香港中文大學
The Chinese University of Hong Kong

8.2 The Rate-Distortion Function

Definition of a Rate-Distortion Code

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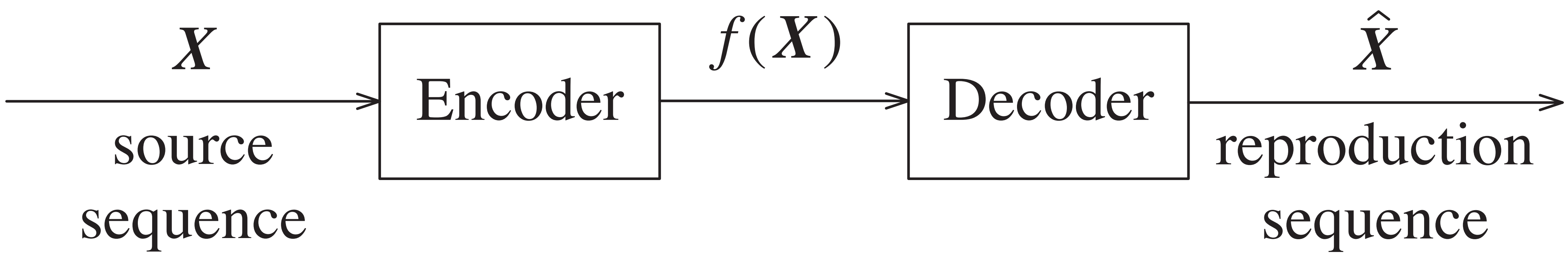
