香港中文大學
The Chinese University of Hong Kong

Chapter 7
Discrete Memoryless Channels

The Chinese University of Hong Kong

In this chapter

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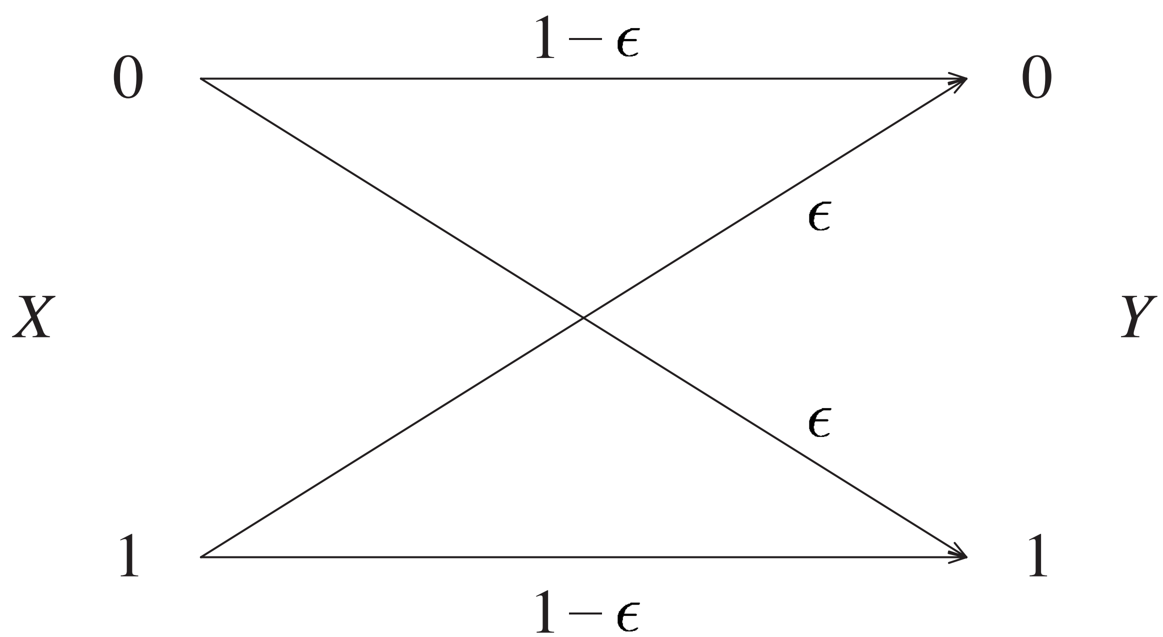
- The capacity when there is feedback
- Separation theorem for source coding and channel coding

An Informal Discussion

Binary Symmetric Channel (BSC)



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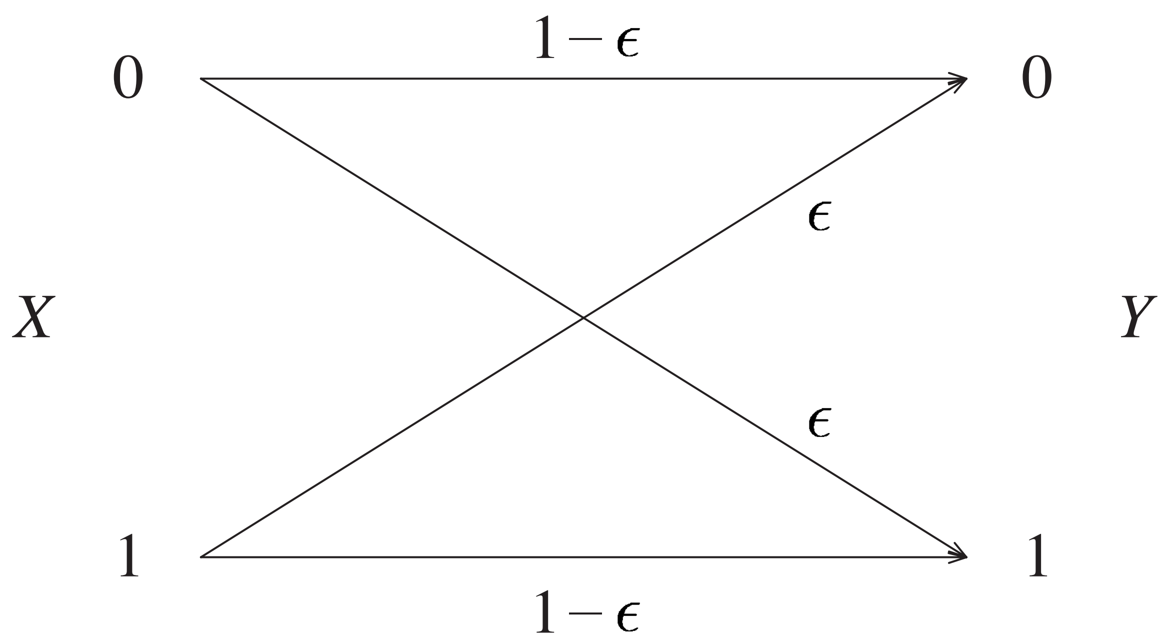

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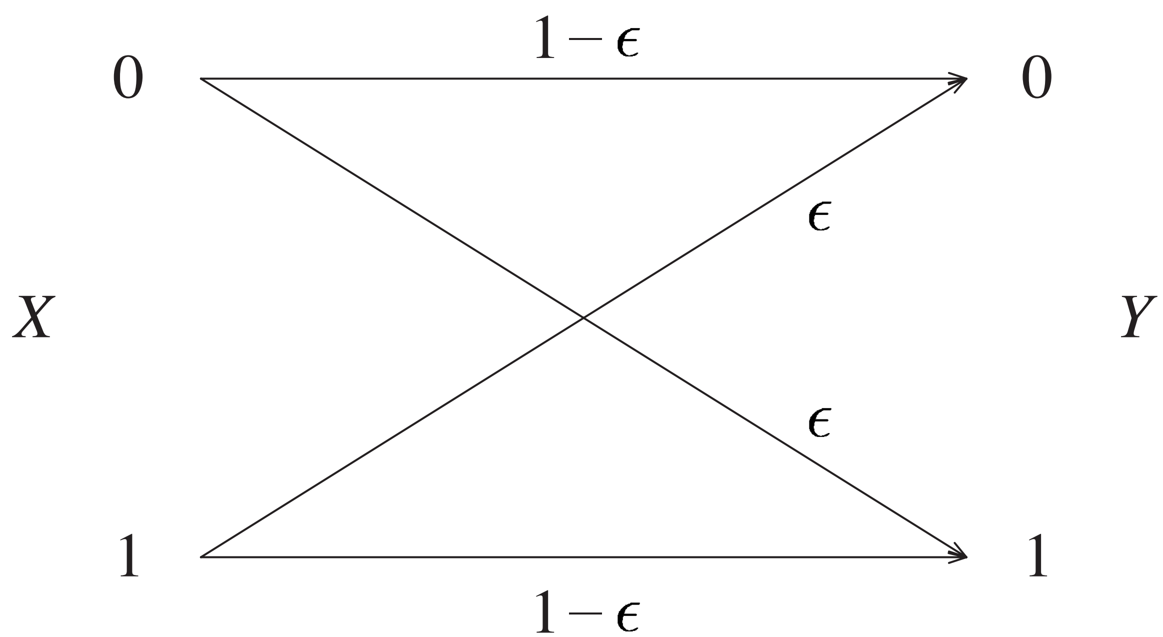

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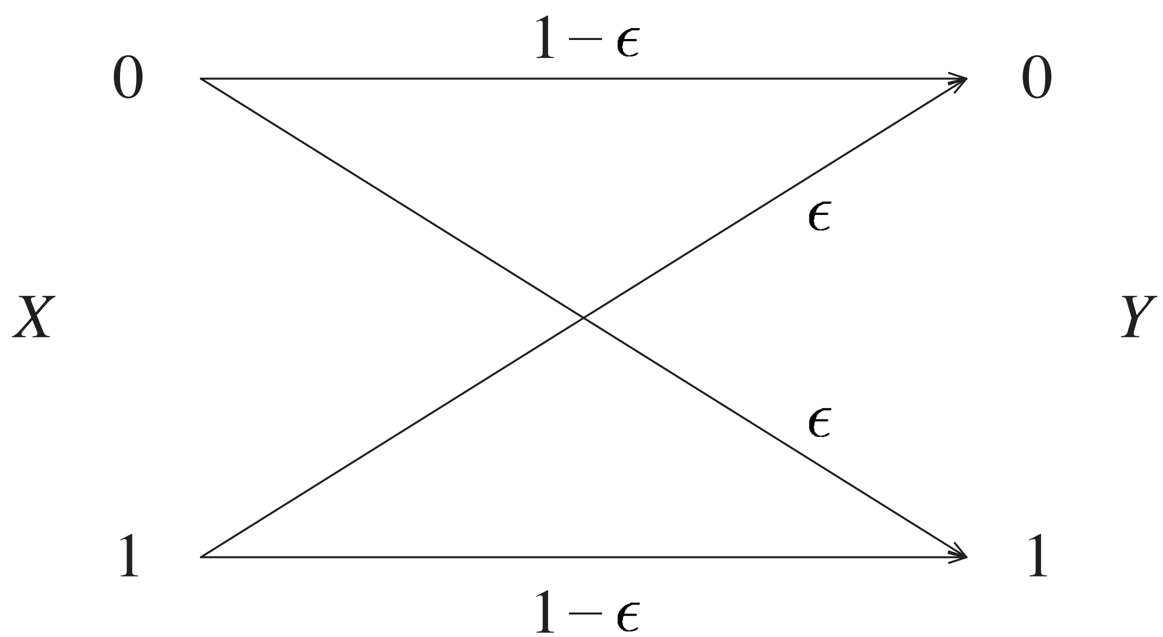

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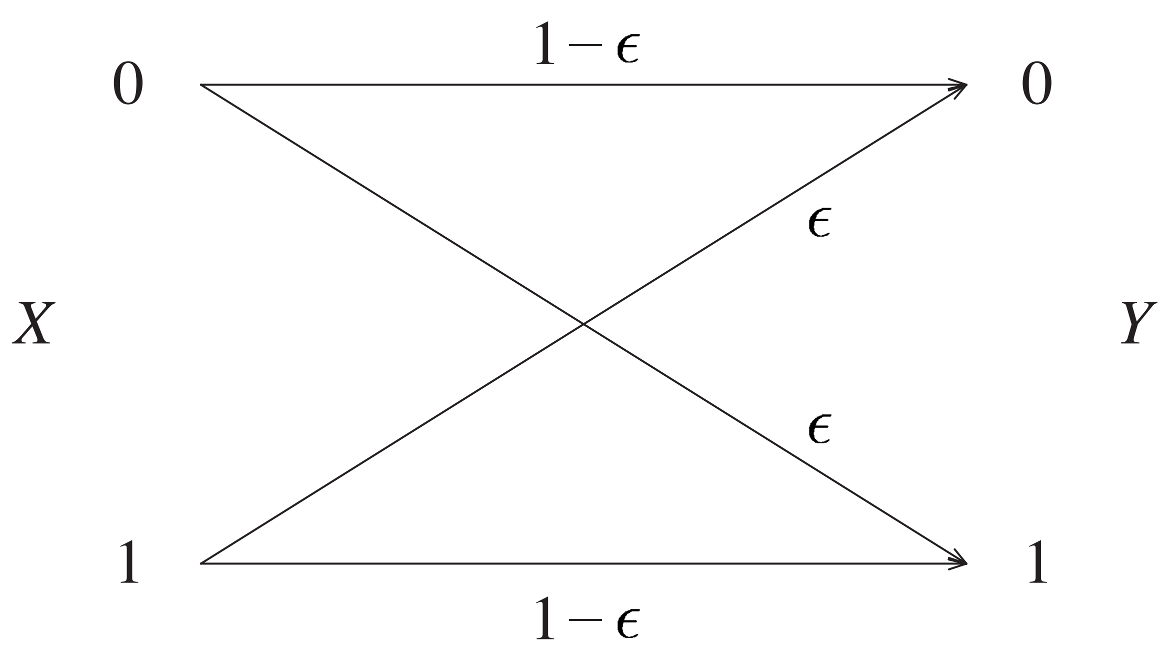

- the simplest channel model
- input alphabet $\mathcal{X} = \{0, 1\}$
- output alphabet $\mathcal{Y} = \{0, 1\}$
- Crossover probability $= \epsilon$, $0 \leq \epsilon \leq 1$

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- **Coding Scheme 1**

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- A decoding error if and only if a crossover occurs. Therefore, $P_e = \epsilon$.

