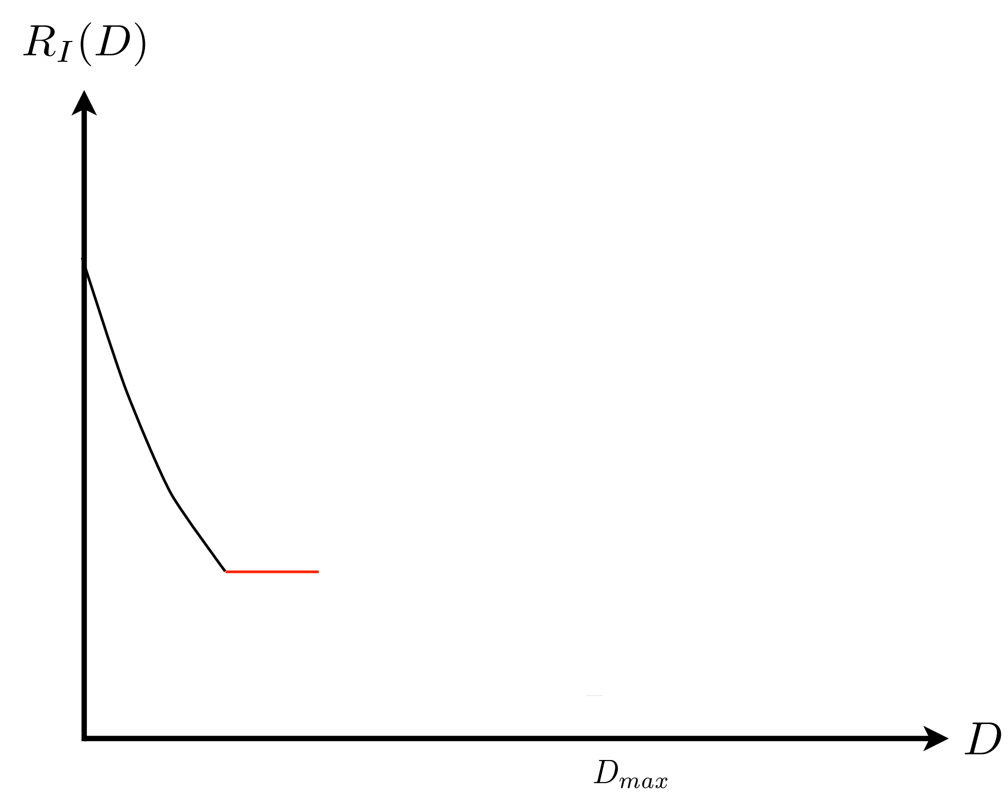

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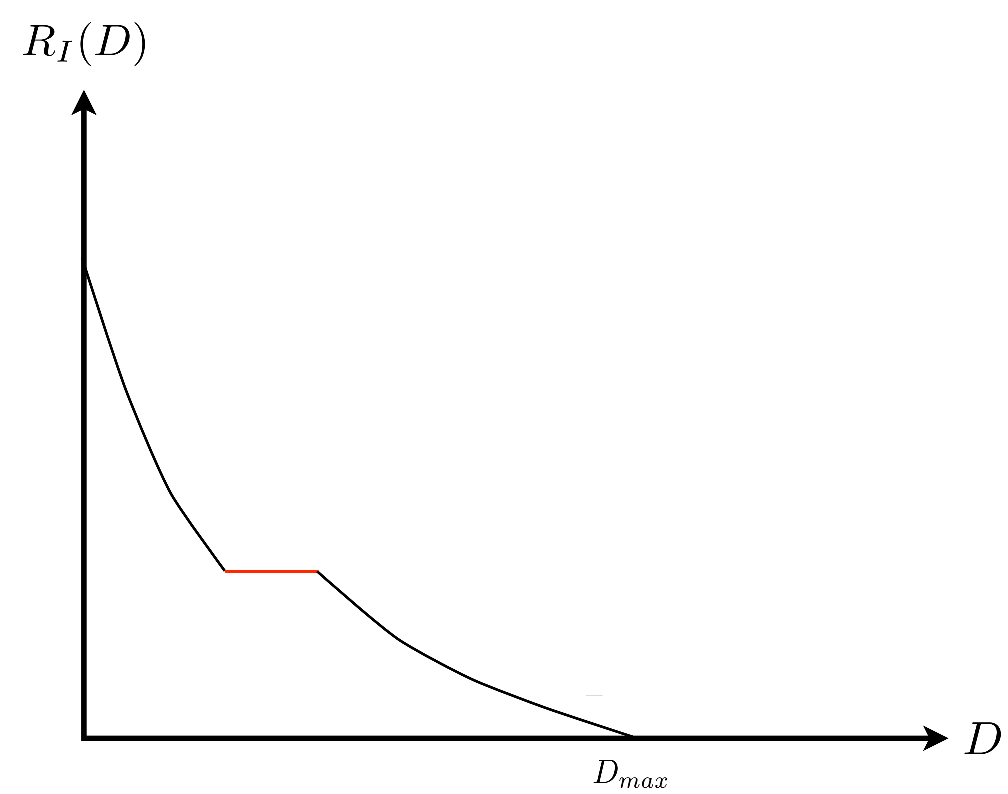

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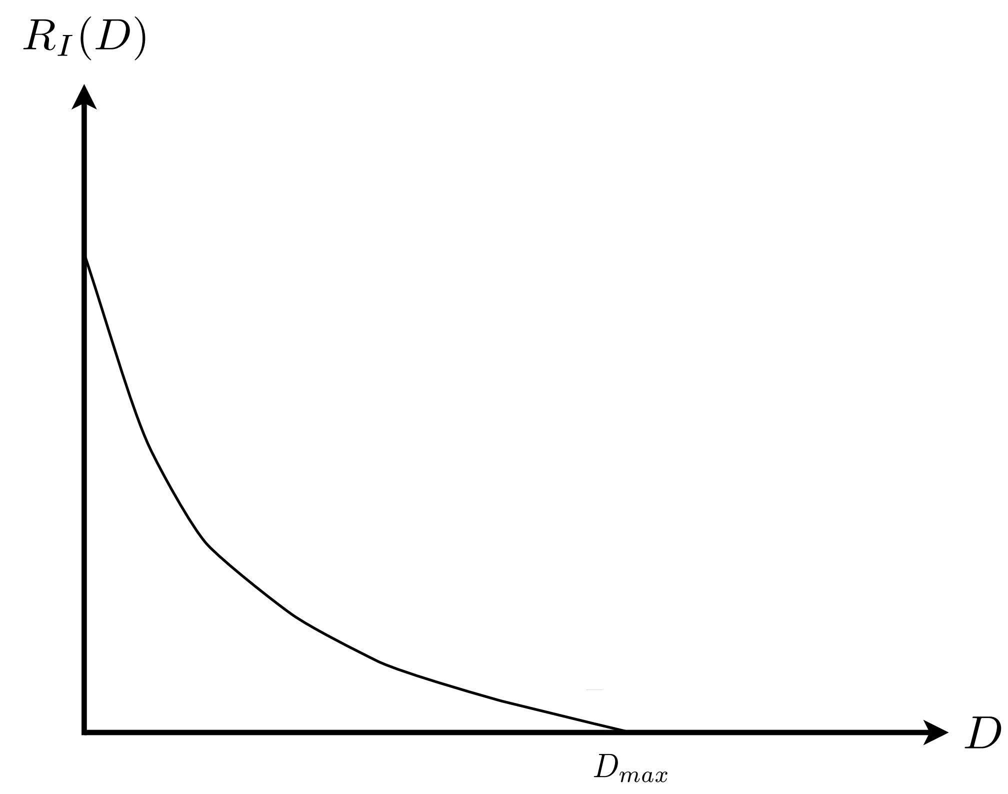