- **Codebook** the set of all codewords

A Rate-Distortion Code



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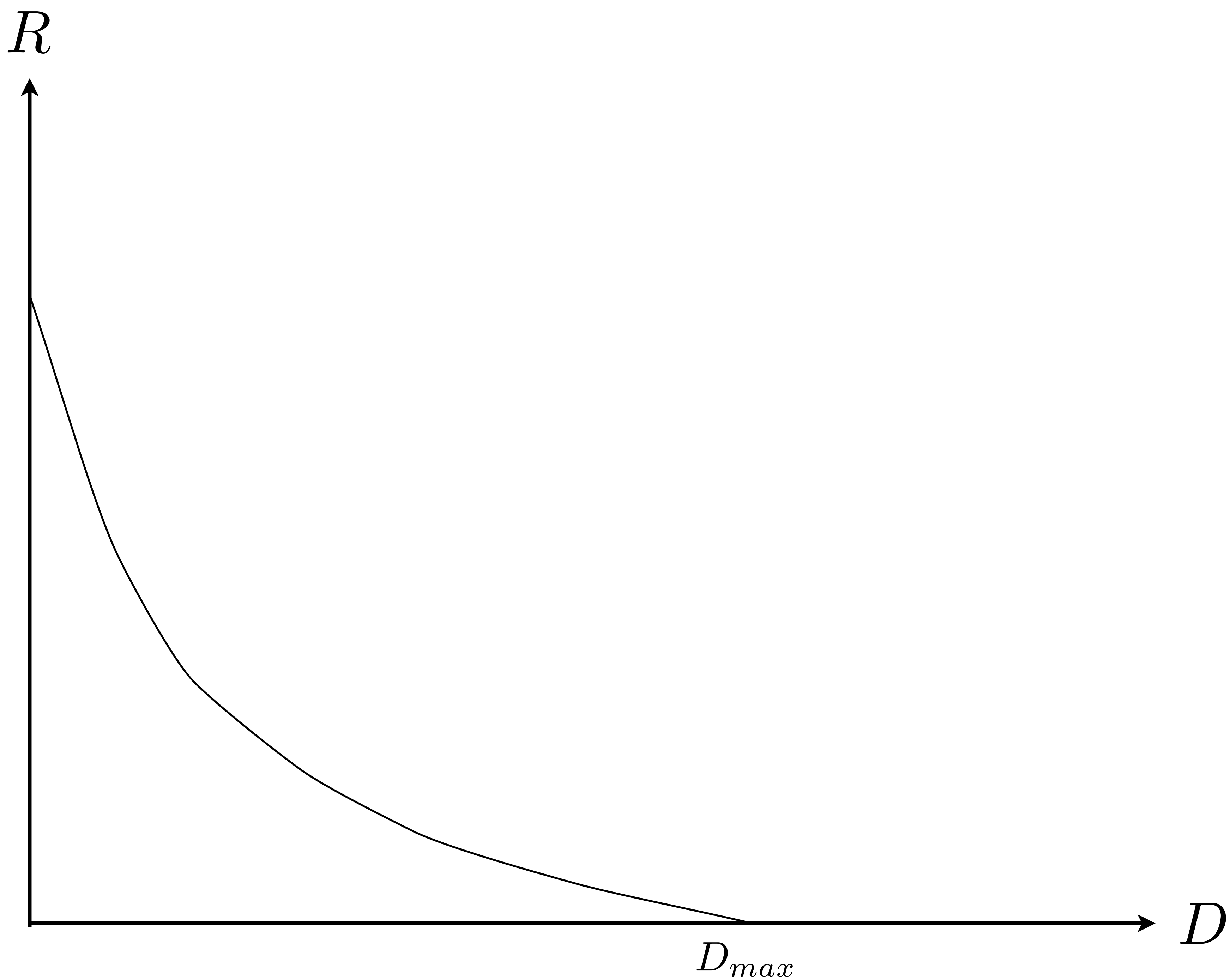
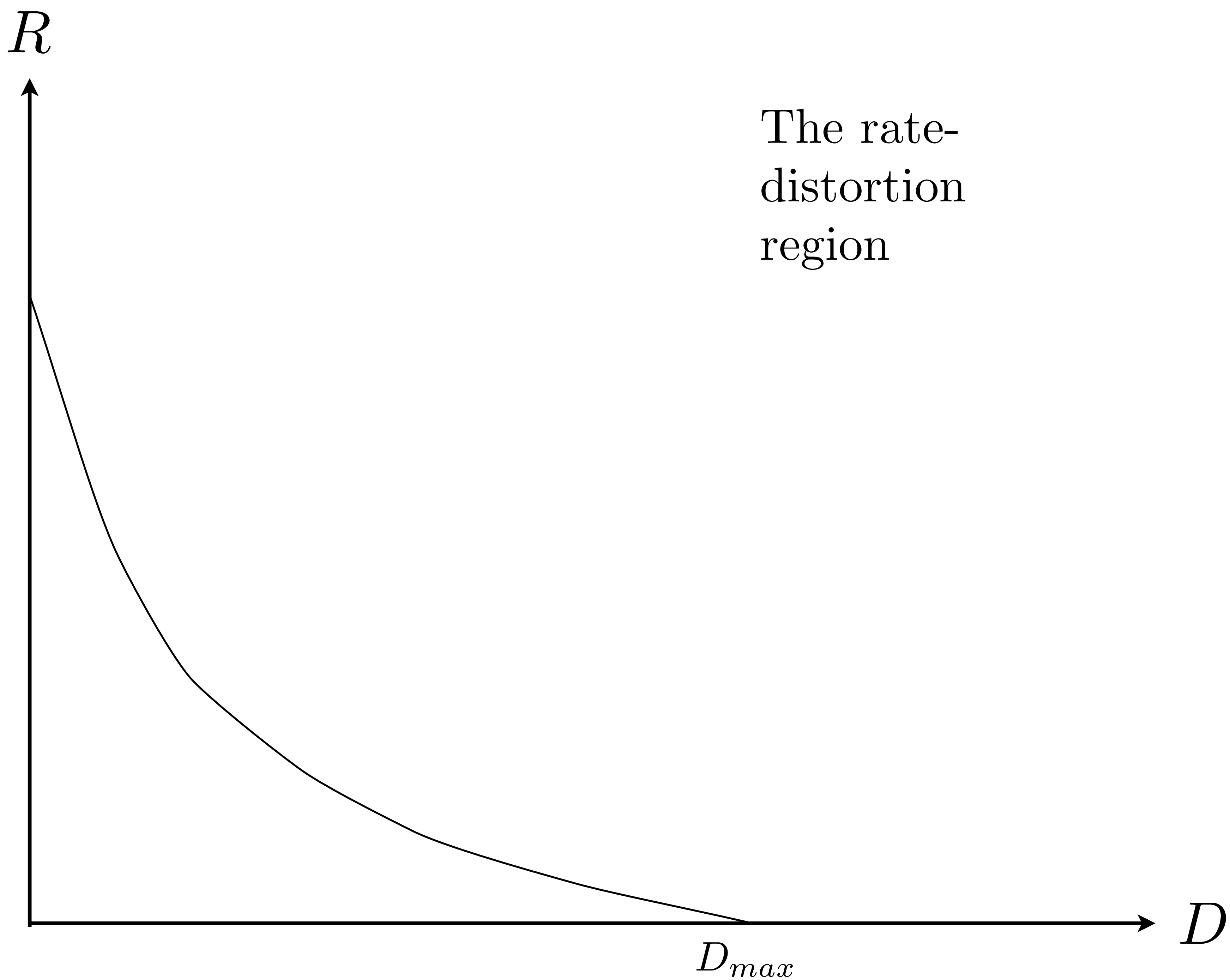
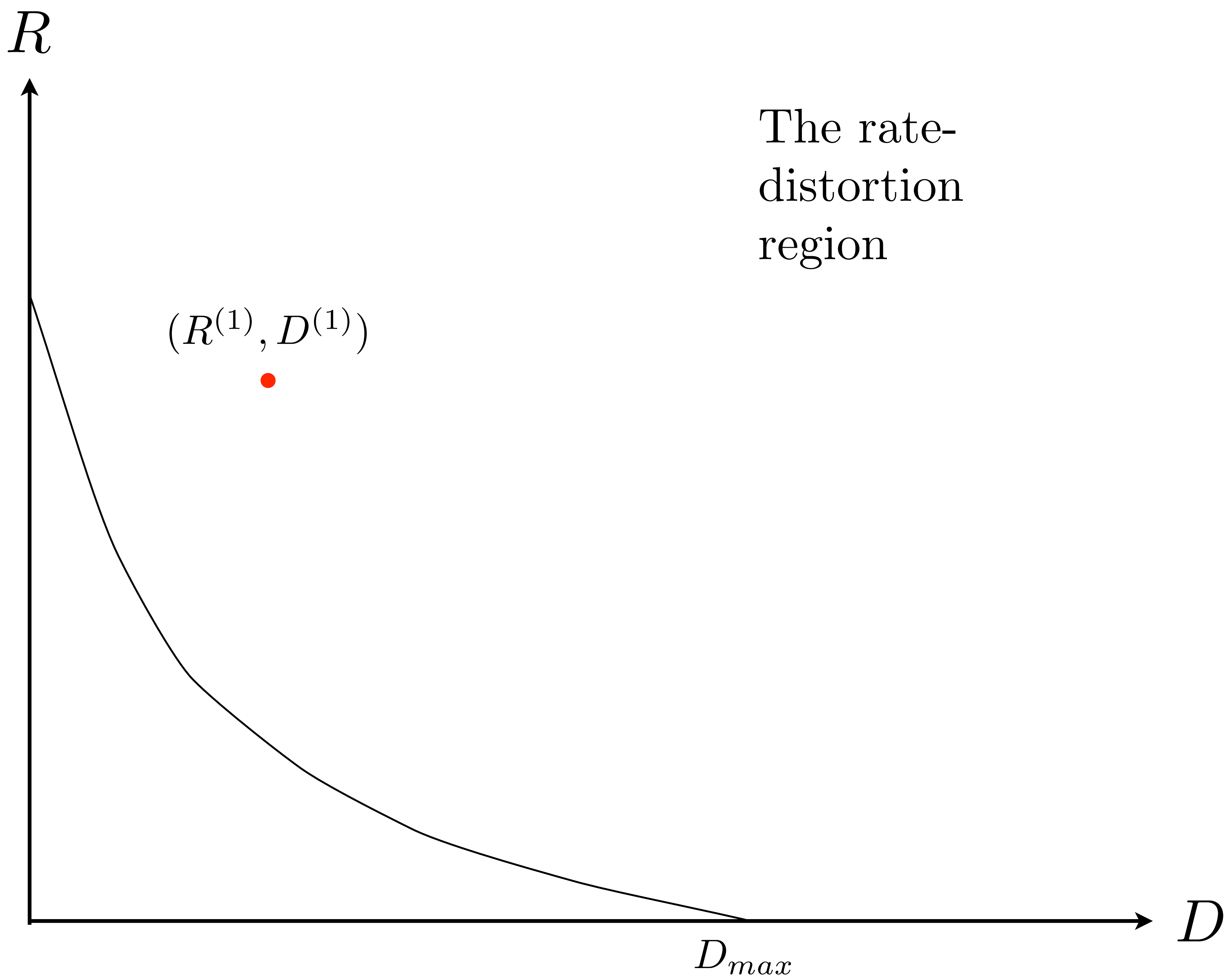