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- Then $N_0 > N_1$ because $\epsilon < 0.5$.
- Therefore decode correctly w.p. $\rightarrow 1$ if message is A (similarly for B).
- However, $R = \frac{1}{n}\log 2 \rightarrow 0$ as $n \rightarrow \infty$. :(

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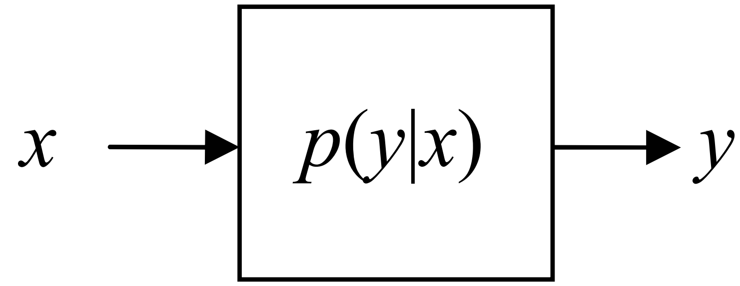
7.1 Definition and Capacity

Definition 7.1 (Discrete Channel I)



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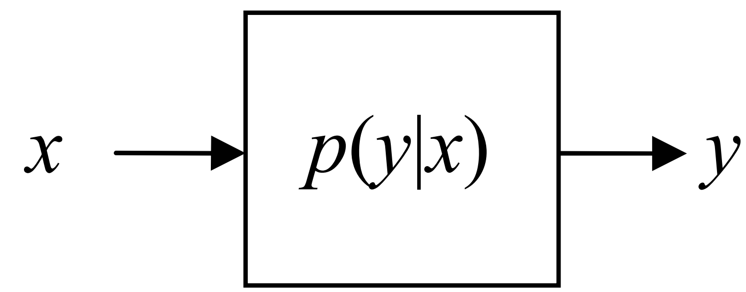

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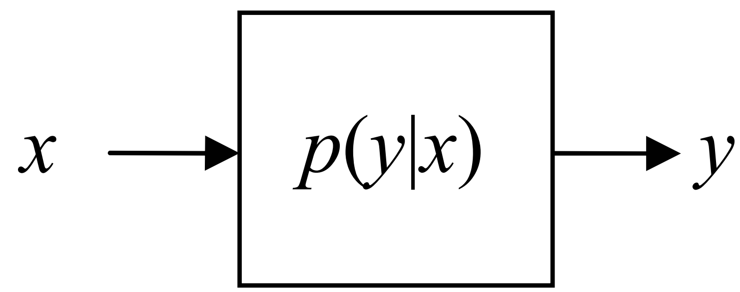

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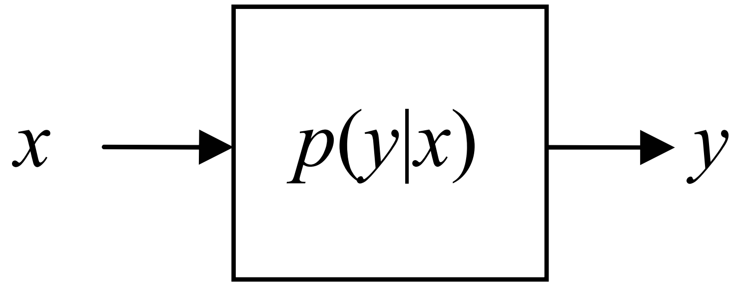

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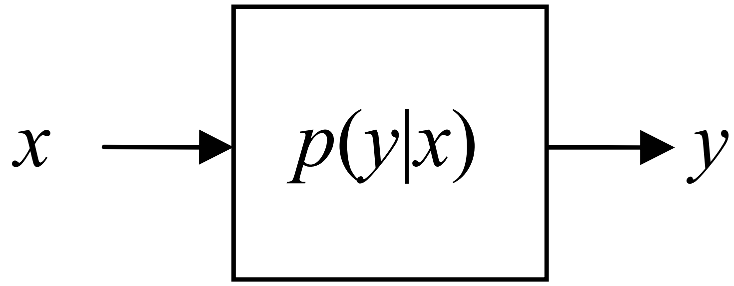

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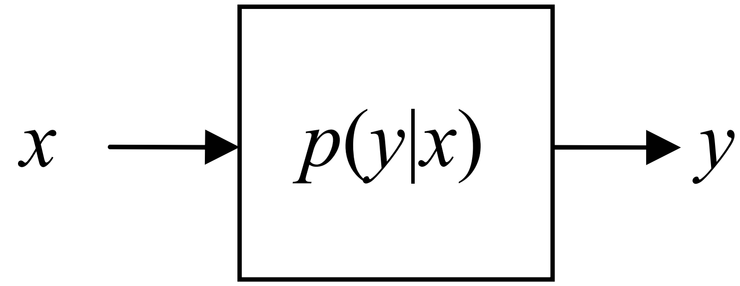

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- BSC with crossover probability ϵ:

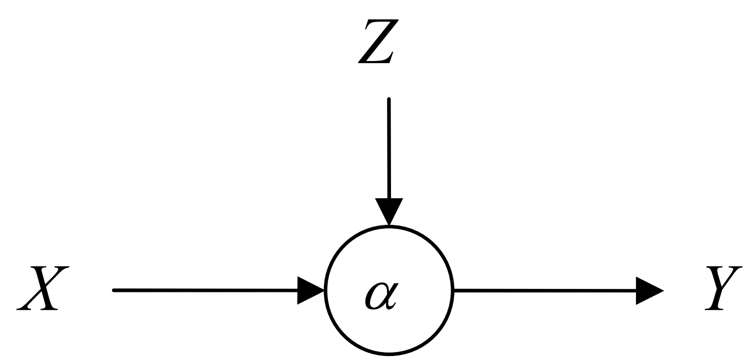
$$[\,p(y|x)] = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}$$

Definition 7.1 (Discrete Channel I) Let $\mathcal{X}$ and $\mathcal{Y}$ be discrete alphabets, and $p(y|x)$ be a transition matrix from $\mathcal{X}$ to $\mathcal{Y}$. A discrete channel $p(y|x)$ is a single-input single-output system with input random variable X taking values in $\mathcal{X}$ and output random variable Y taking values in $\mathcal{Y}$ such that

$$\Pr\{X = x, Y = y\} = \Pr\{X = x\}\, p(y|x)$$

for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Definition 7.2 (Discrete Channel II)



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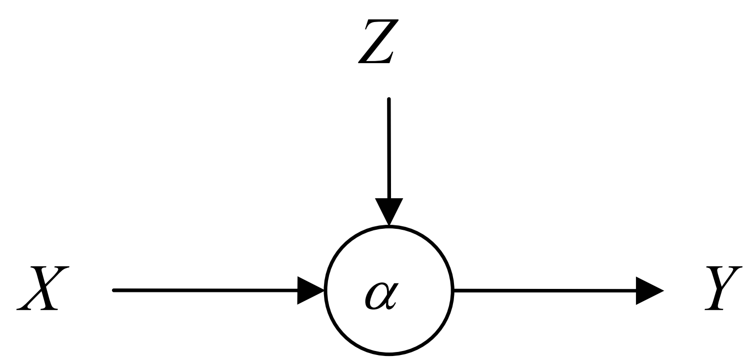

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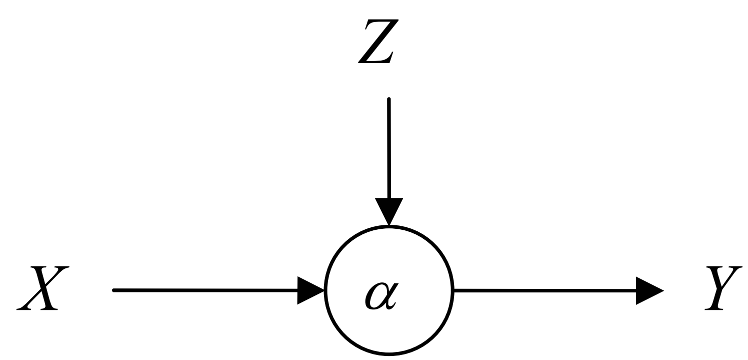

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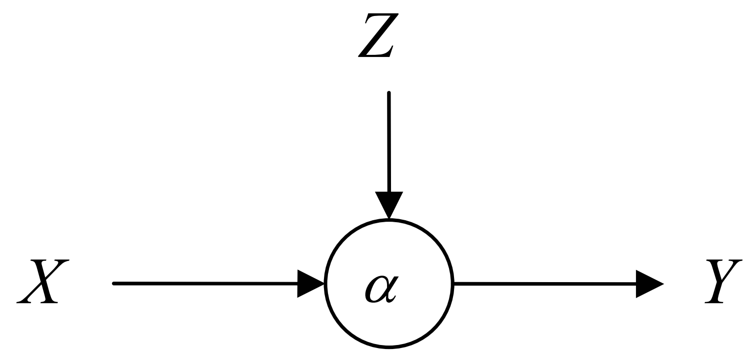

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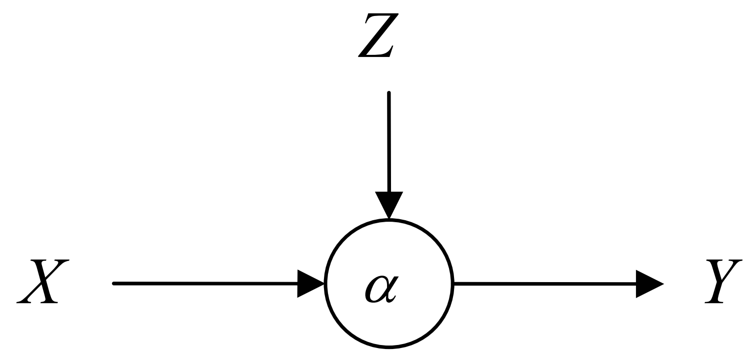

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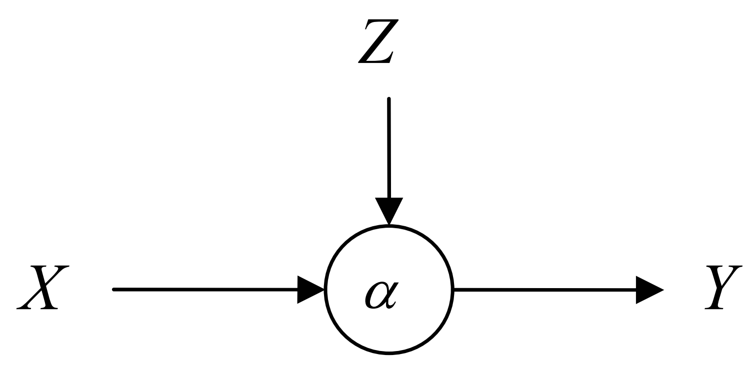

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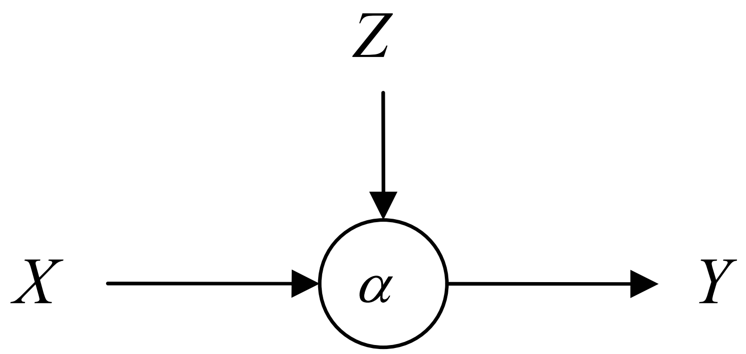

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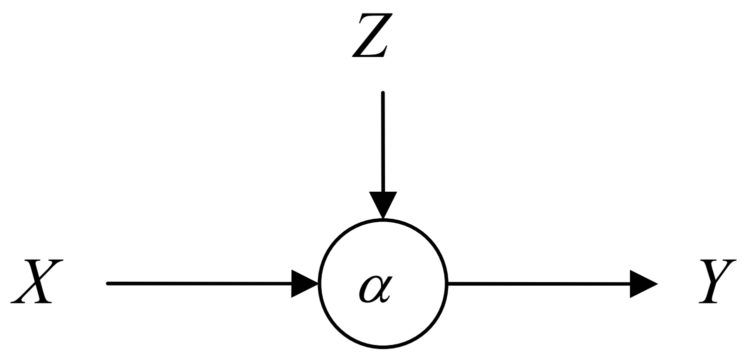

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- Input-output relation: $Y = \alpha(X, Z)$.

Definition 7.2 (Discrete Channel II) Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be discrete alphabets. Let $\alpha : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$, and Z be a random variable taking values in $\mathcal{Z}$, called the noise variable. A discrete channel (α, Z) is a single-input single-output system with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$. For any input random variable X, the noise variable Z is independent of X, and the output random variable Y is given by

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&= \Pr\{X = x\}p(y|x)
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$$

Definition 7.3 Two discrete channels $p(y|x)$ and (α, Z) defined on the same input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ are equivalent if

$$\Pr\{\alpha(x, Z) = y\} = p(y|x)$$

for all x and y.

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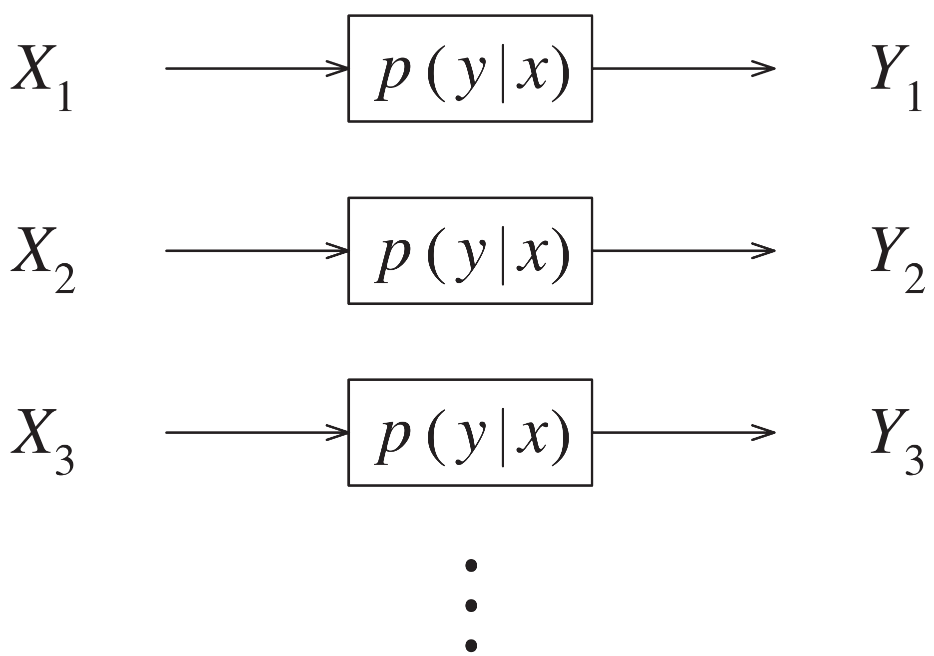
- In such a system, random variables are generated sequentially in discrete-time.
- More than one random variable may be generated instantaneously but sequentially at a particular time index.

Definition 7.4 (DMC I)



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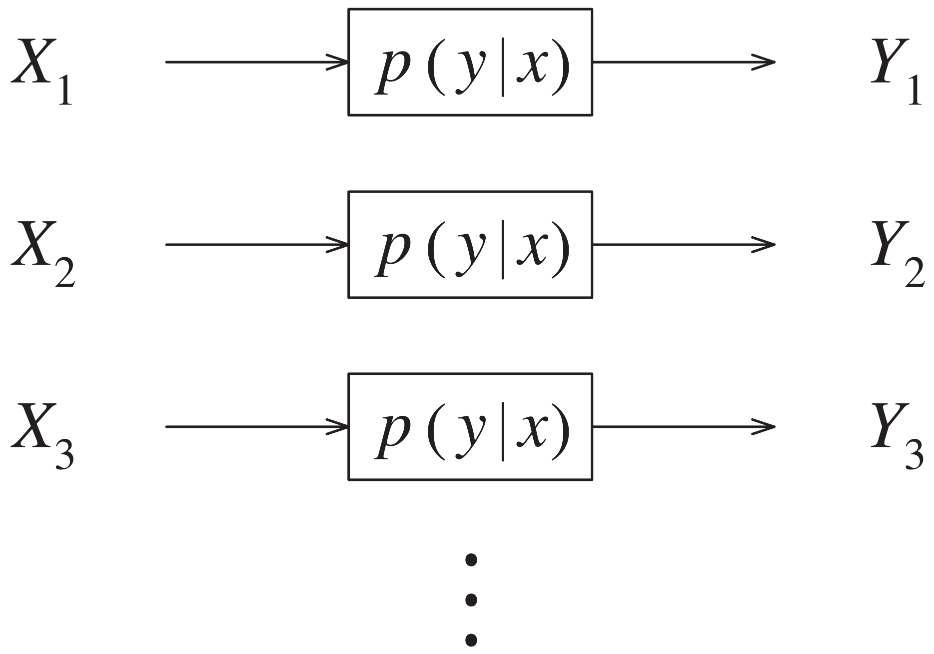

- A DMC specified by $p(y|x)$ is a sequence of replicates of a generic discrete channel $p(y|x)$.