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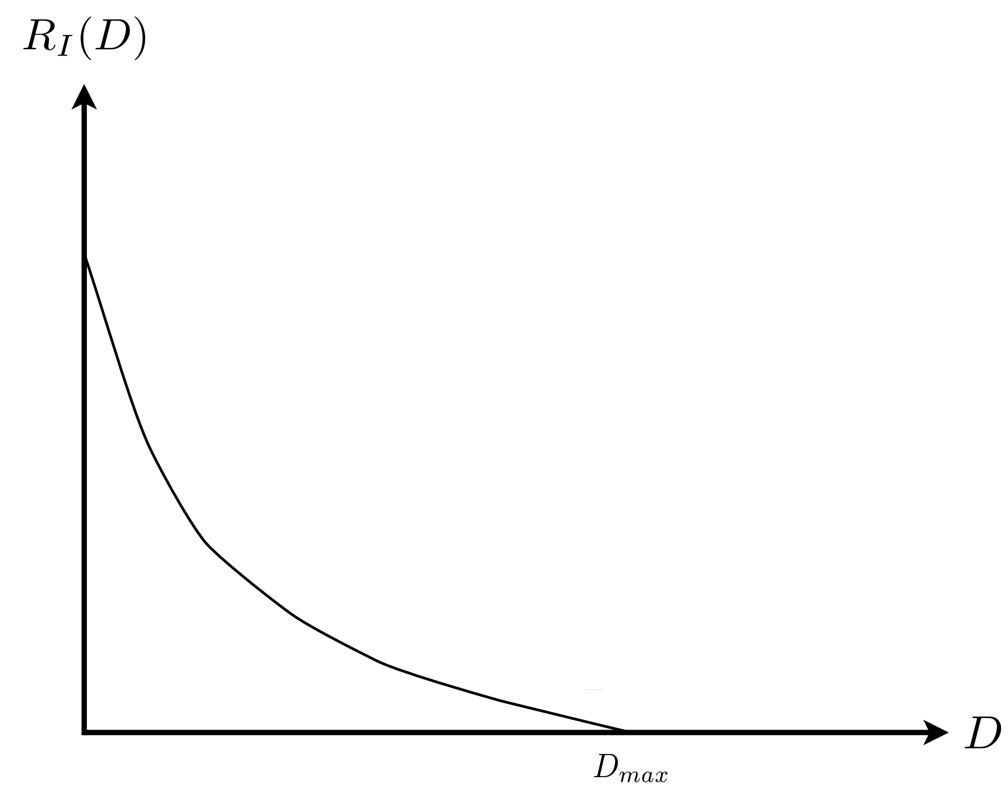

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$$\Pr\{X = 0\} = 1 - \gamma \quad \text{and} \quad \Pr\{X = 1\} = \gamma.$$

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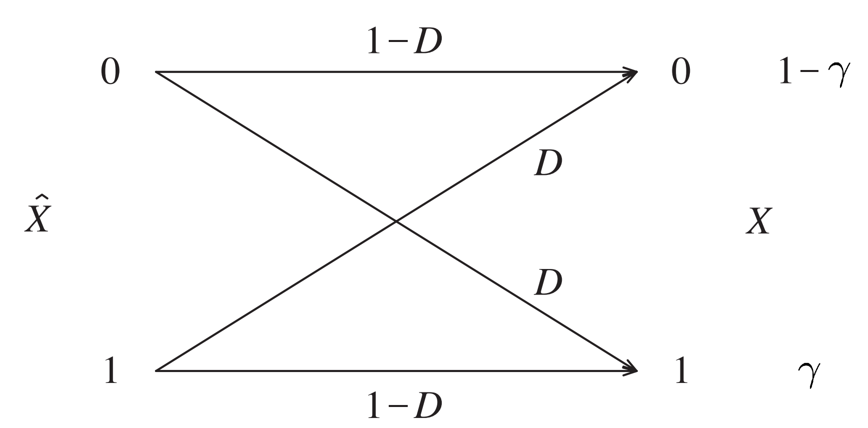

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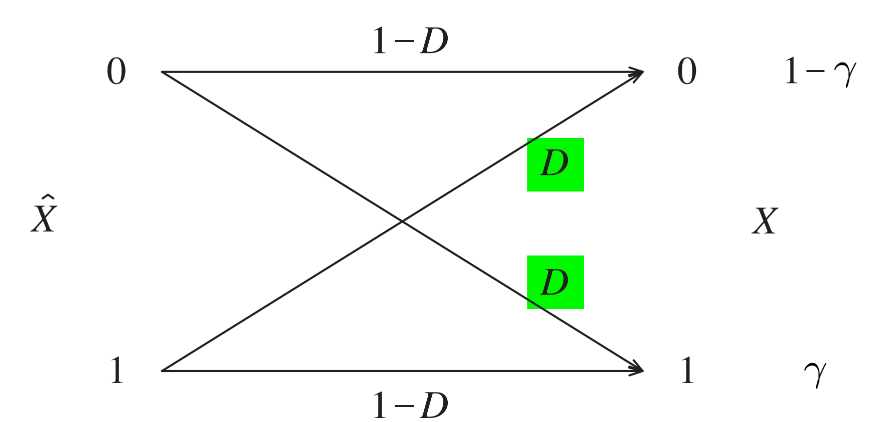

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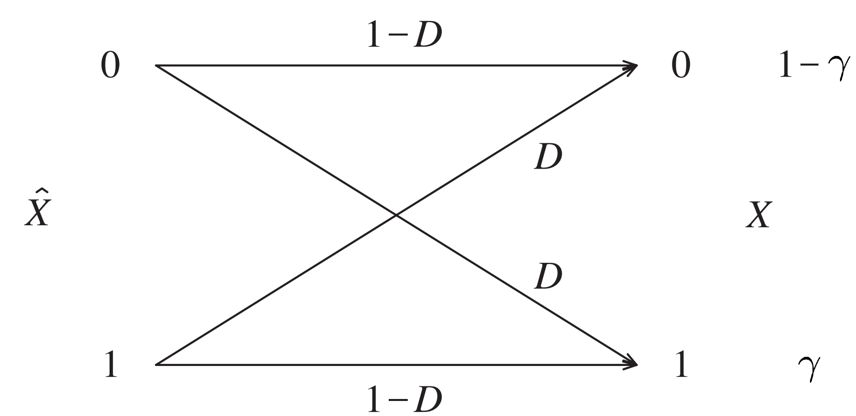