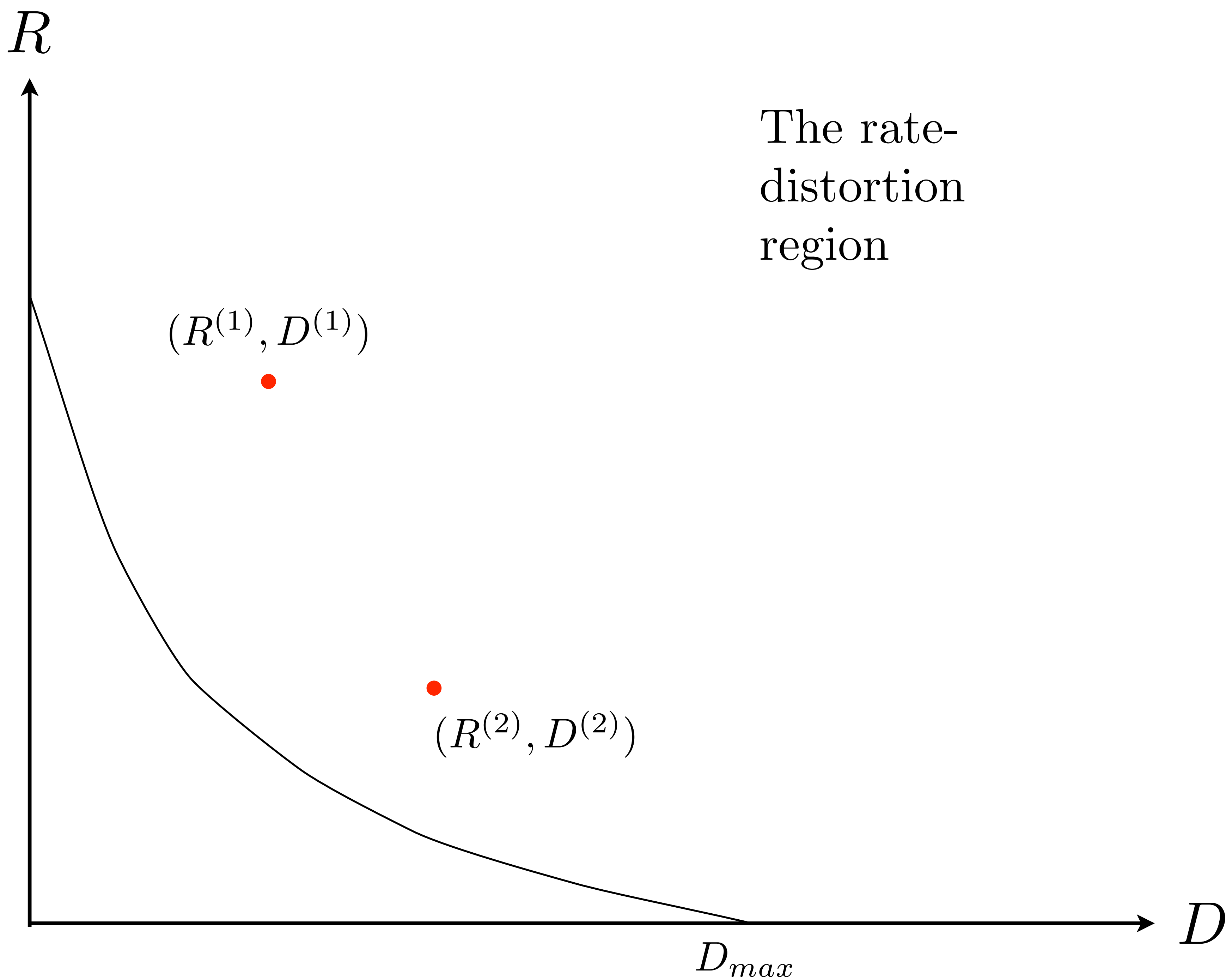

R
D
D_{max}


R
The rate-
distortion
region
D
D_{max}


R
The rate-distortion region
$(R^{(1)}, D^{(1)})$
D
D_{max}


R
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