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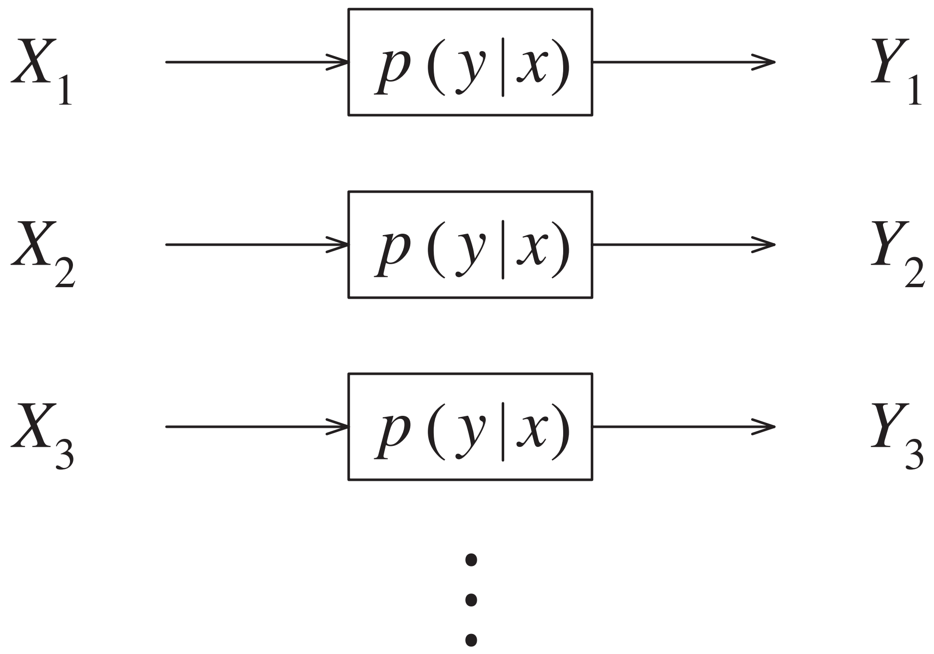

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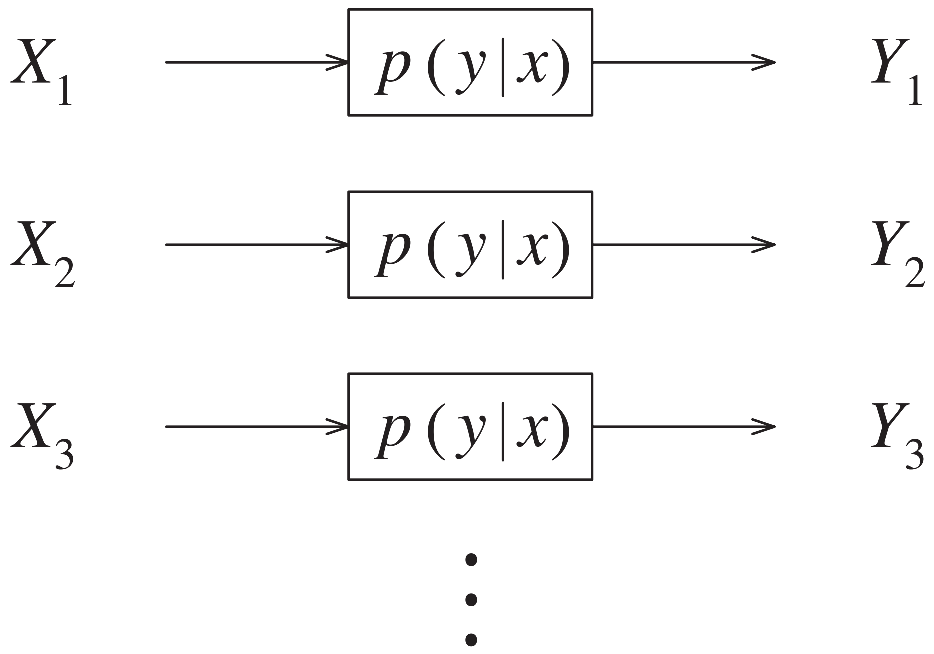

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$$\begin{aligned} &\Pr\{Y_i = y, X_i = x, T_{i-} = t\} \\ &= \Pr\{X_i = x, T_{i-} = t\} p(y|x) \end{aligned}$$

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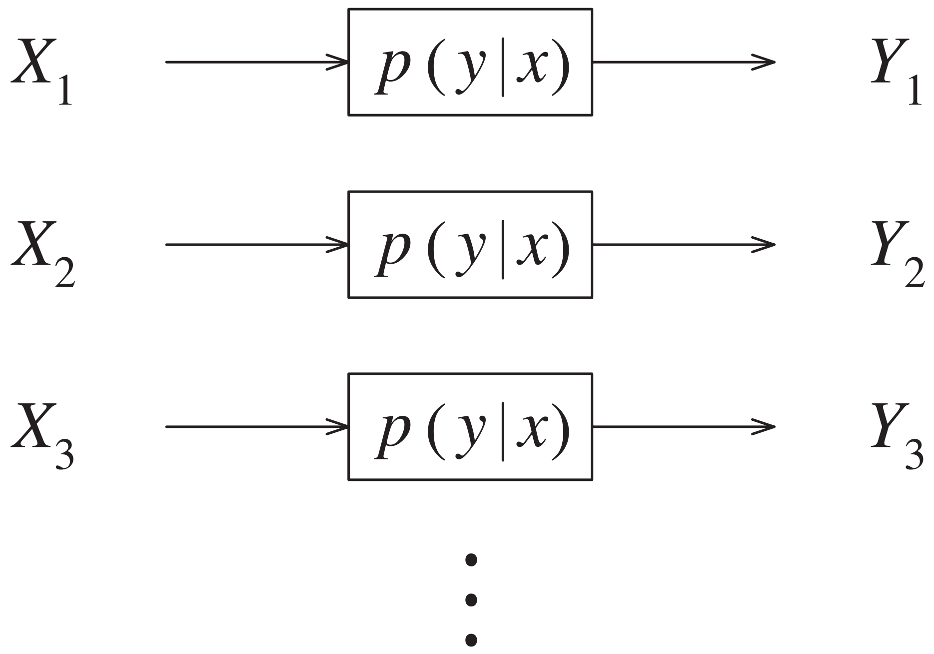

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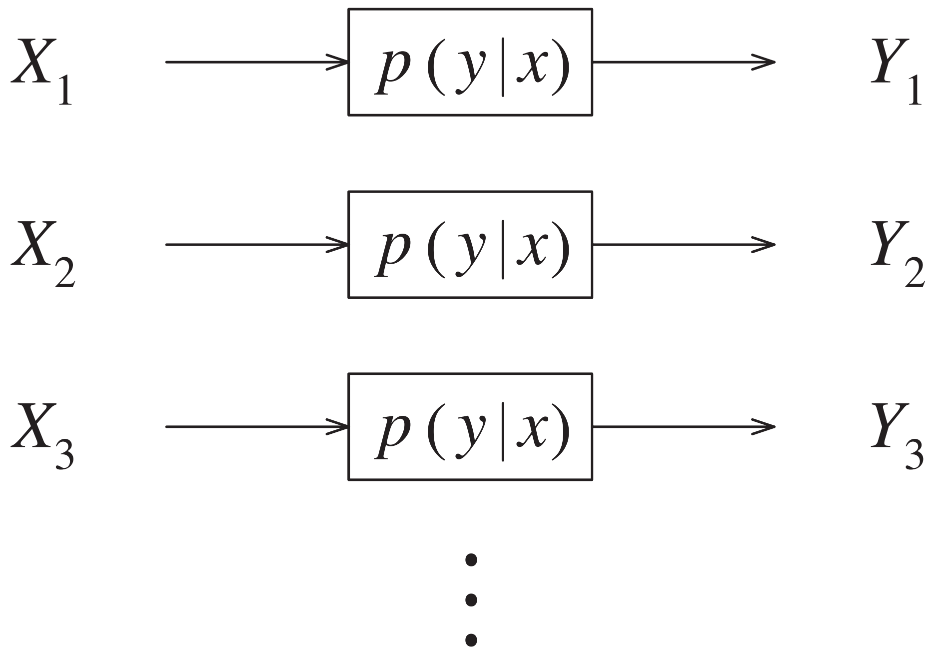

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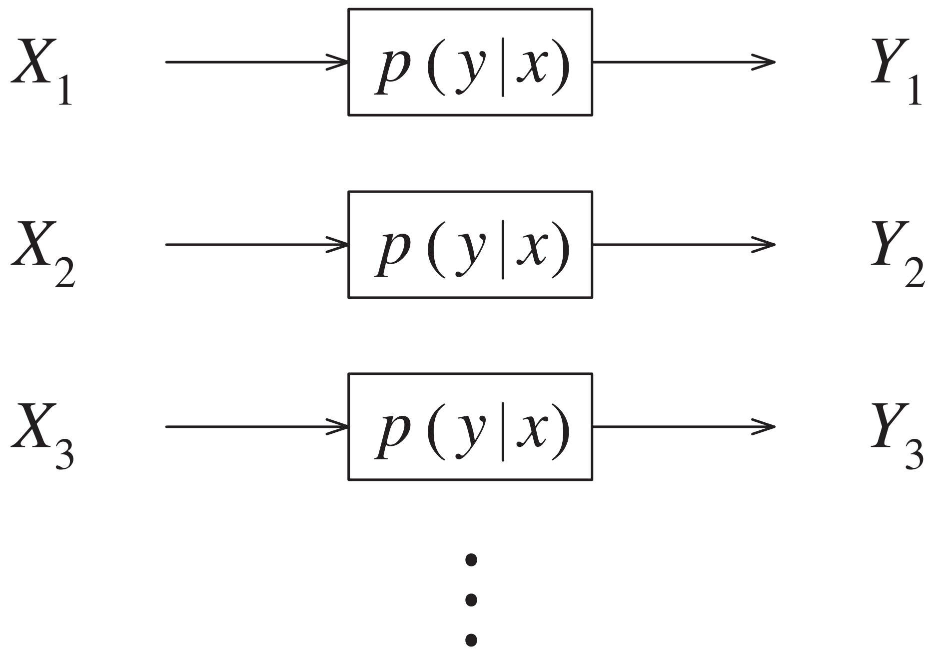

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- Equivalently, $T_{i-} \to X_i \to Y_i$, or

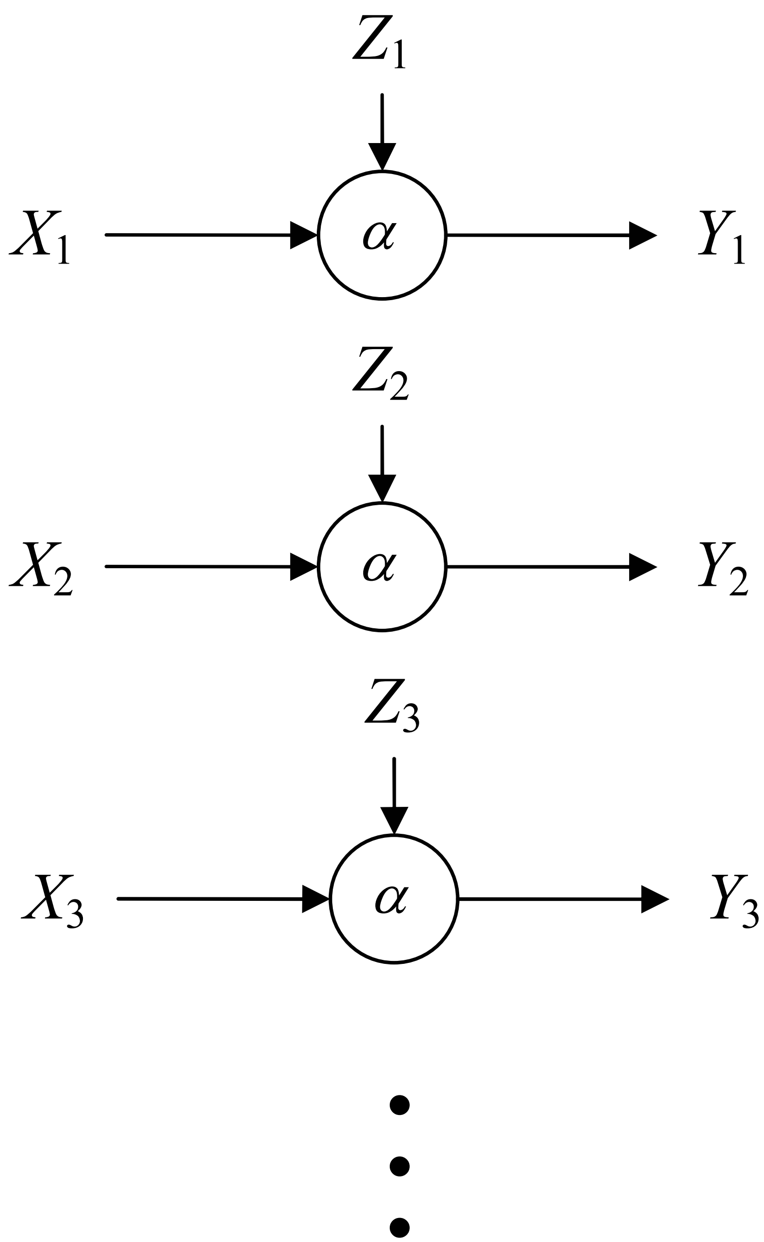
Given X_i, Y_i is independent of everything in the past.

