香港中文大學
The Chinese University of Hong Kong

7.6 Feedback Capacity

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- Can feedback increase the channel capacity?
- Not for DMC, even with complete feedback!

Definition 7.18 An (n, M) code with complete feedback for a discrete memoryless channel with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ is defined by encoding functions

$$f_i : \{1, 2, \cdots, M\} \times \mathcal{Y}^{i-1} \to \mathcal{X}$$

for $1 \leq i \leq n$ and a decoding function

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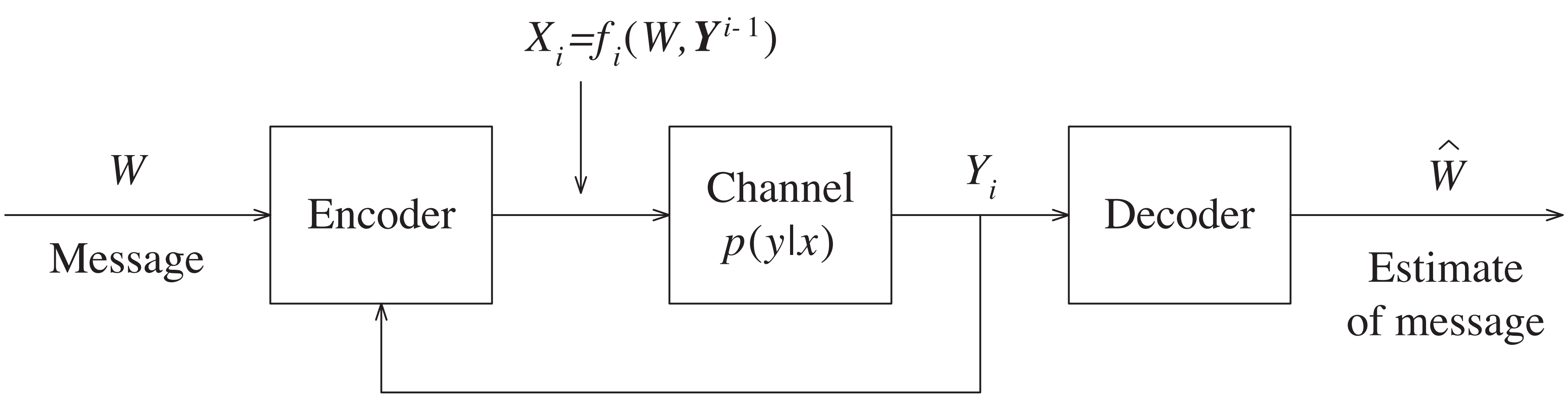

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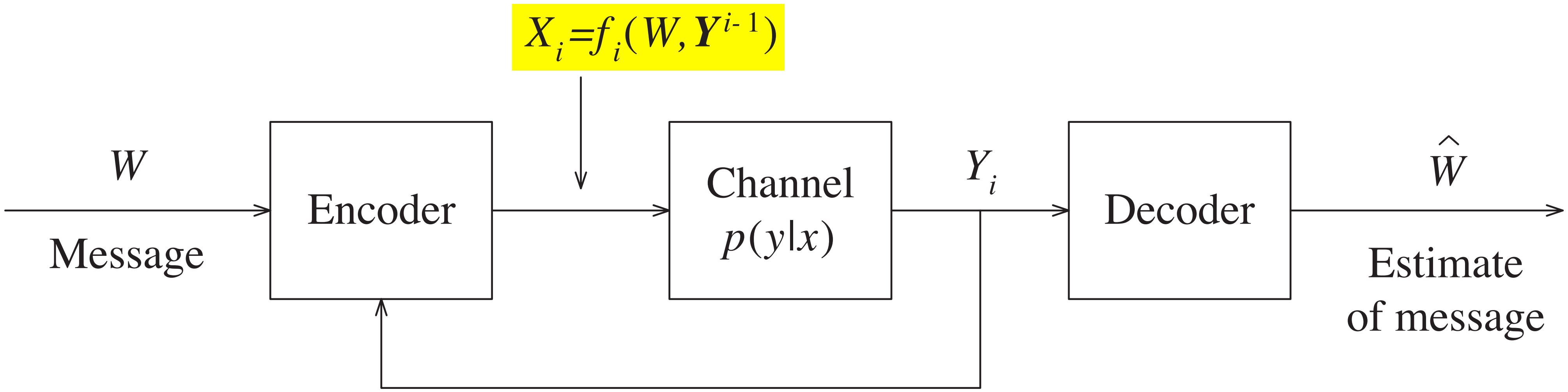