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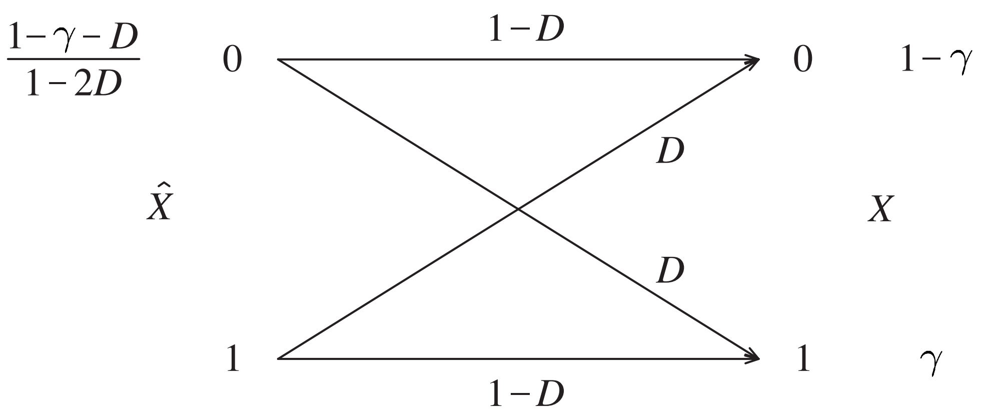

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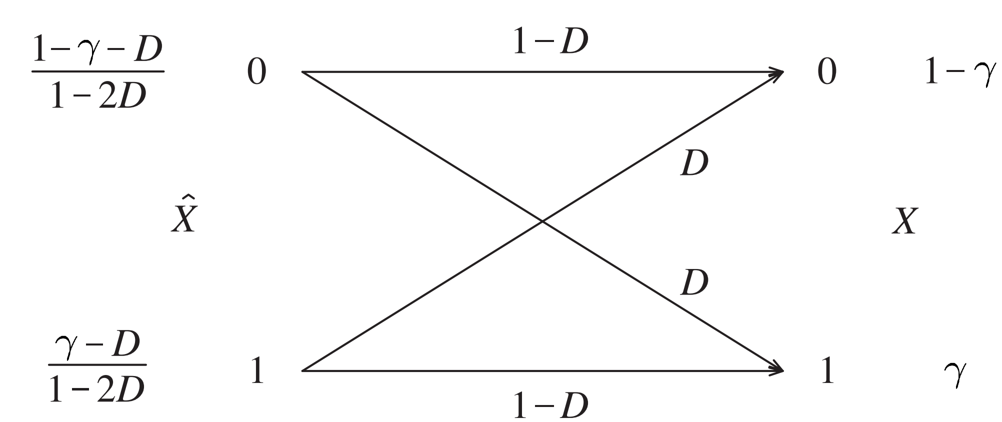

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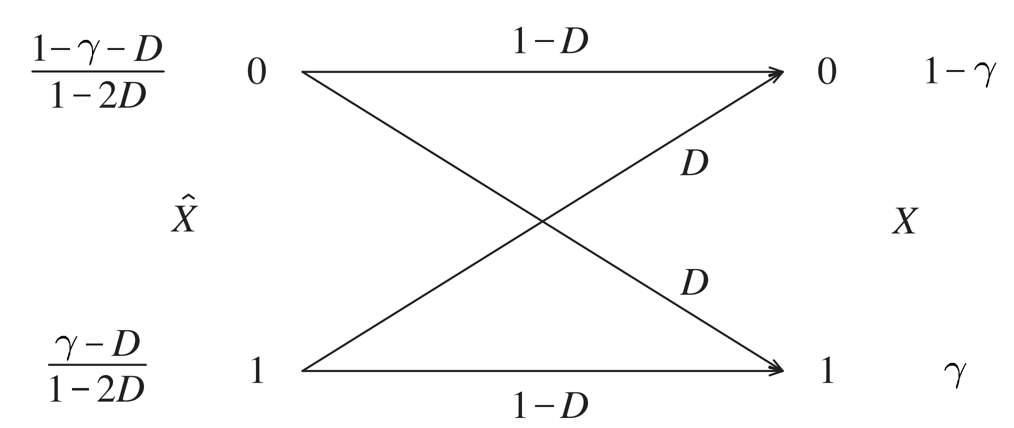

Note that

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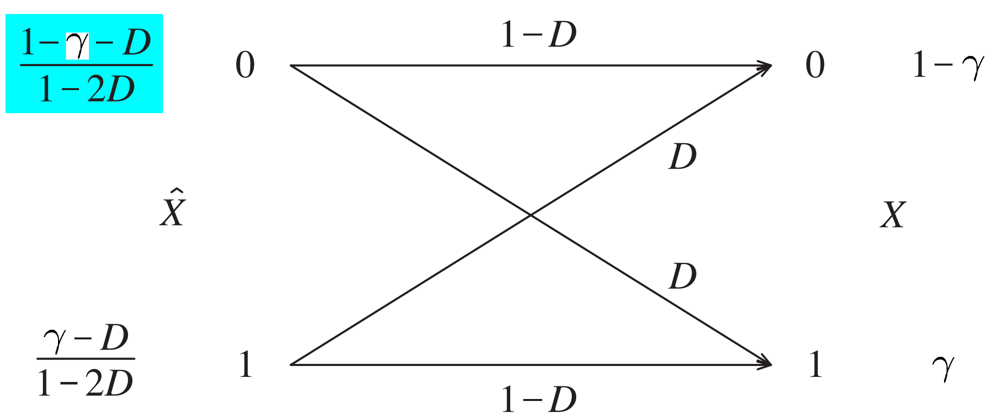

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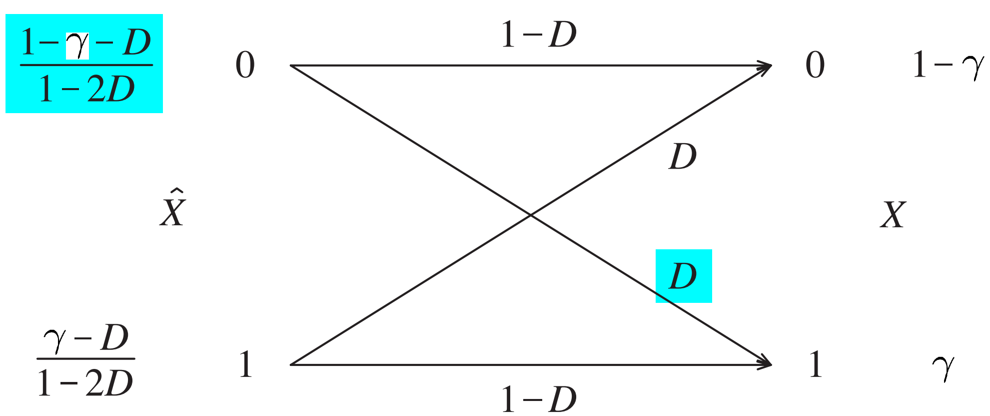