Definition 7.4 (DMC I) A discrete memoryless channel (DMC) $p(y|x)$ is a sequence of replicates of a generic discrete channel $p(y|x)$. These discrete channels are indexed by a discrete-time index i, where $i \geq 1$, with the ith channel being available for transmission at time i. Transmission through a channel is assumed to be instantaneous. Let X_i and Y_i be respectively the input and the output of the DMC at time i, and let T_{i-} denote all the random variables that are generated in the system before X_i. The equality

$$\Pr\{Y_i = y, X_i = x, T_{i-} = t\} = \Pr\{X_i = x, T_{i-} = t\} p(y|x)$$

holds for all $(x, y, t) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{T}_{i-}$.

Definition 7.5 (DMC II)



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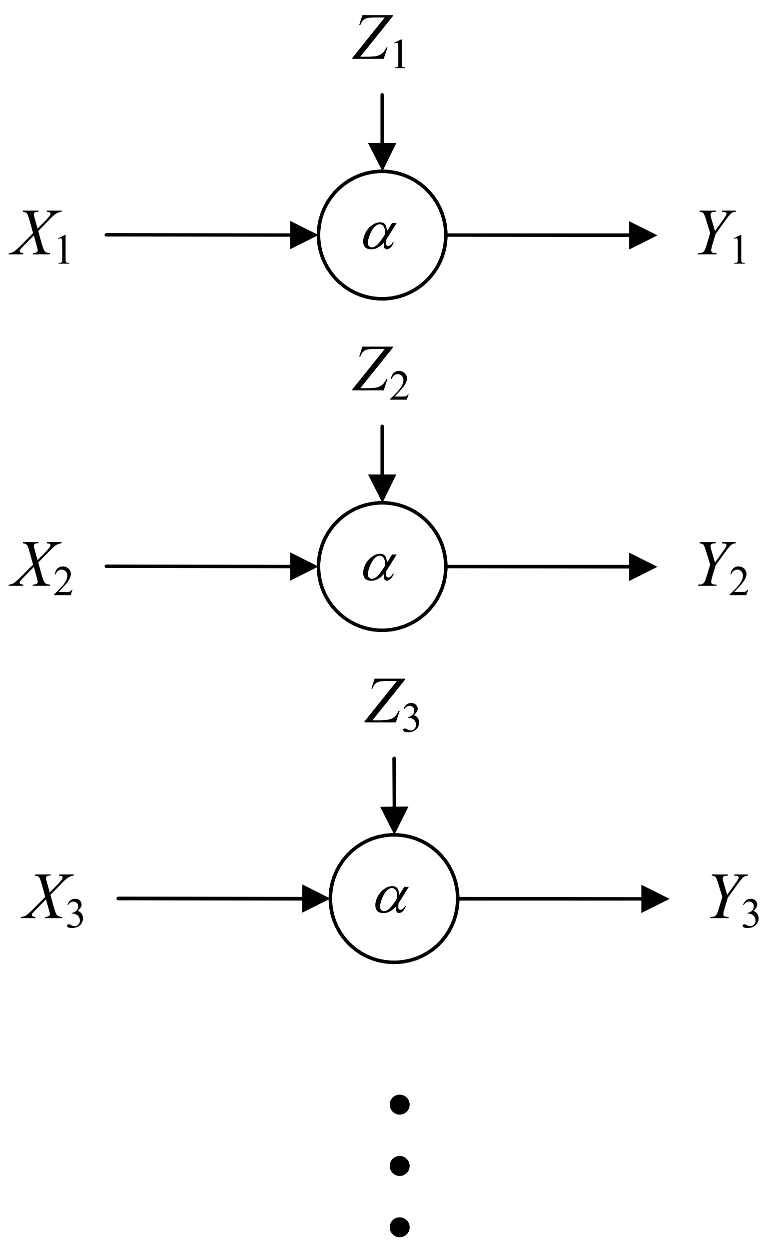

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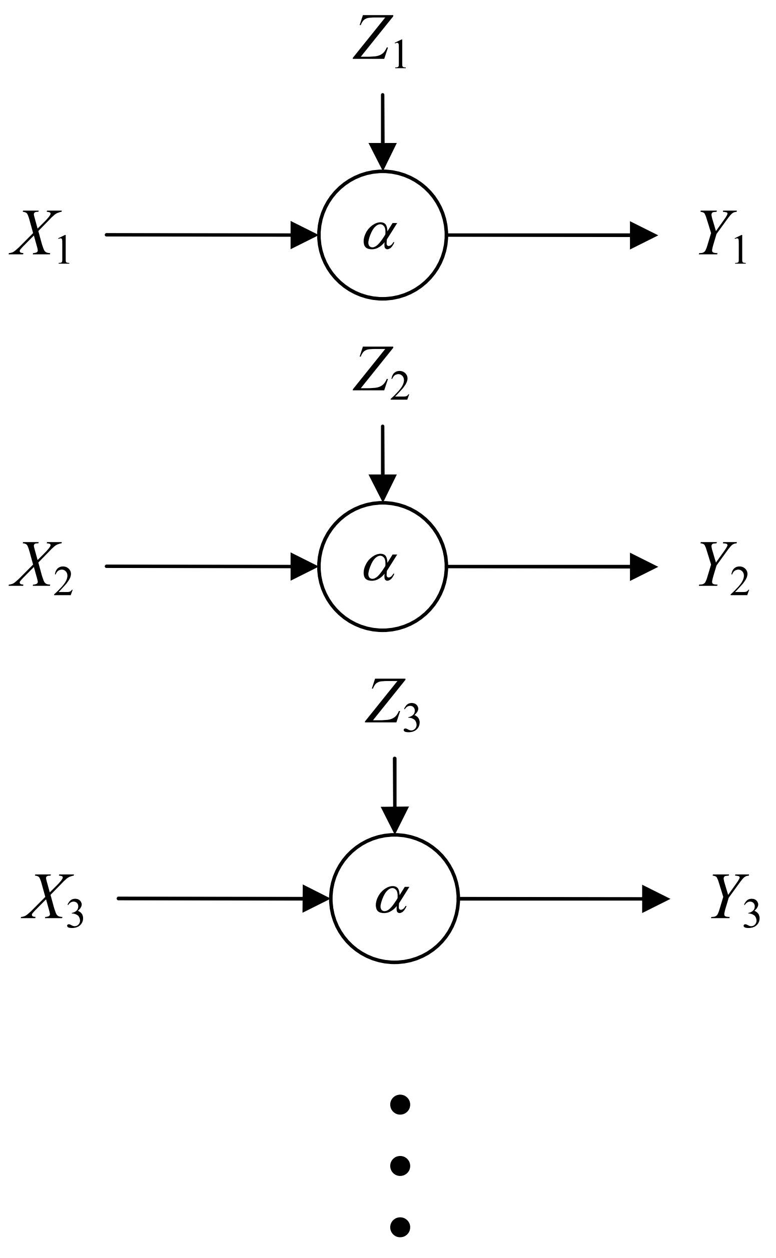

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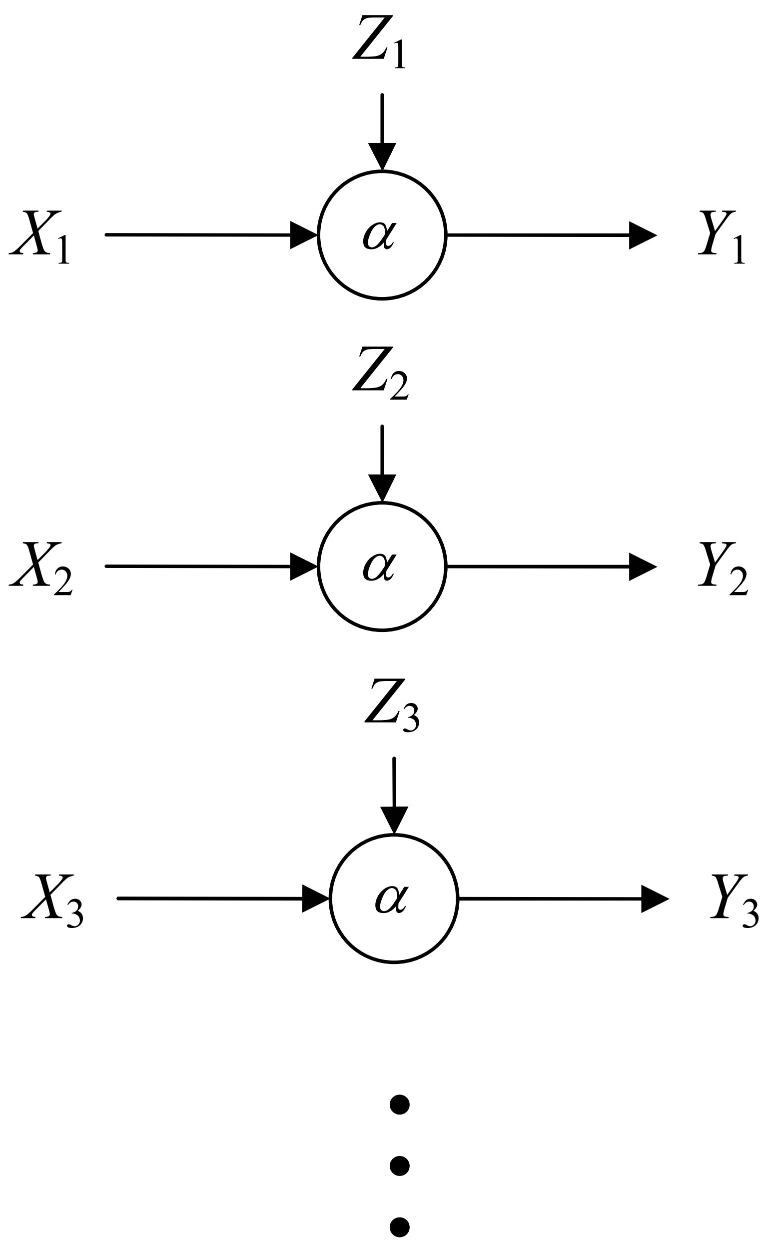

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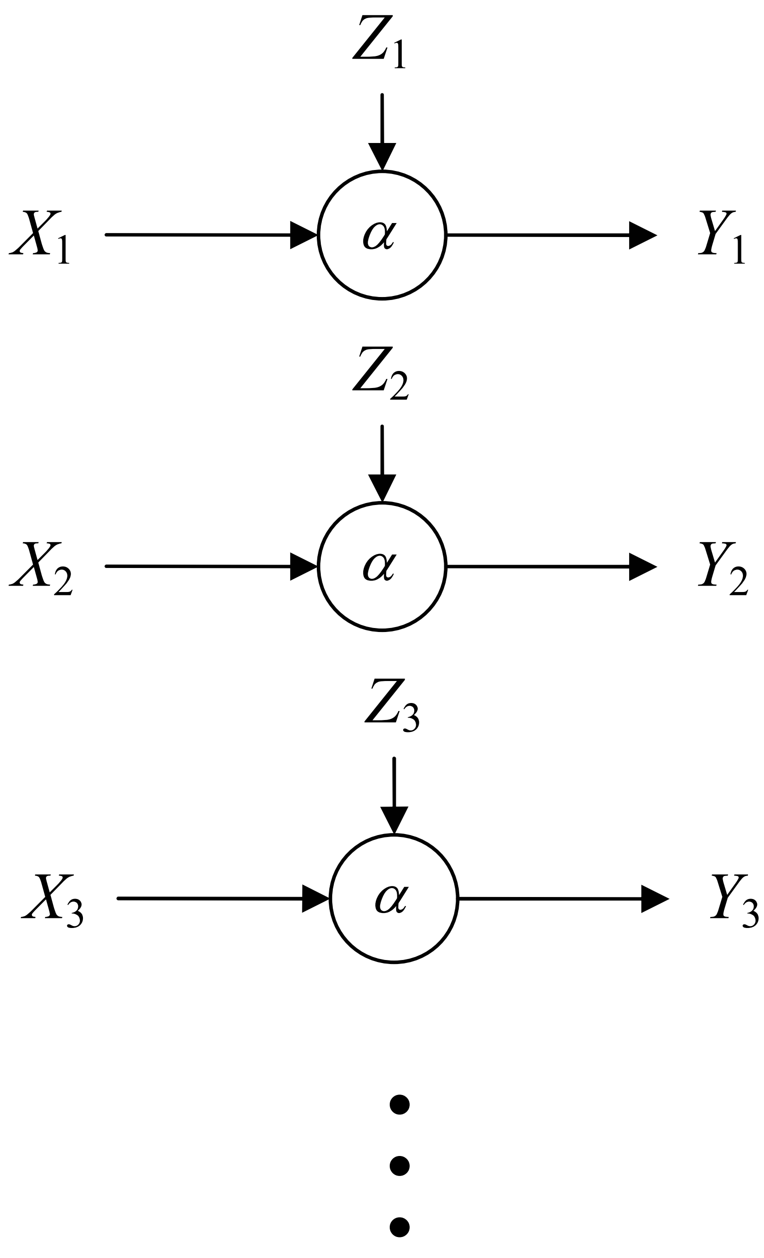

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$$Z_i \text{ is independent of } (X_i, T_{i-}).$$

- That is,

The noise for transmission at time i is independent of both X_i and everything in the past.

Definition 7.5 (DMC II) A discrete memoryless channel (α, Z) is a sequence of replicates of a generic discrete channel (α, Z). These discrete channels are indexed by a discrete-time index i, where $i \geq 1$, with the ith channel being available for transmission at time i. Transmission through a channel is assumed to be instantaneous. Let X_i and Y_i be respectively the input and the output of the DMC at time i, and let T_{i-} denote all the random variables that are generated in the system before X_i. The noise variable Z_i for the transmission at time i is a copy of the generic noise variable Z, and is independent of (X_i, T_{i-}). The output of the DMC at time i is given by

$$Y_i = \alpha(X_i, Z_i).$$

Remark: The equivalence of Definitions 7.4 and 7.5 can be shown. See textbook.