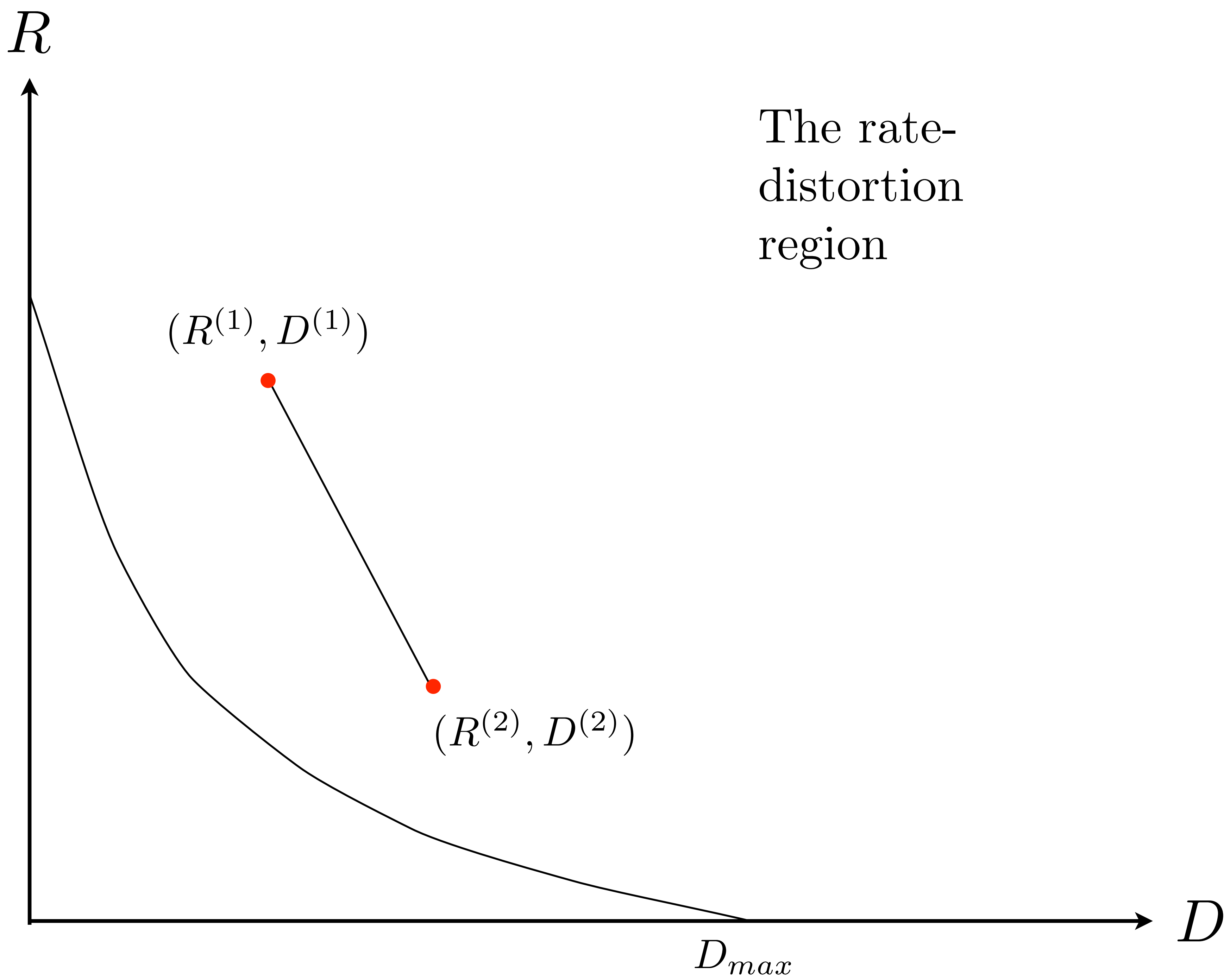
$(R^{(2)}, D^{(2)})$
D
D_{max}



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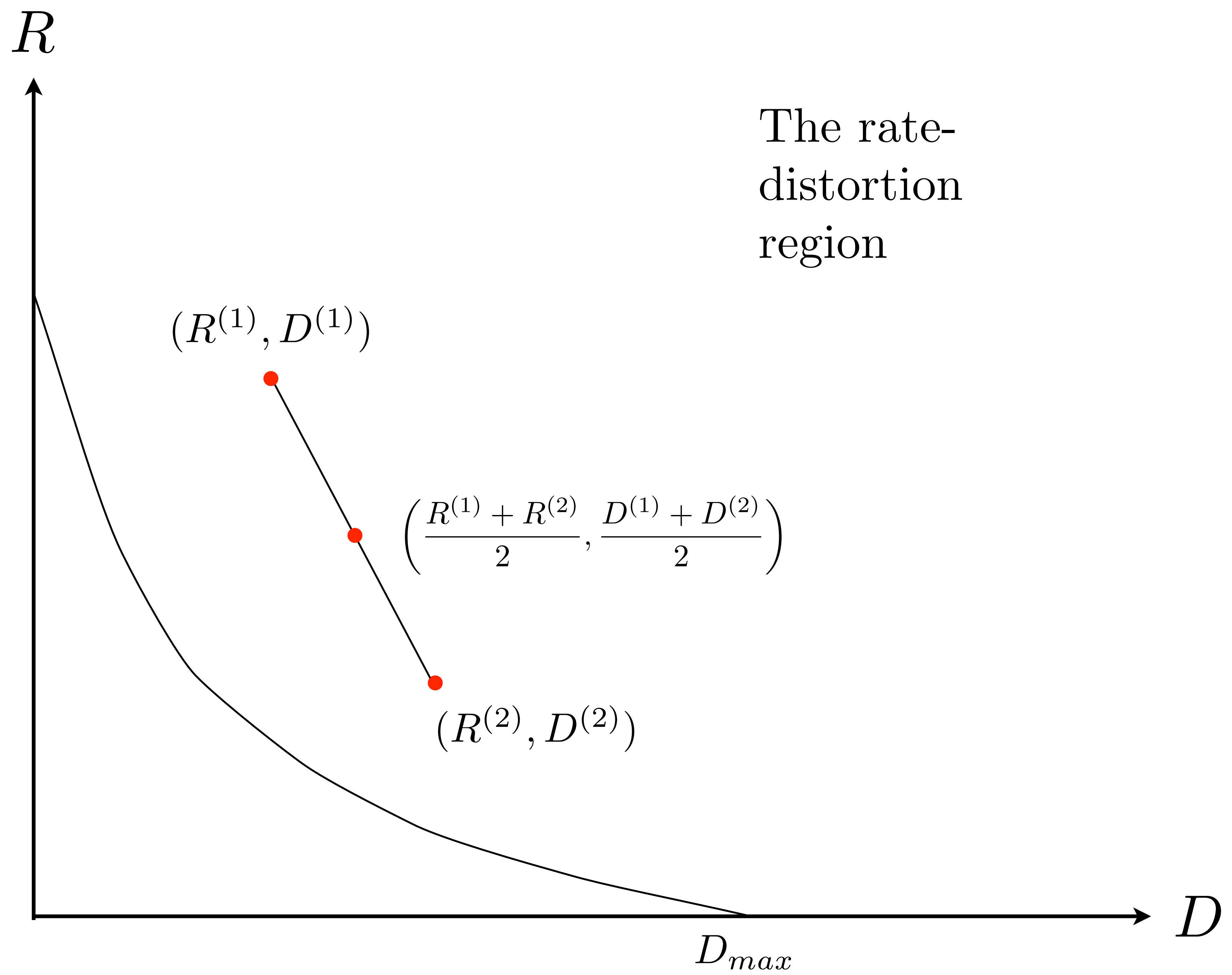

R
The rate-distortion region
$(R^{(1)}, D^{(1)})$