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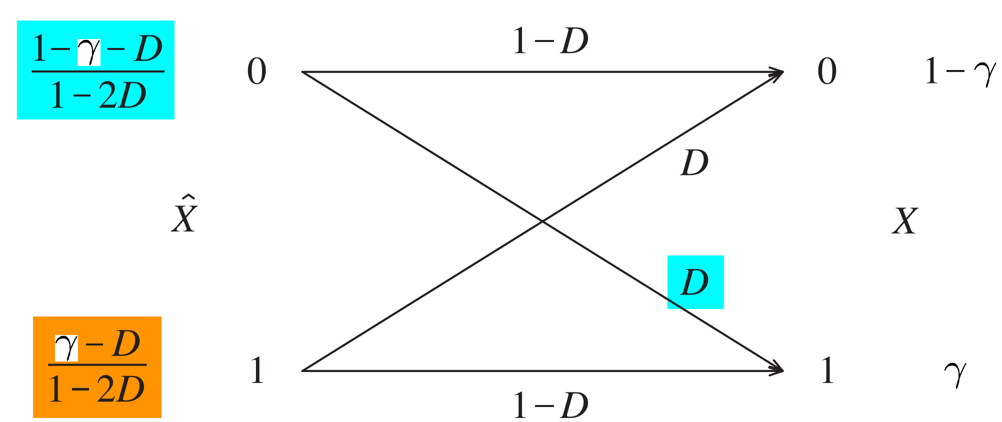

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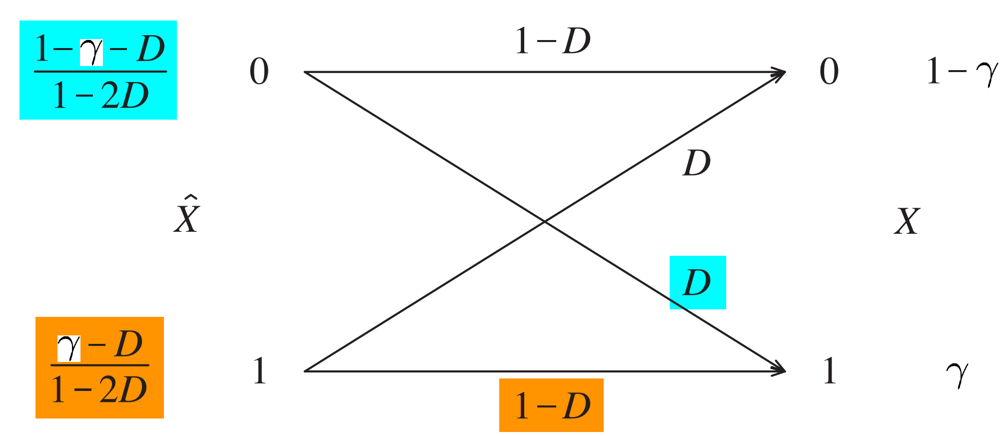

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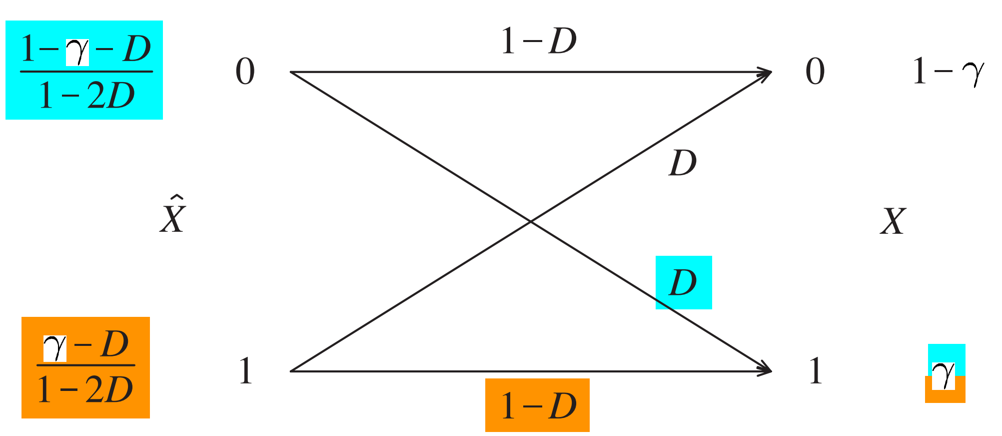

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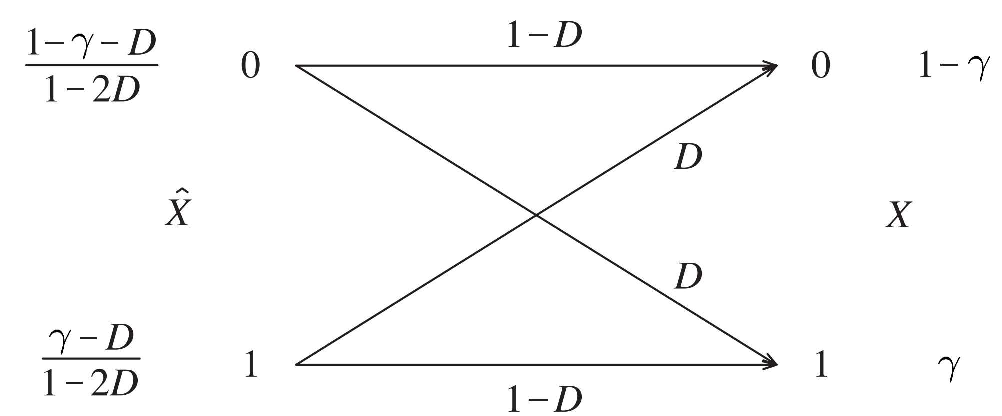