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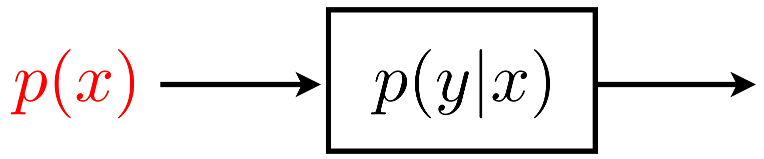

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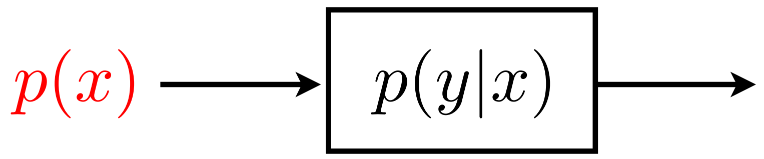

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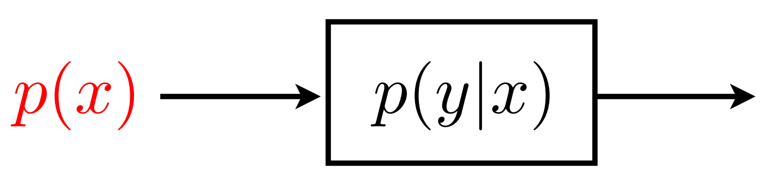

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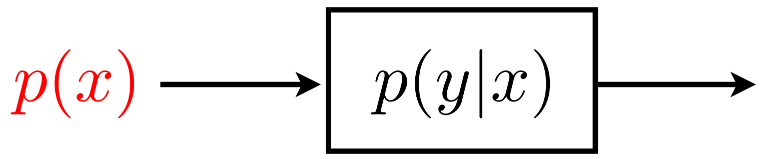

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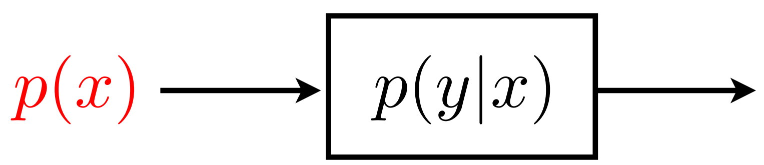

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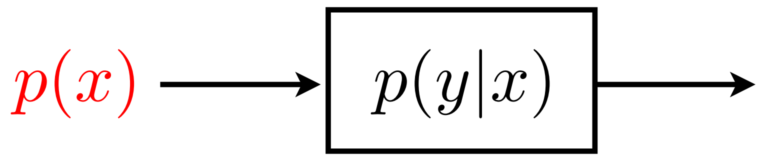

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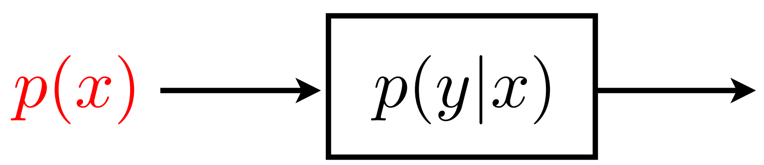

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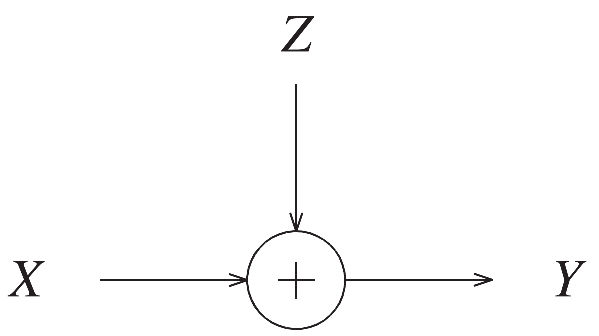
- Will see that C is in fact the maximum rate at which information can be communicated reliably through a DMC.
- Can communicate through a channel at a positive rate while $P_e \to 0$!

Example 7.7 (BSC)



Alternative representation of a BSC with crossover probability ϵ:

$$Y = X + Z \bmod 2$$

with

$$\Pr\{Z = 0\} = 1 - \epsilon \quad \text{and} \quad \Pr\{Z = 1\} = \epsilon$$

and Z is independent of X.

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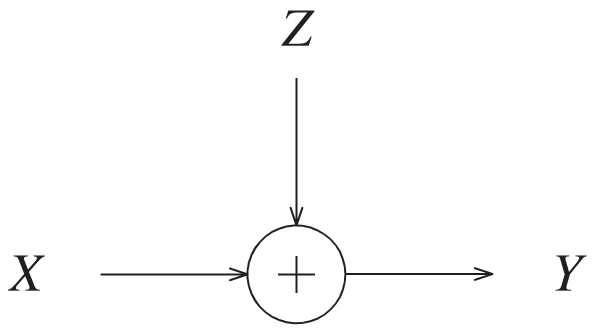

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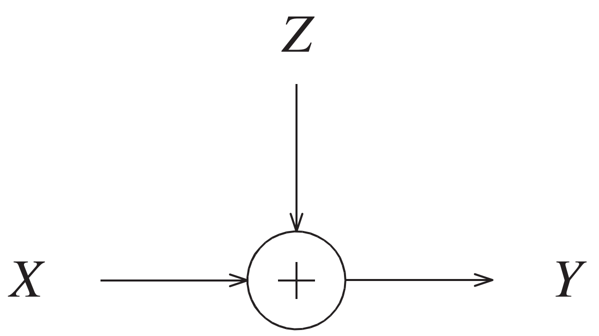

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Determination of C

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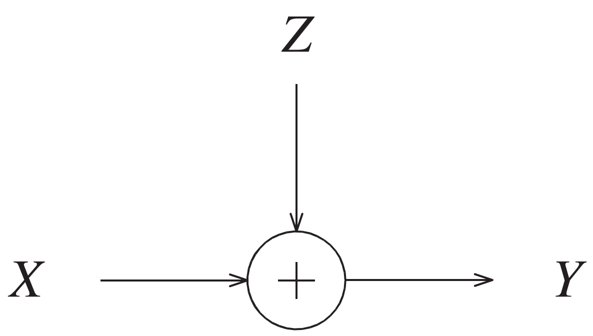

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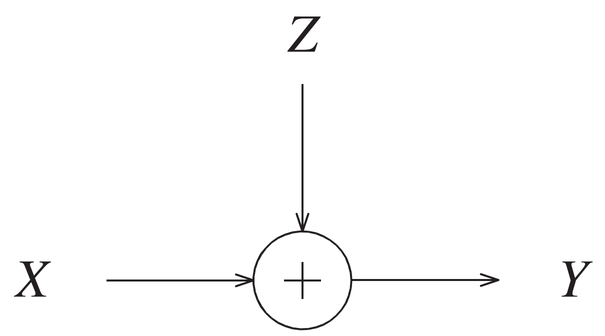

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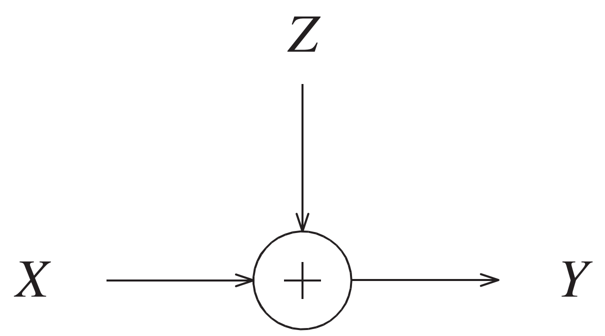

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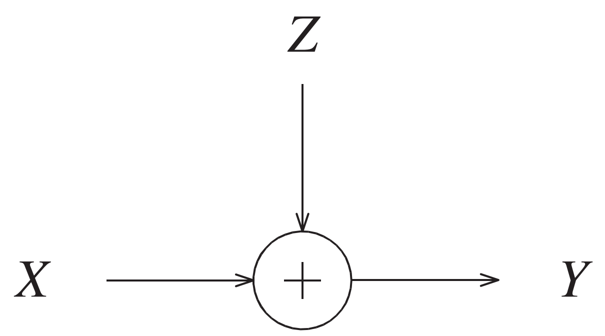

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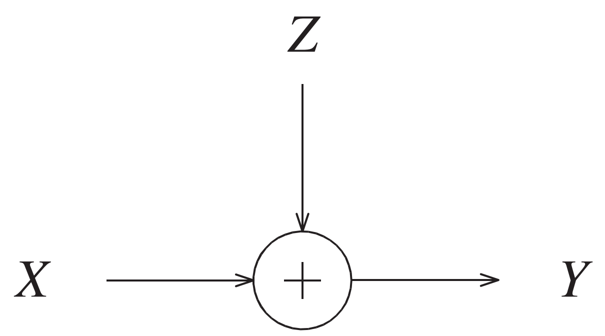

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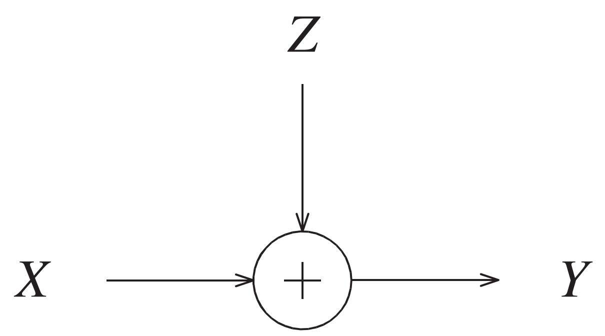

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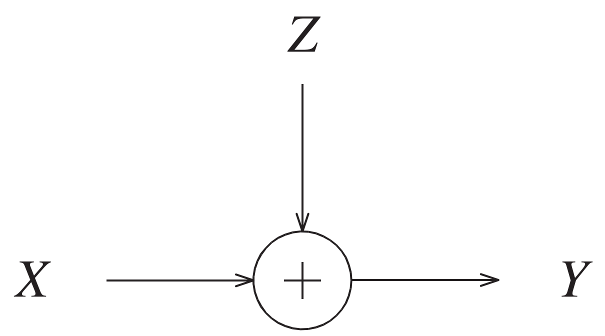

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2. So,

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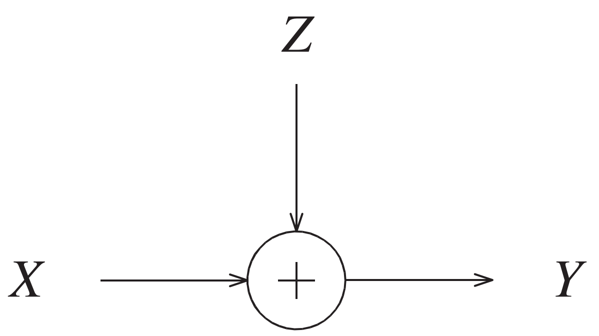

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$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - \sum_x p(x) H(Y|X = x) \\ &= H(Y) - \sum_x p(x) h_b(\epsilon) \\ &= H(Y) - h_b(\epsilon) \\ &\leq 1 - h_b(\epsilon). \end{aligned}$$

2. So,

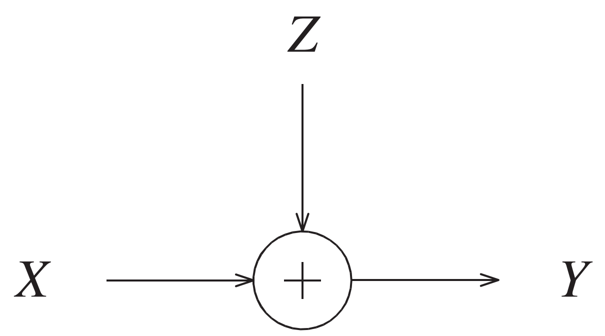
$$C = \max_{p(x)} I(X;Y) \leq 1 - h_b(\epsilon).$$

3. The upper bound on $I(X;Y)$ is tight if

$$H(Y) = 1.$$

This can be achieved by taking the uniform input distribution.

Example 7.7 (BSC)



Alternative representation of a BSC with crossover probability ϵ:

$$Y = X + Z \bmod 2$$

with

$$\Pr\{Z = 0\} = 1 - \epsilon \quad \text{and} \quad \Pr\{Z = 1\} = \epsilon$$

and Z is independent of X.

Determination of C

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3. The upper bound on $I(X;Y)$ is tight if

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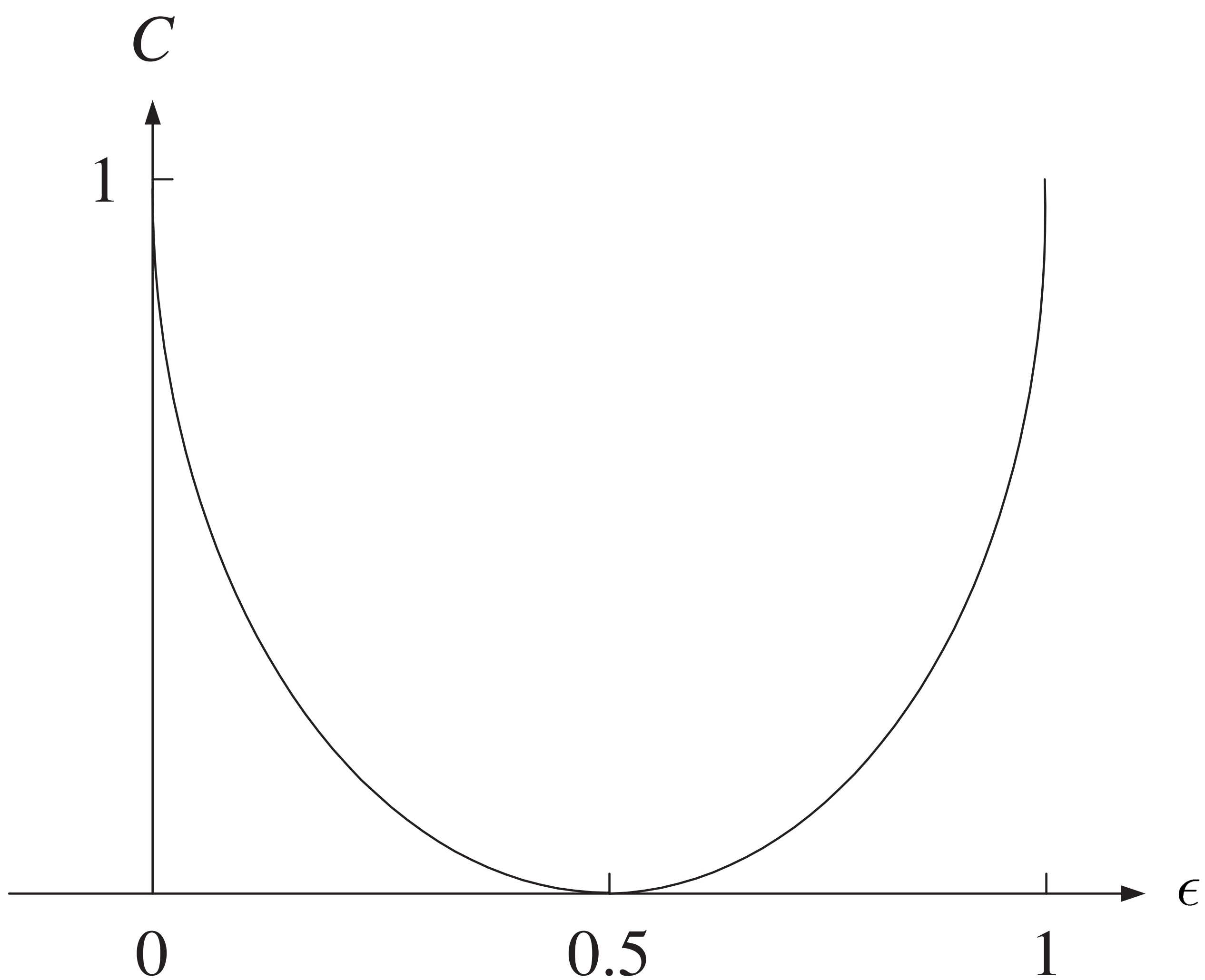
4. Therefore, $C = 1 - h_b(\epsilon)$ bit per use.