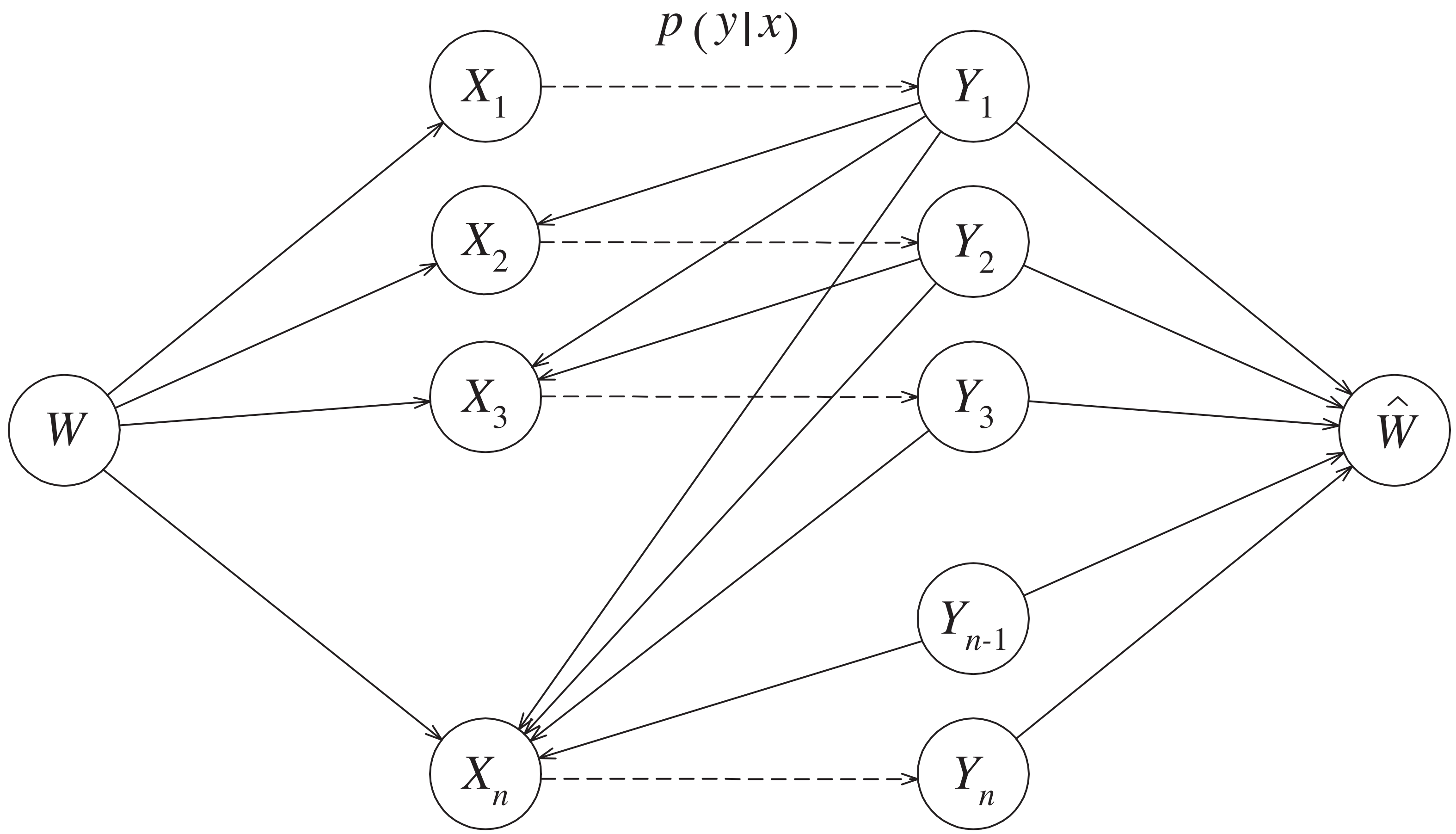

- The above is the dependency graph for a channel code with feedback, which is equivalent to

$$q(w, \mathbf{x}, \mathbf{y}, \hat{w}) = q(w) \left(\prod_{i=1}^{n} q(x_i | w, \mathbf{y}^{i-1}) \right) \left(\prod_{i=1}^{n} p(y_i | x_i) \right) q(\hat{w} | \mathbf{y})$$

  for all $(w, \mathbf{x}, \mathbf{y}, \hat{w}) \in \mathcal{W} \times \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{W}$ such that $q(w, \mathbf{y}^{i-1}), q(x_i) > 0$ for $1 \leq i \leq n$ and $q(\mathbf{y}) > 0$, where $\mathbf{y}^i = (y_1, y_2, \cdots, y_i)$.

Definition 7.19 A rate R is achievable with complete feedback for a discrete memoryless channel $p(y|x)$ if for any $\epsilon > 0$, there exists for sufficiently large n an (n, M) code with complete feedback such that

$$\frac{1}{n} \log M > R - \epsilon$$

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Remark Since a channel code without feedback is a special case of a channel code with complete feedback, $C_{FB} \geq C$.

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forms a Markov chain.

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2. In general, if the channel has memory, feedback can increase the capacity.