$\left(\frac{R^{(1)} + R^{(2)}}{2}, \frac{D^{(1)} + D^{(2)}}{2}\right)$
$(R^{(2)}, D^{(2)})$
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D_{max}

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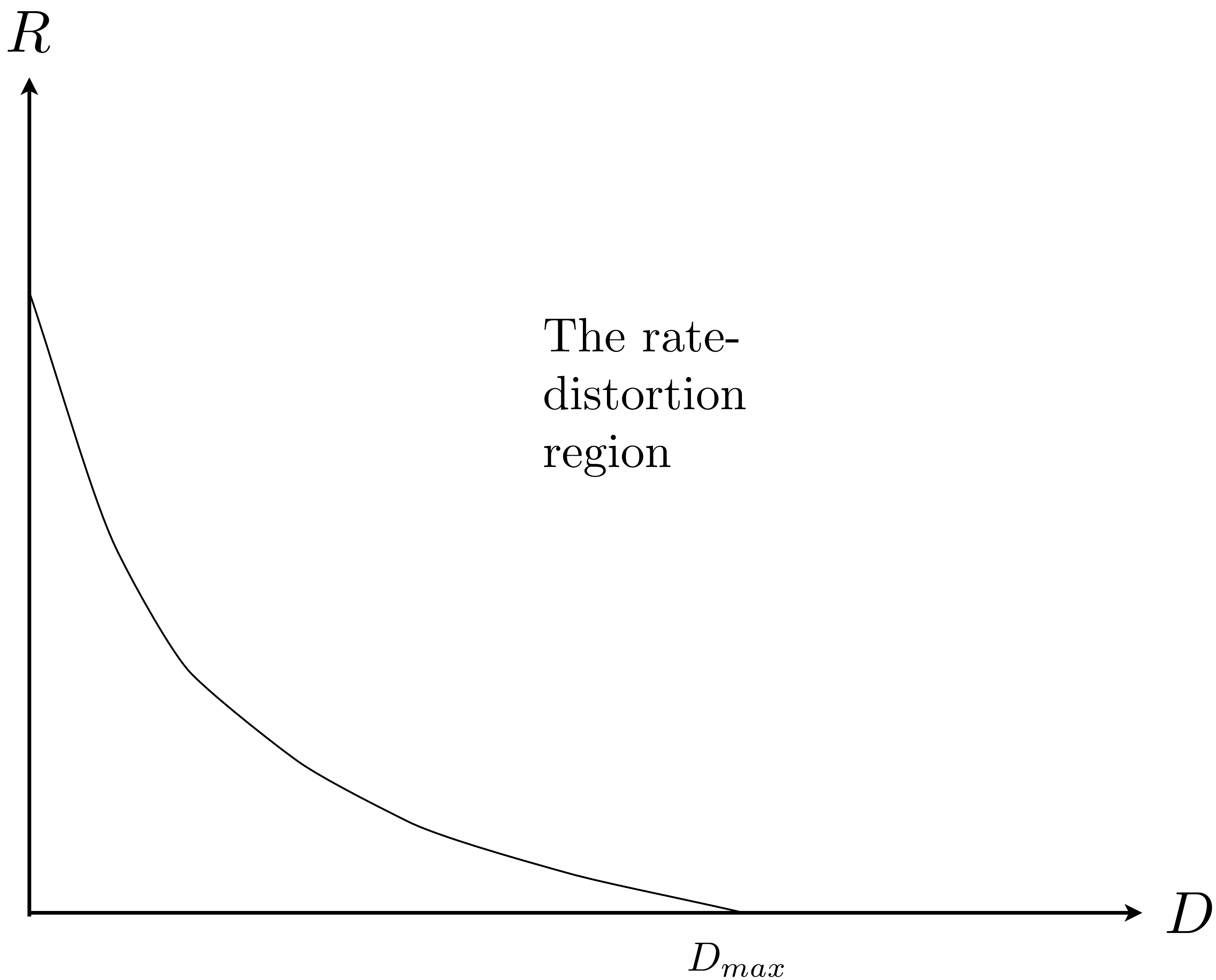