$$C(\epsilon) = 1 - h_b(\epsilon)$$



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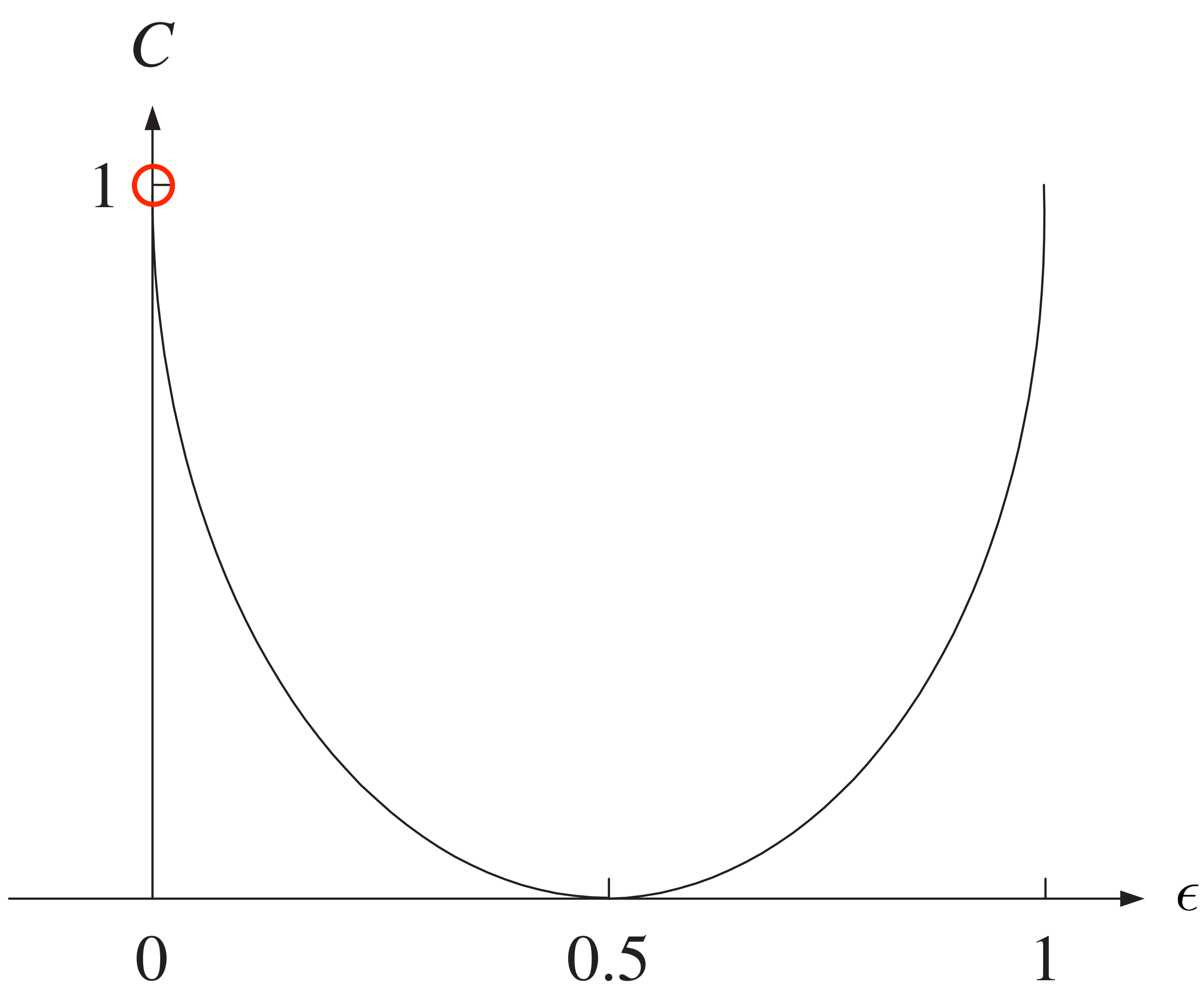
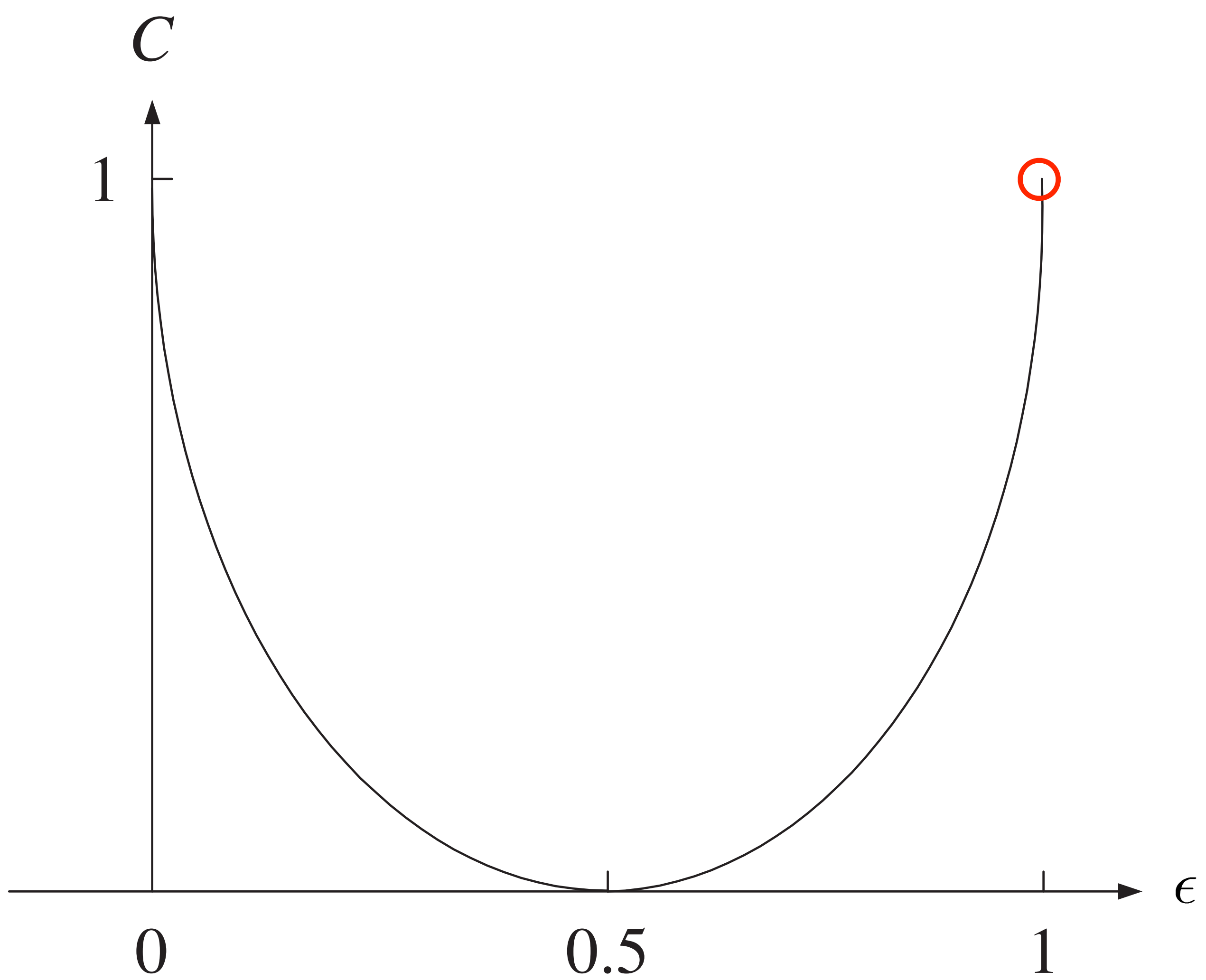


- $C(0) = 1$

$$C(\epsilon) = 1 - h_b(\epsilon)$$



- $C(0) = 1$
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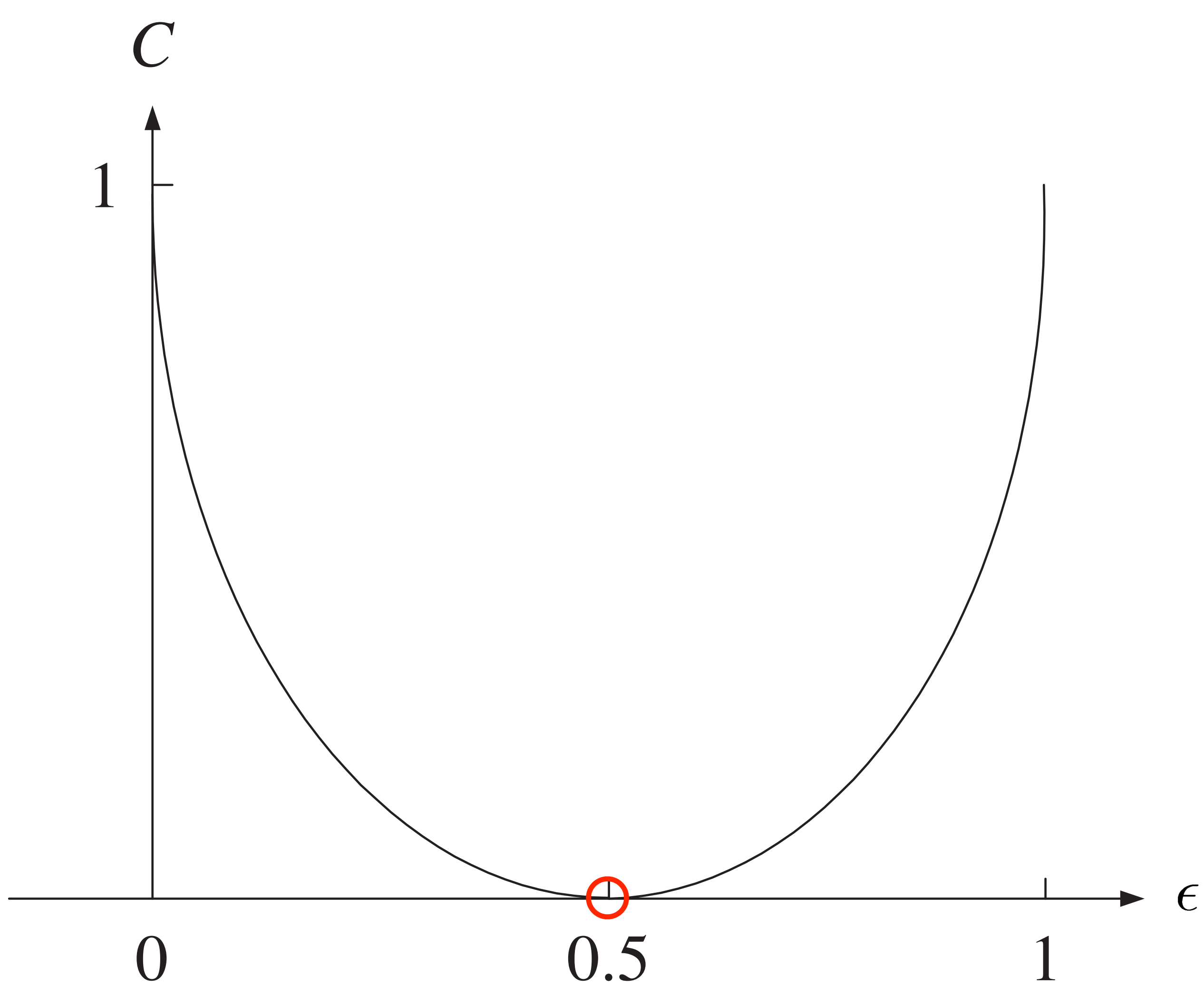