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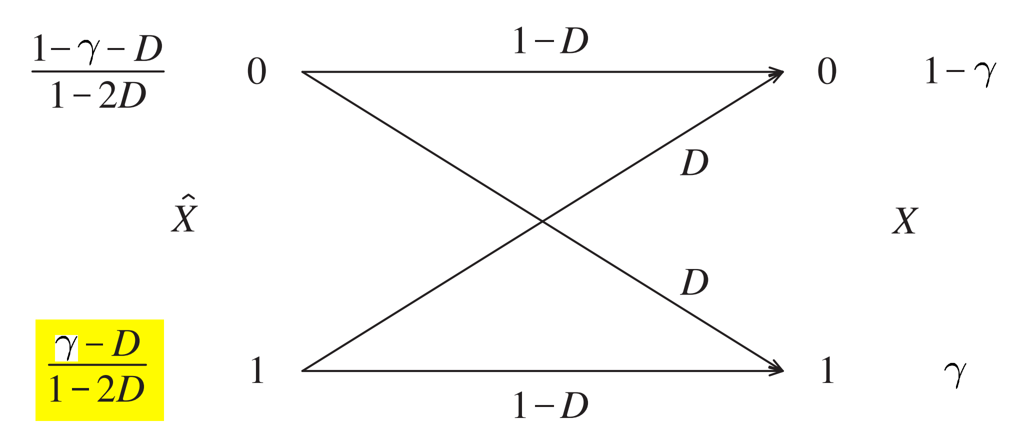

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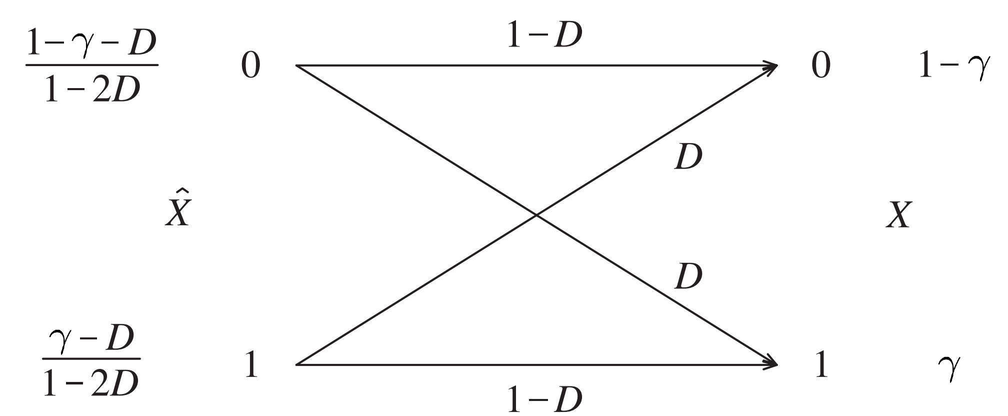

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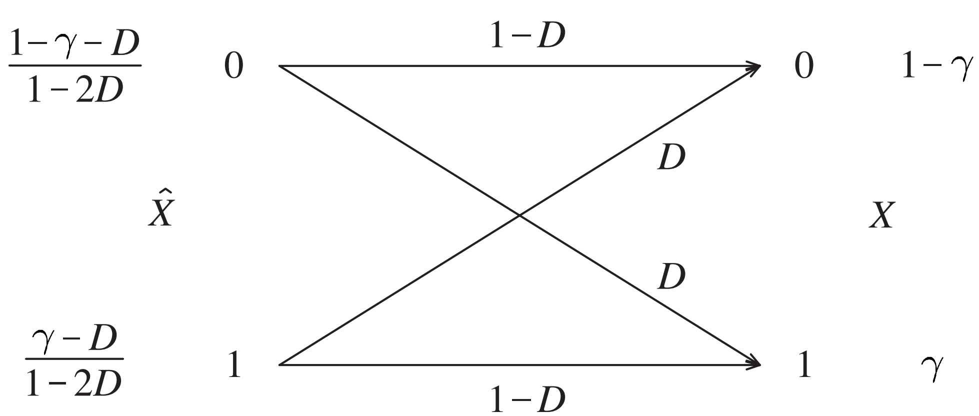

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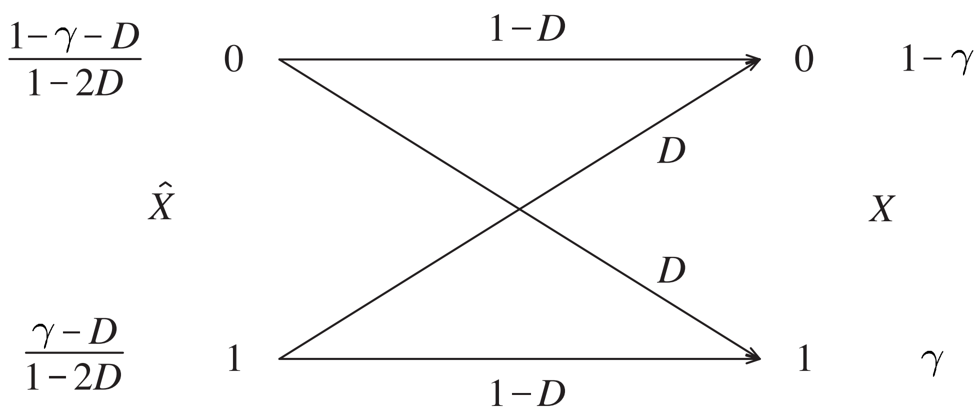

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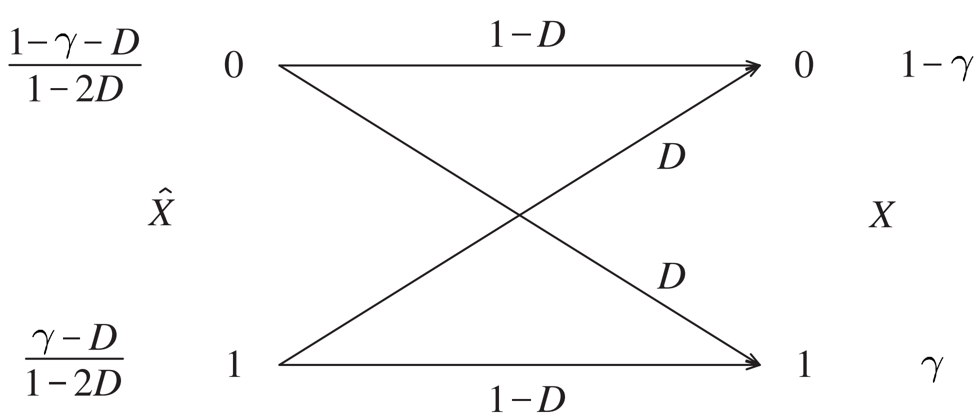

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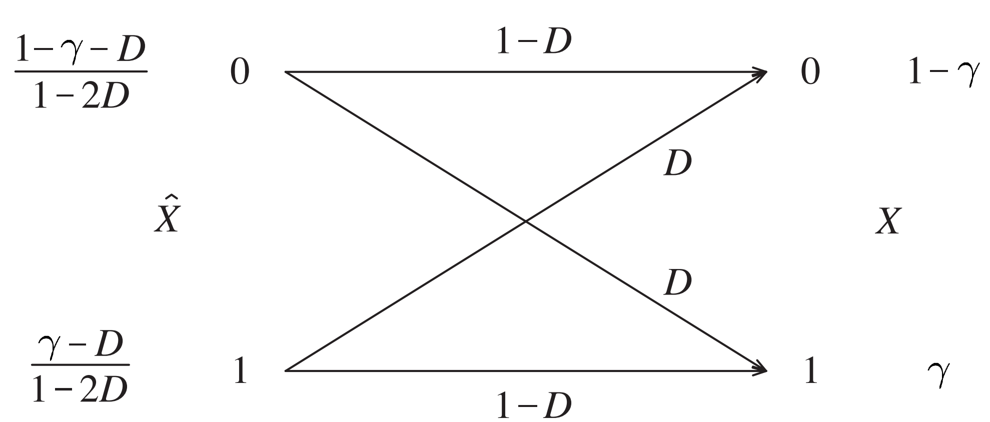