R
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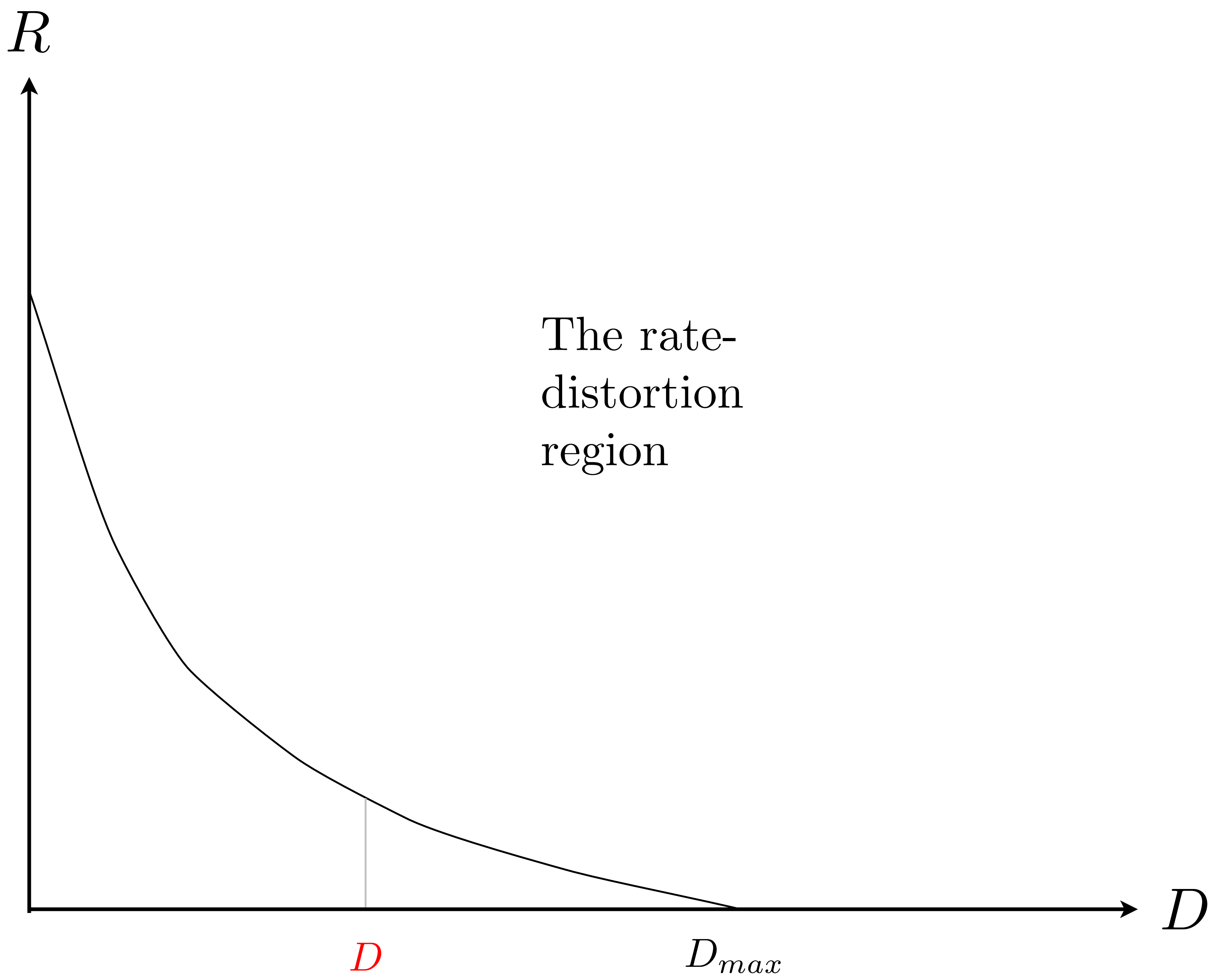
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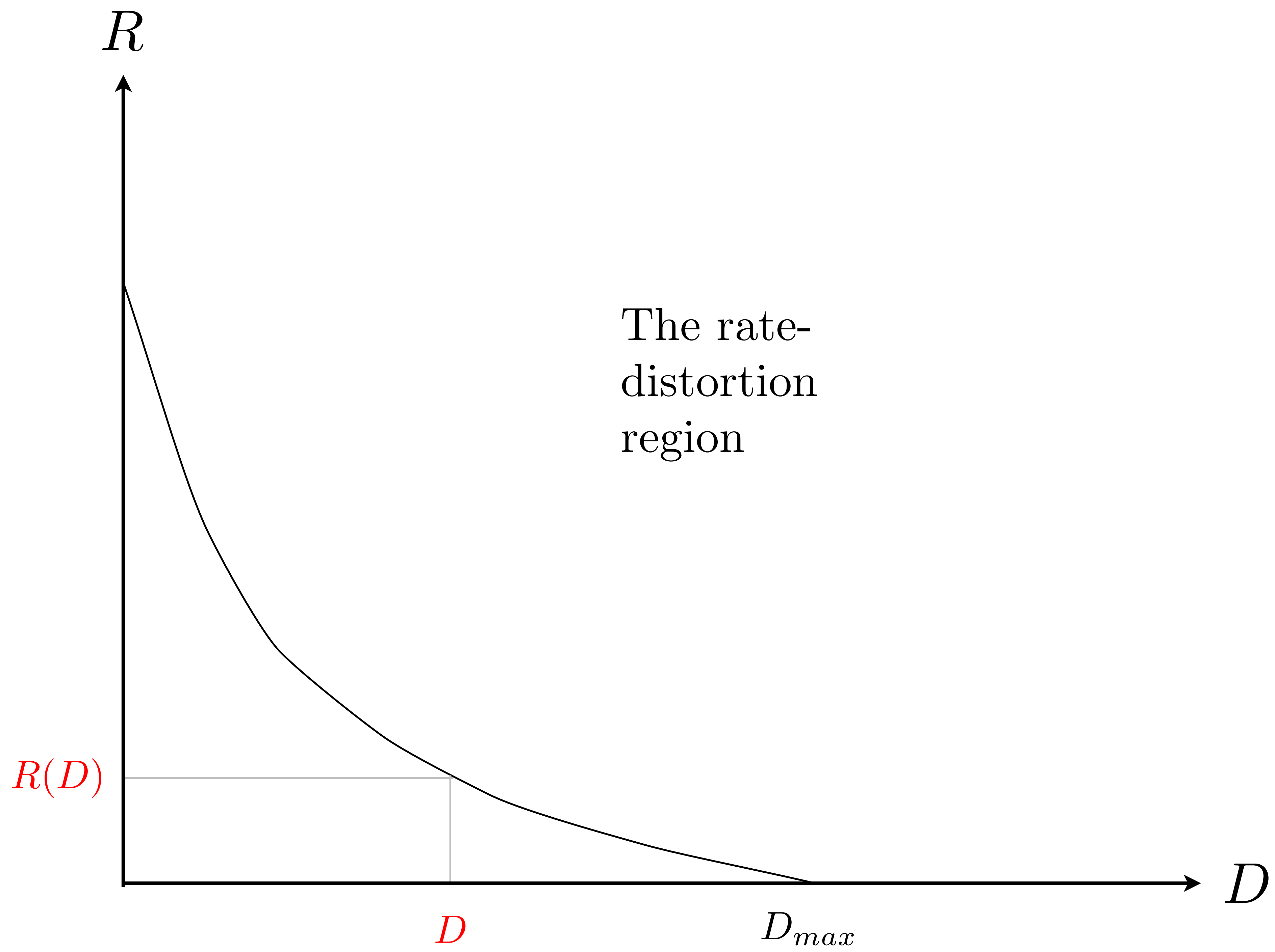
R