- $C(0) = 1$
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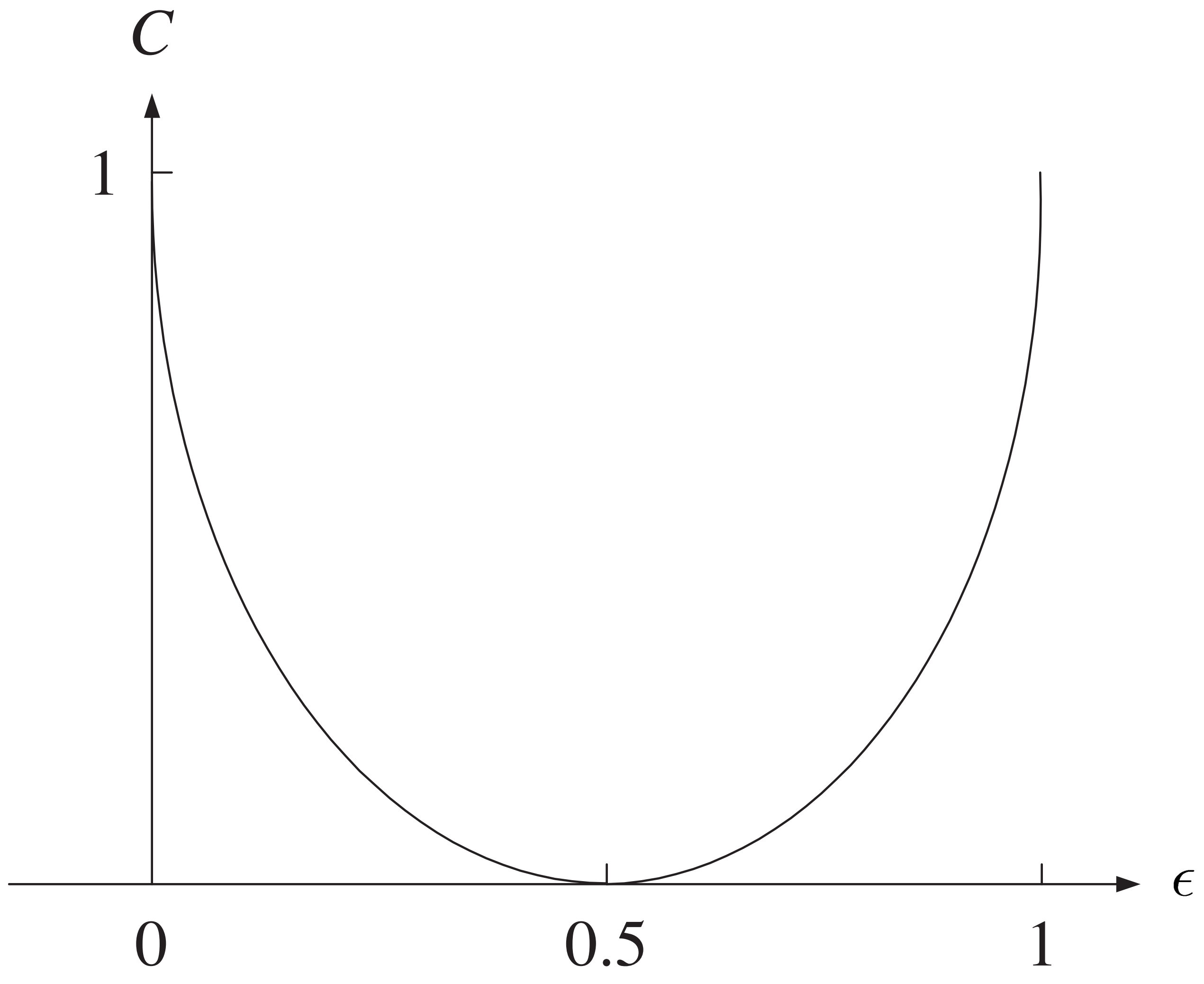

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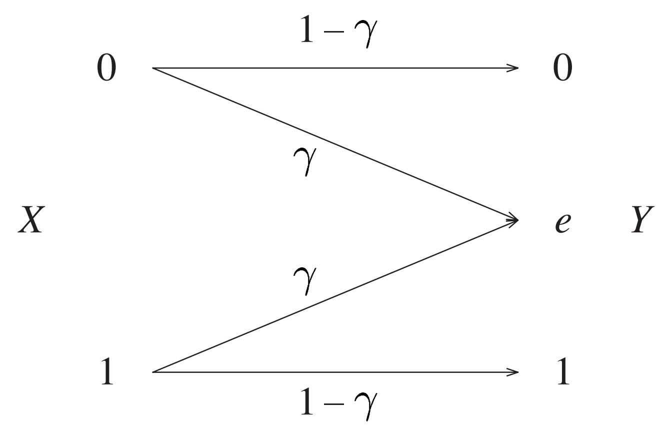
Exercise: Show that X and Y are always independent for $\epsilon = 0.5$.

Example 7.8 (Binary Erasure Channel)



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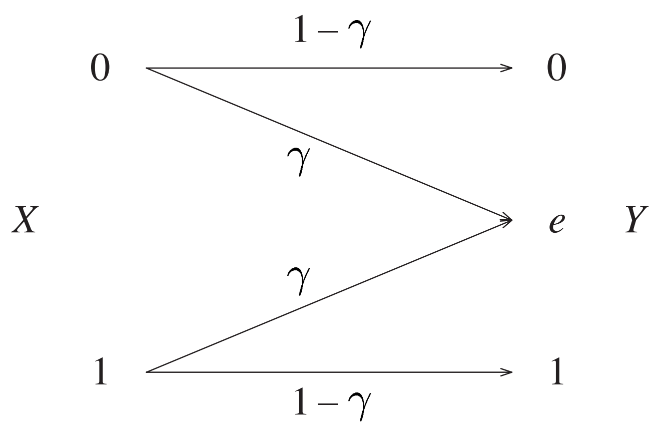

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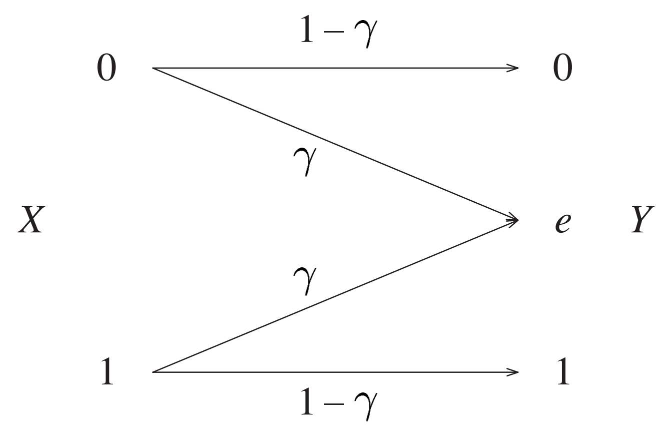

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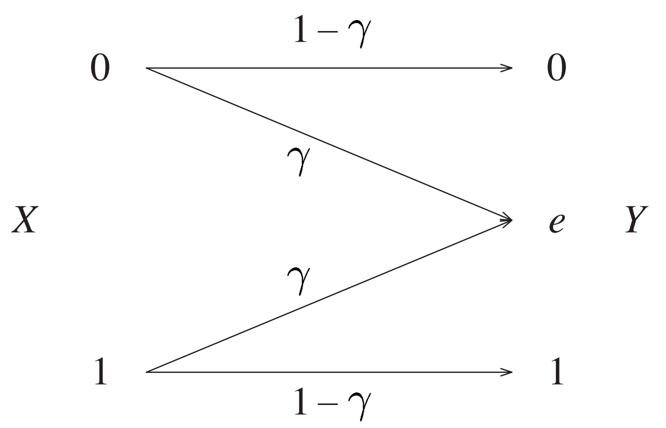

Binary Erasure Channel:

- The output symbol e denotes “erasure”.

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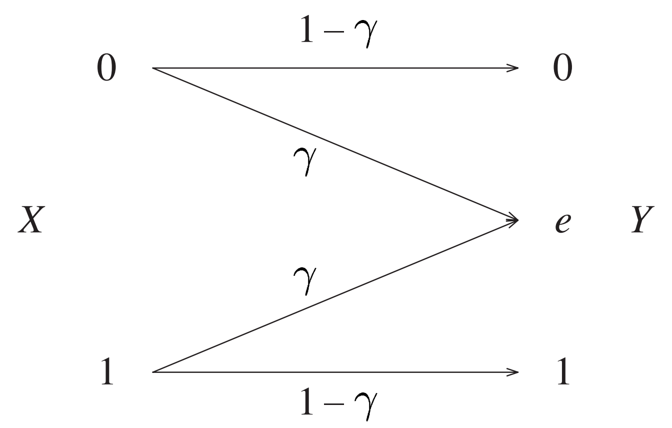

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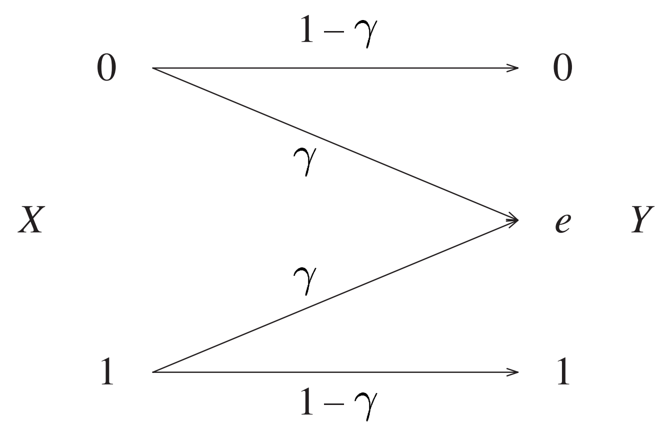

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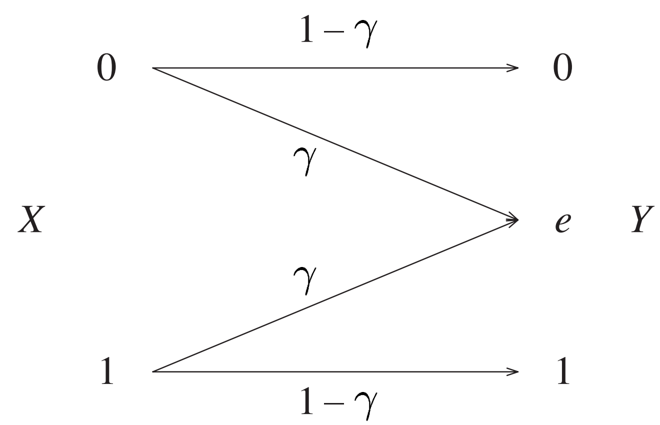

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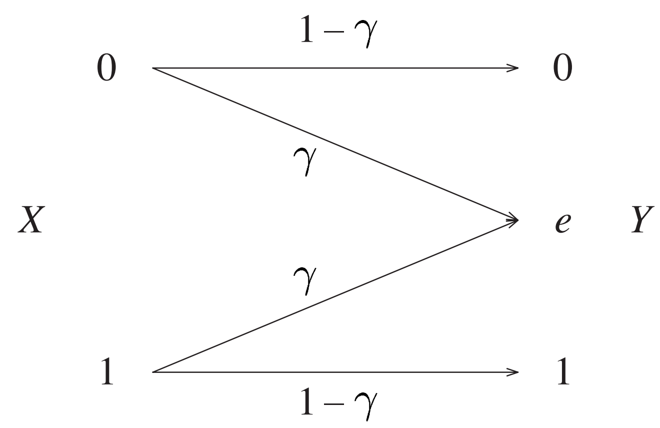

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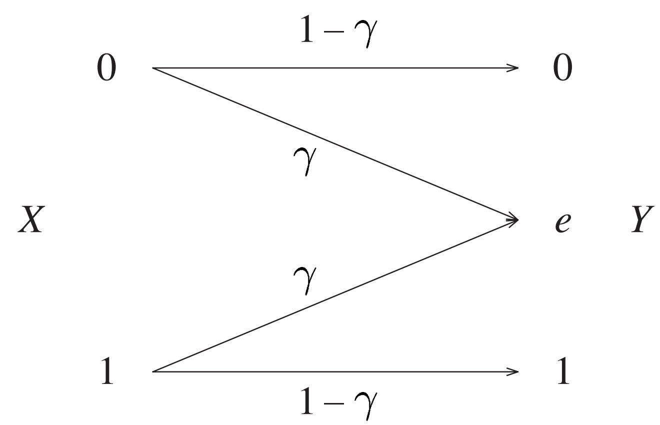

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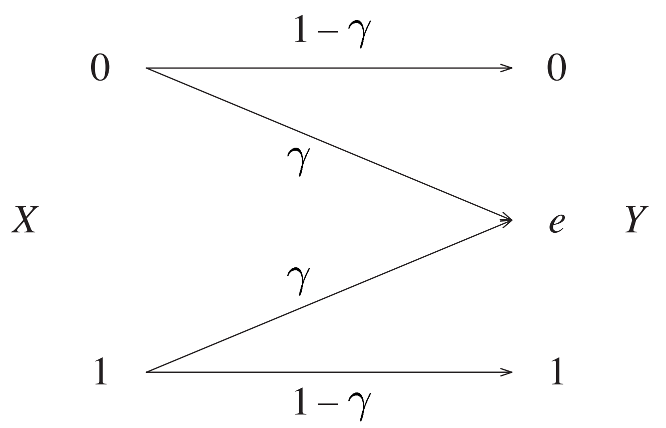

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Determination of C

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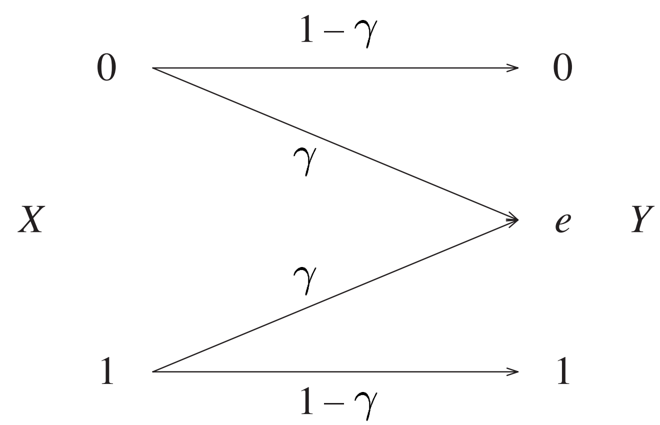

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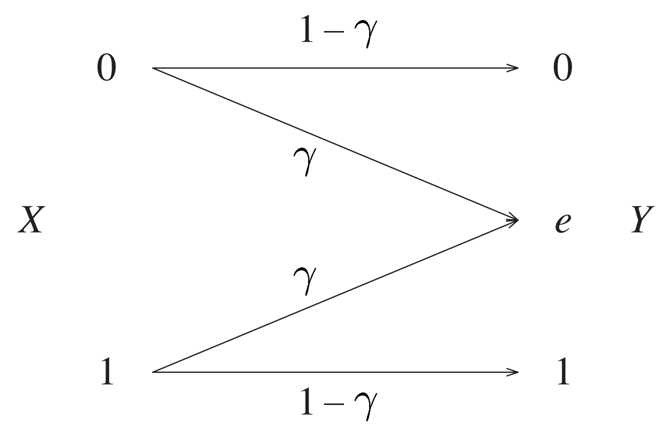

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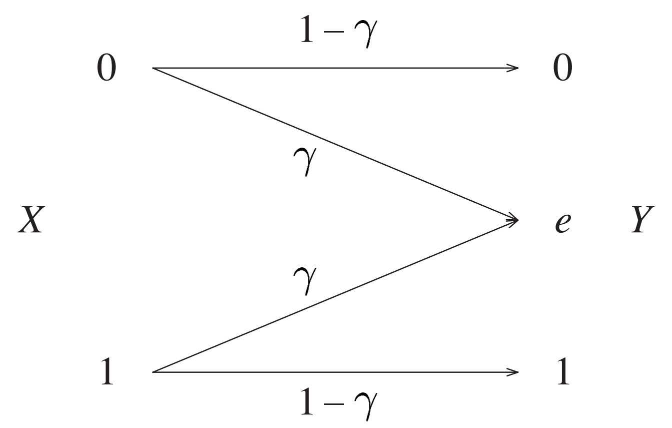

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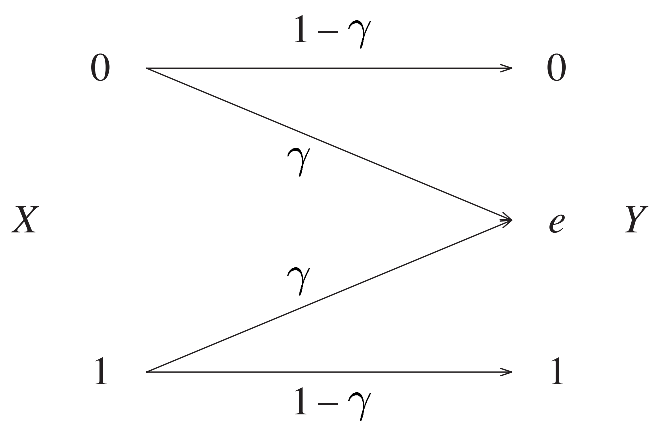

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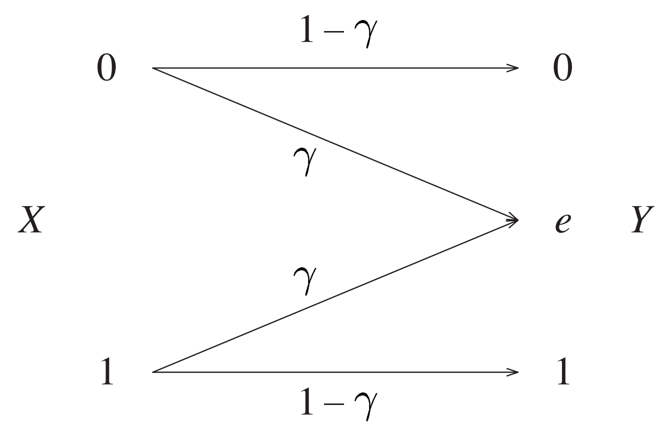

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Thus we only have to maximize $H(Y)$ over all $p(x)$.