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$$R_I(D) = \begin{cases} h_b(1-\gamma) - h_b(D) & \text{if } 0 \le D < 1-\gamma \\ 0 & \text{if } D \ge 1-\gamma. \end{cases}$$

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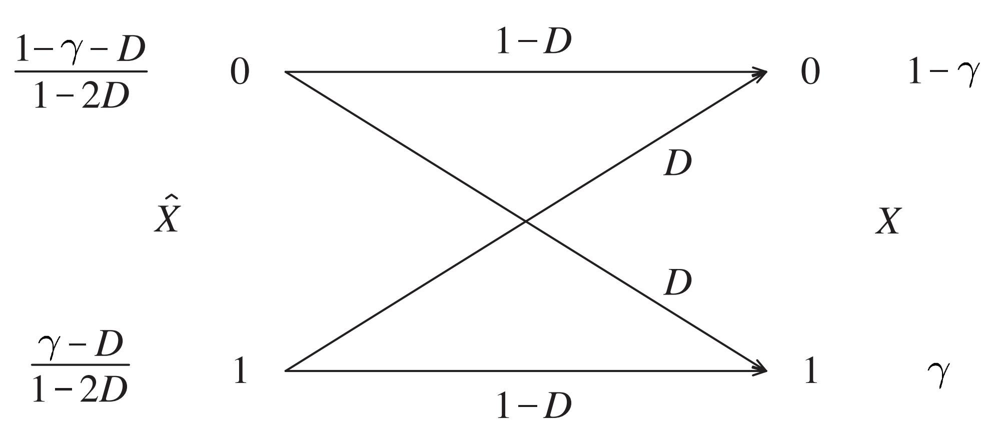

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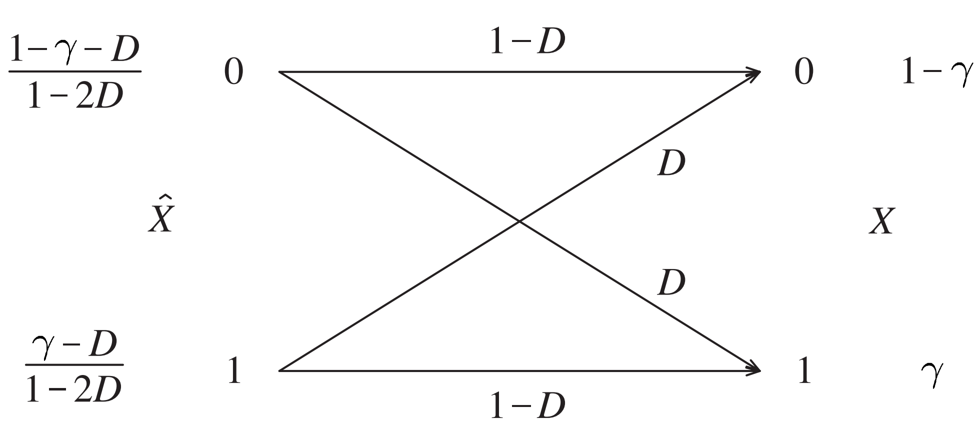

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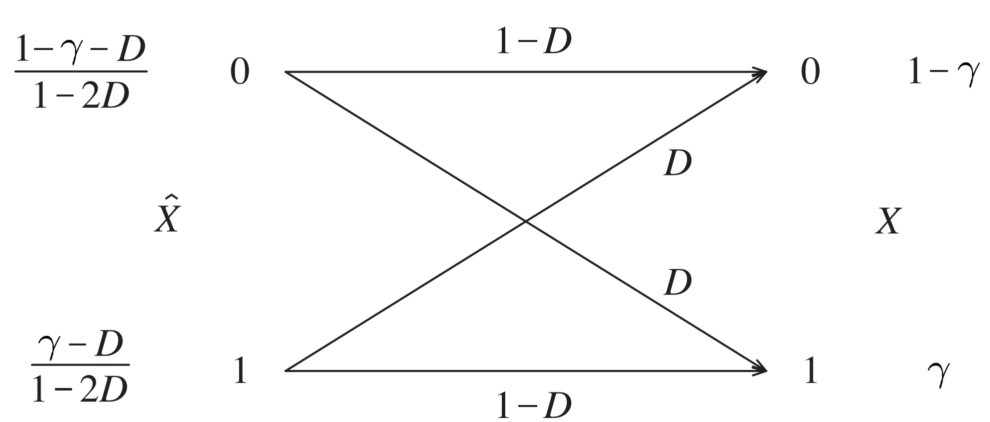

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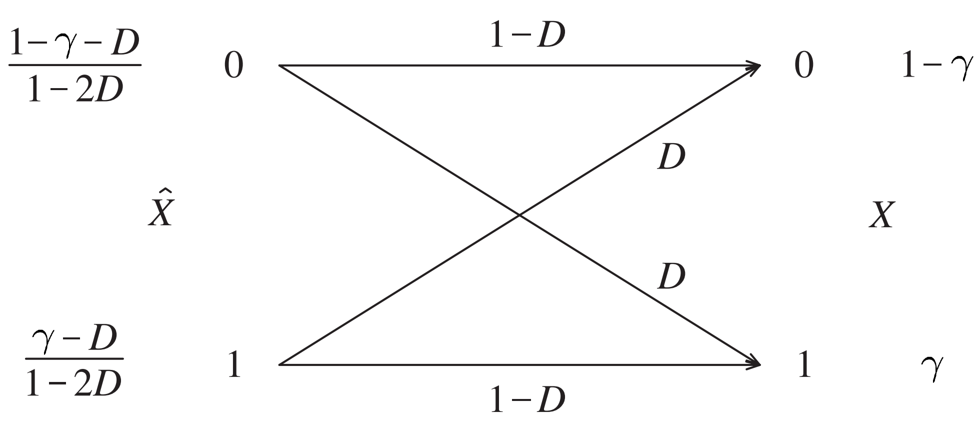