The rate-distortion region
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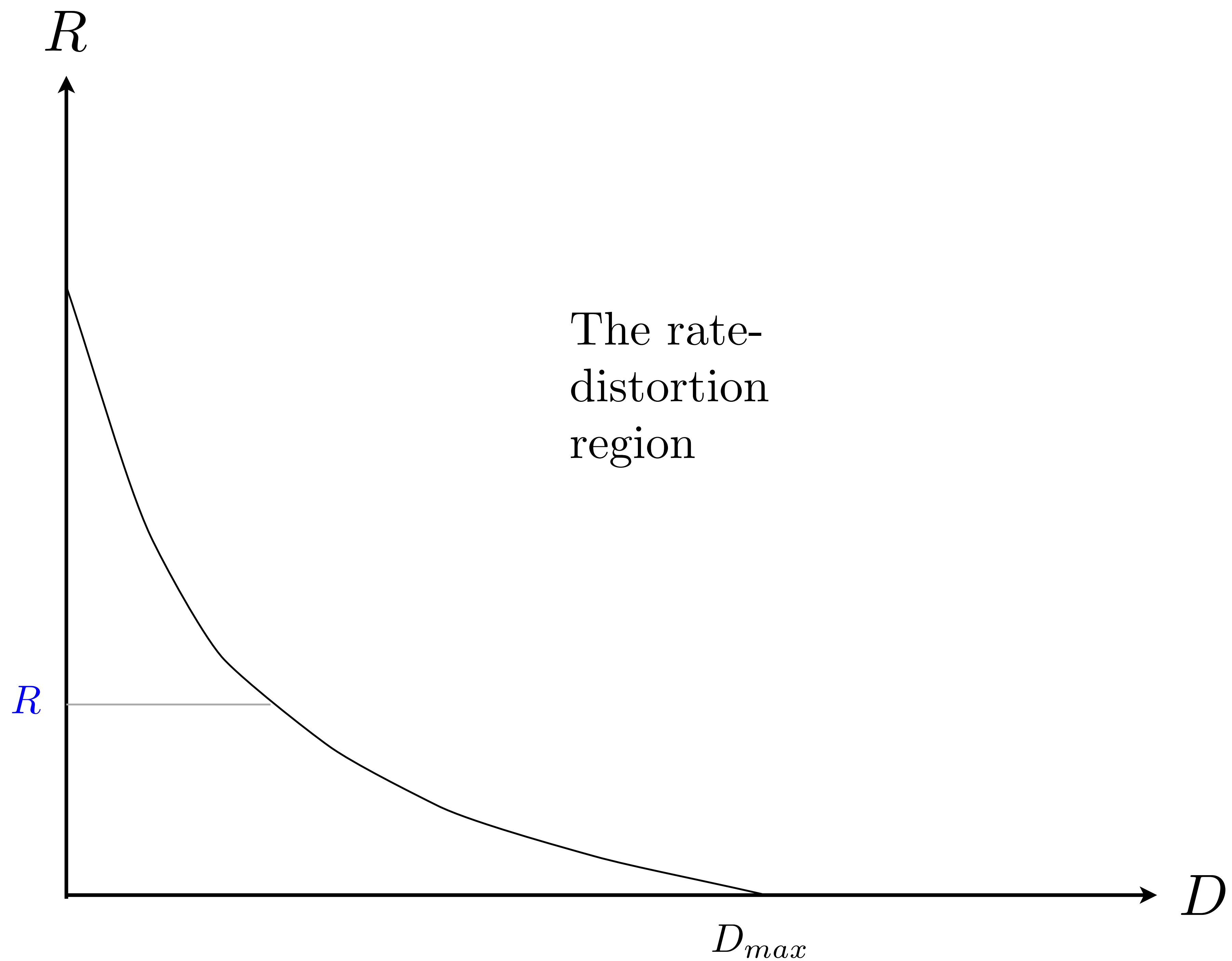
D
D
D_max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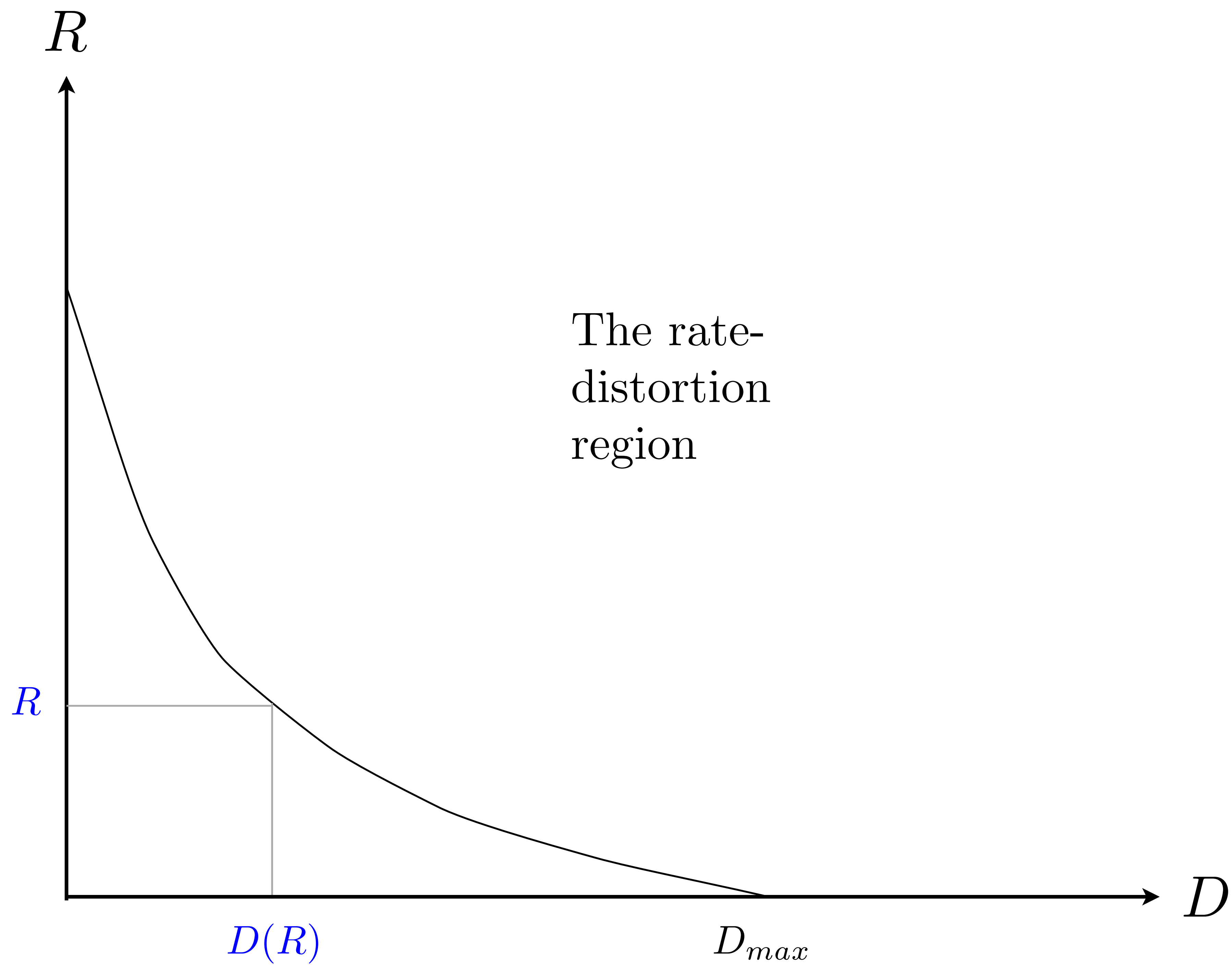