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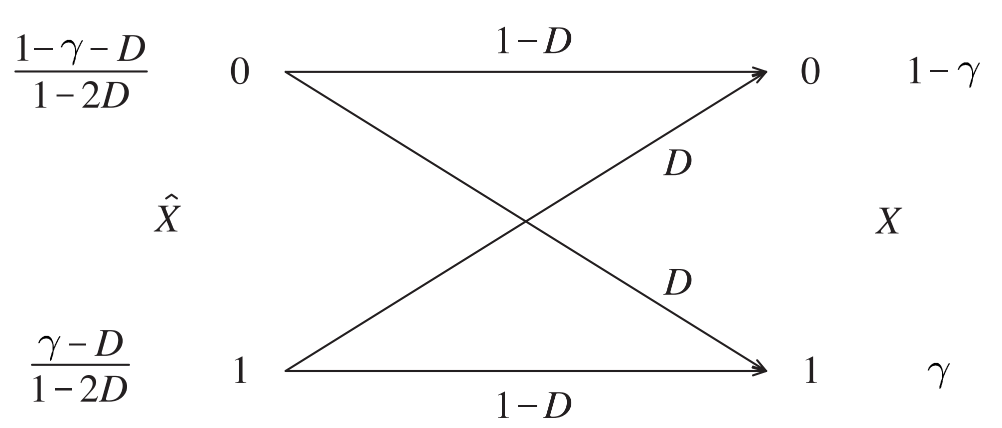

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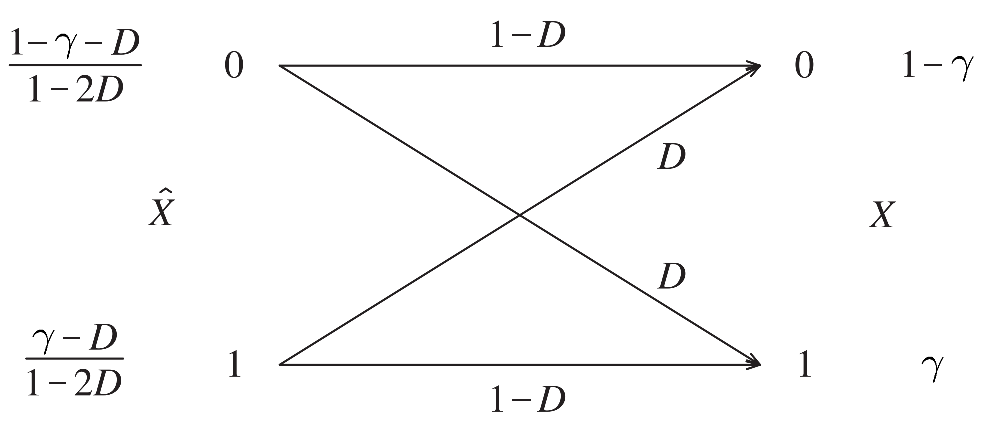

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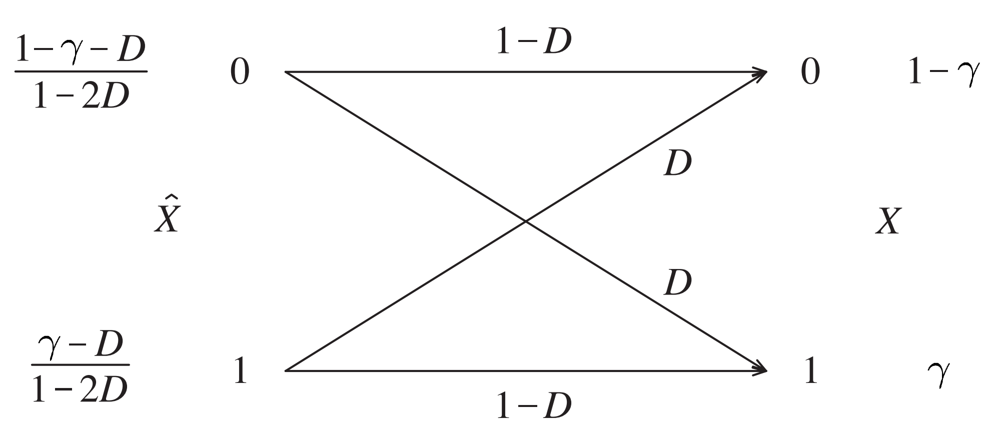

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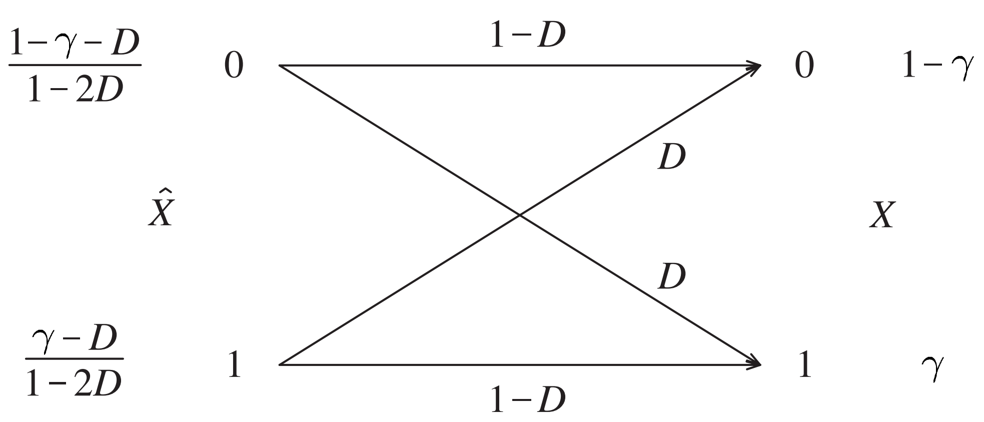

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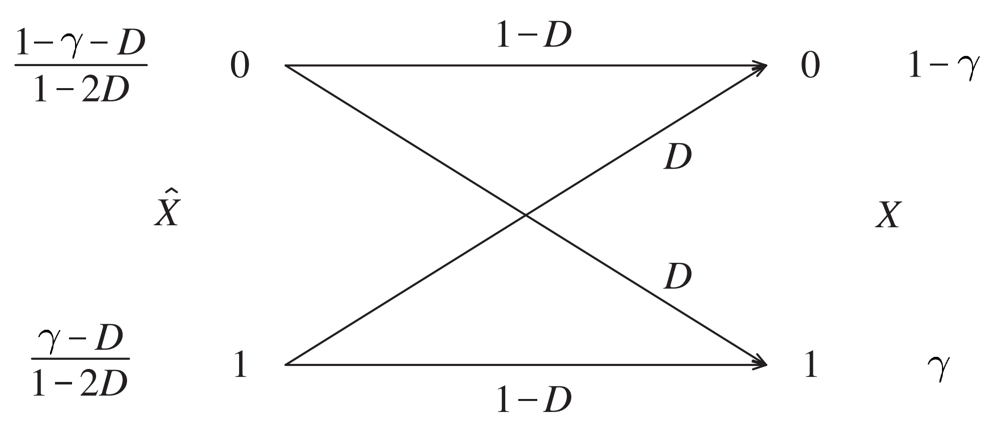

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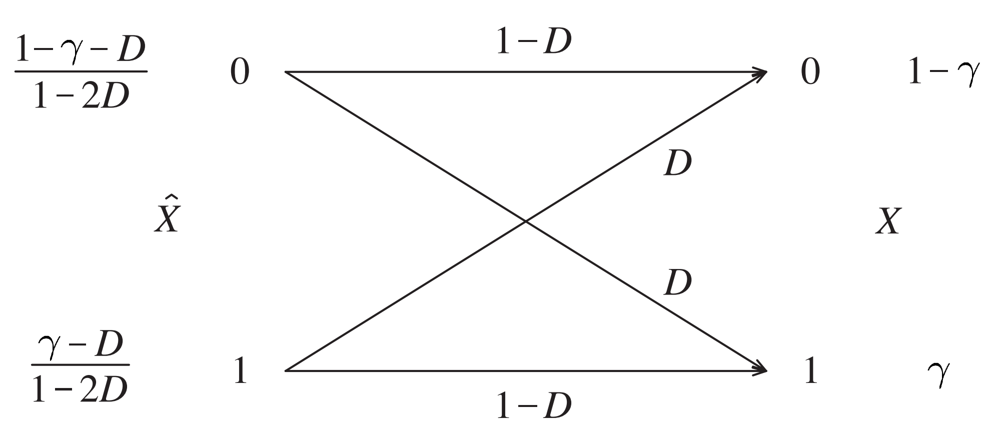

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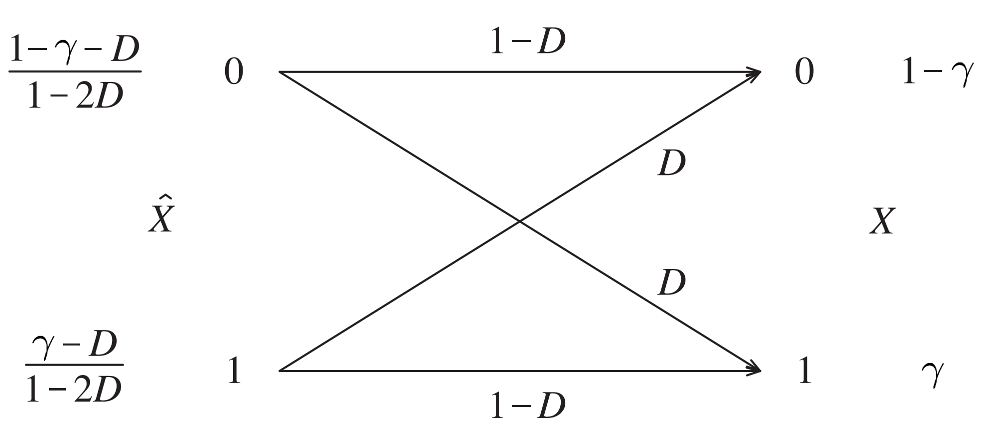

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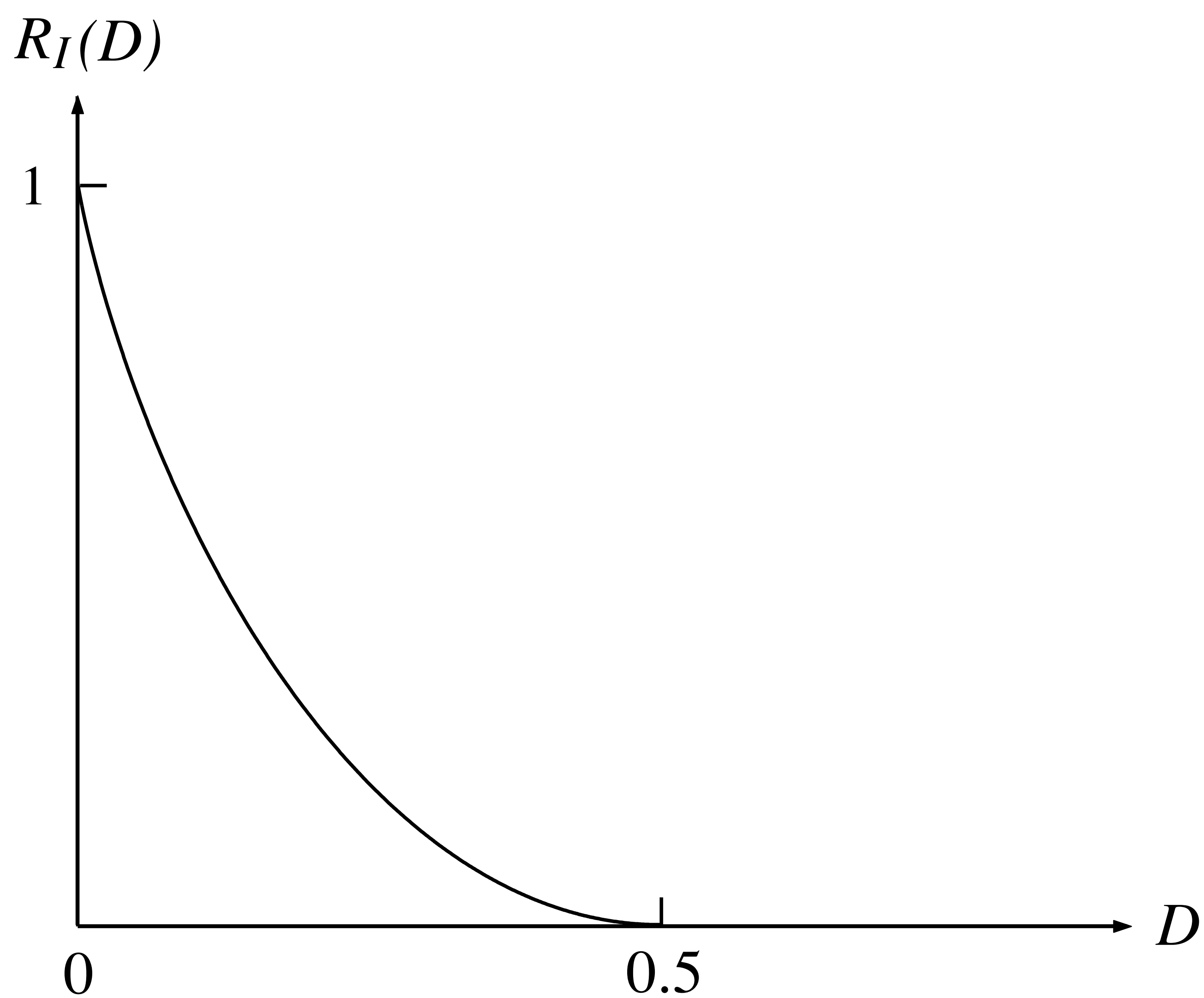

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- However, by the source coding theorem, if $R > H(X)$, the message error probability can be made arbitrarily small, which is much stronger.