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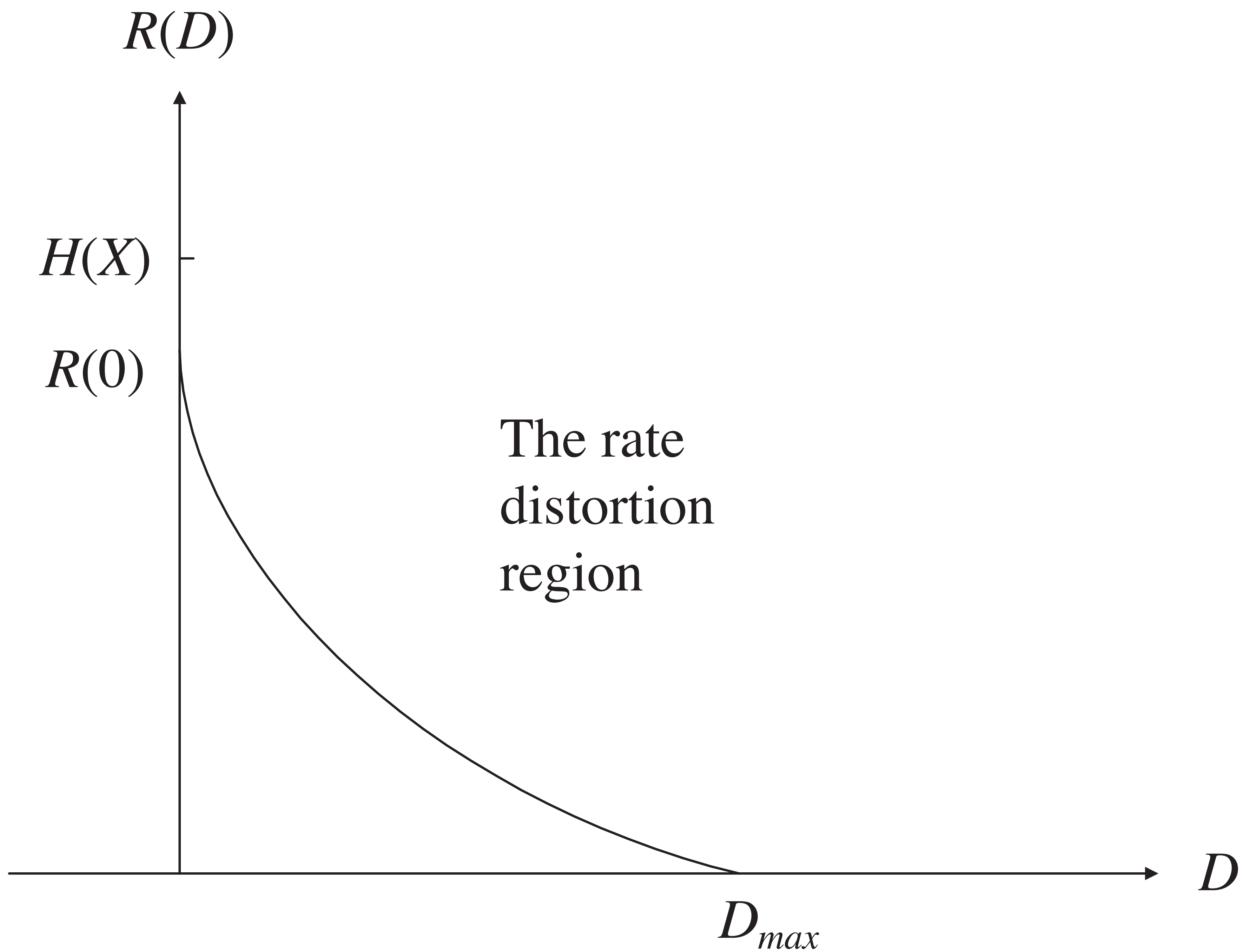
R(D)
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The rate distortion region
D
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