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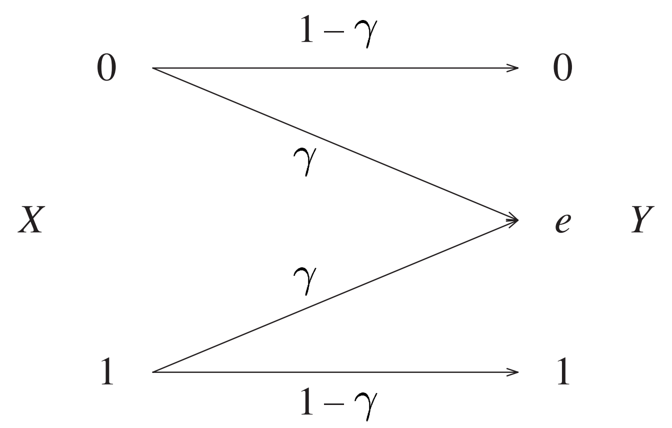

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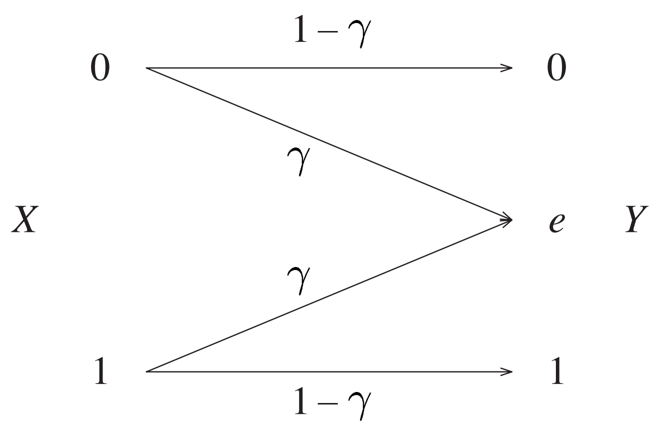

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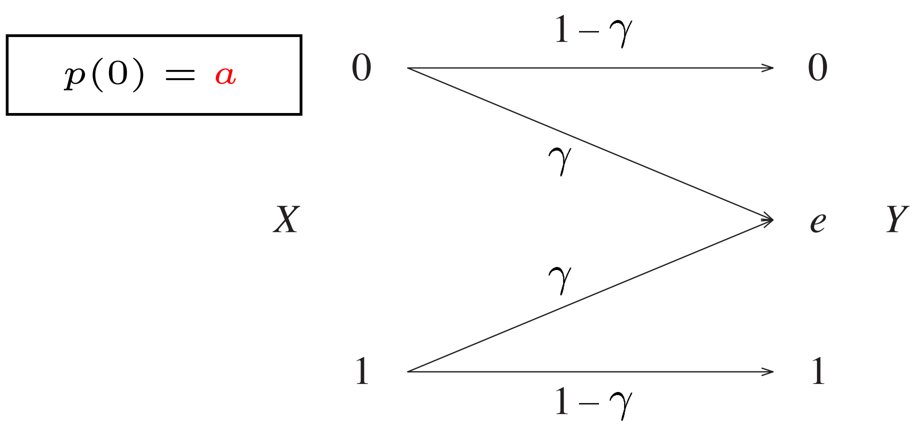

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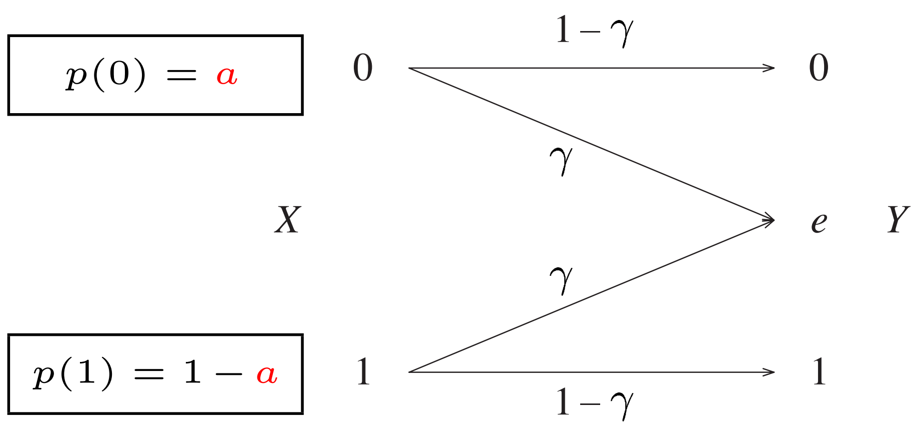

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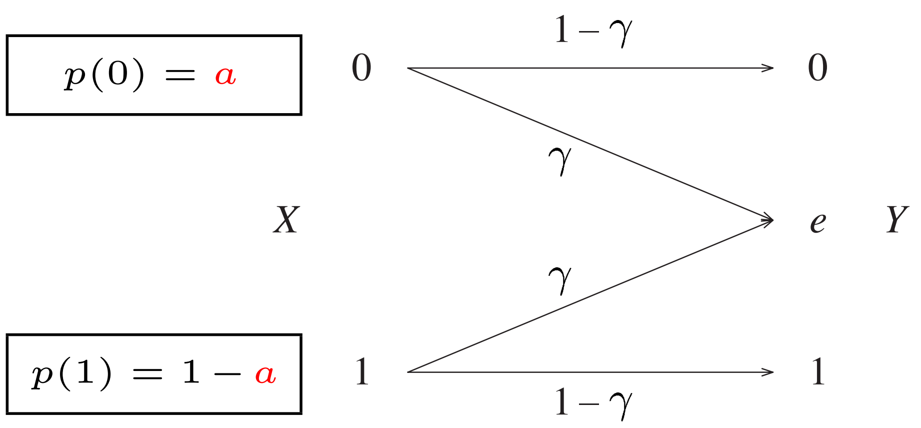

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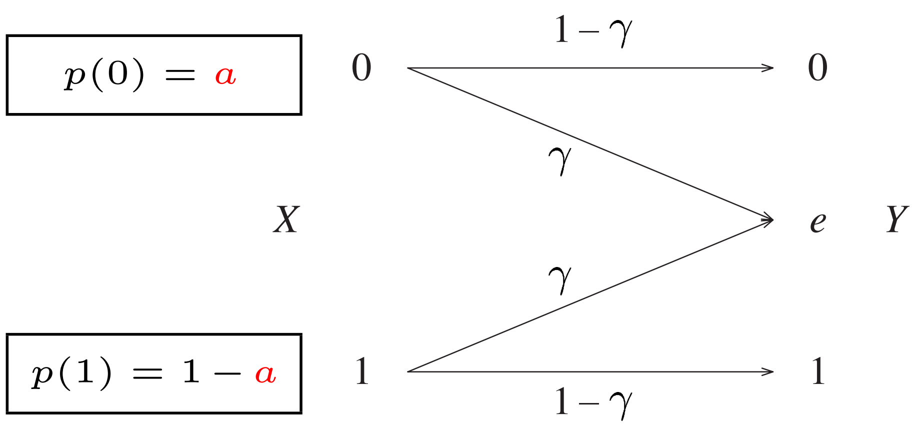

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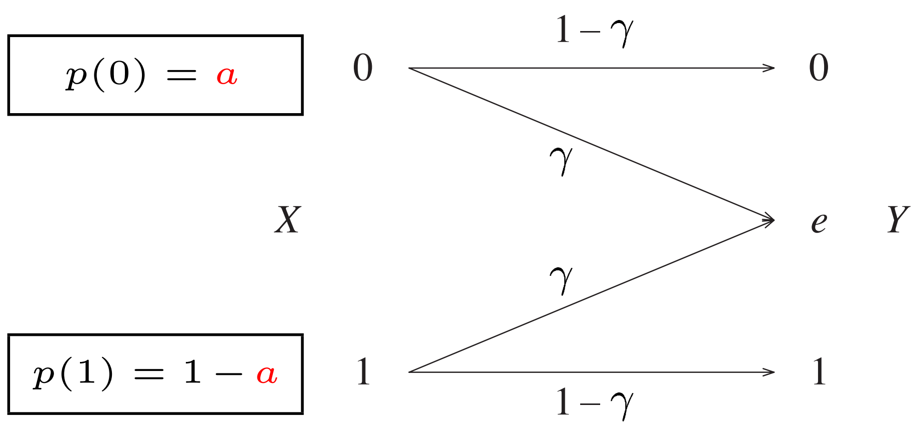

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The random variable E indicates whether an erasure has occurred, and it is a function of Y.

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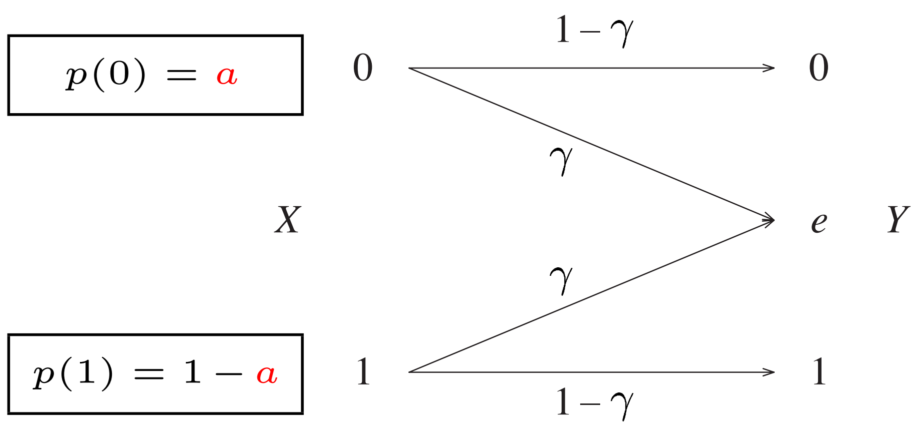

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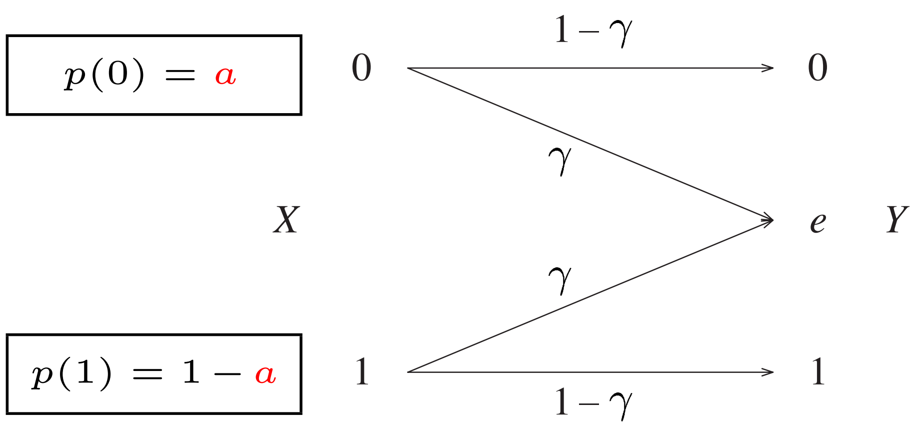

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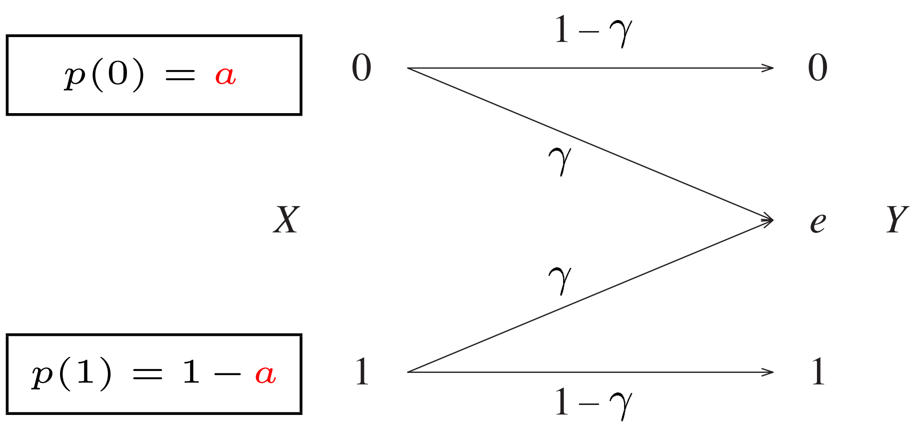

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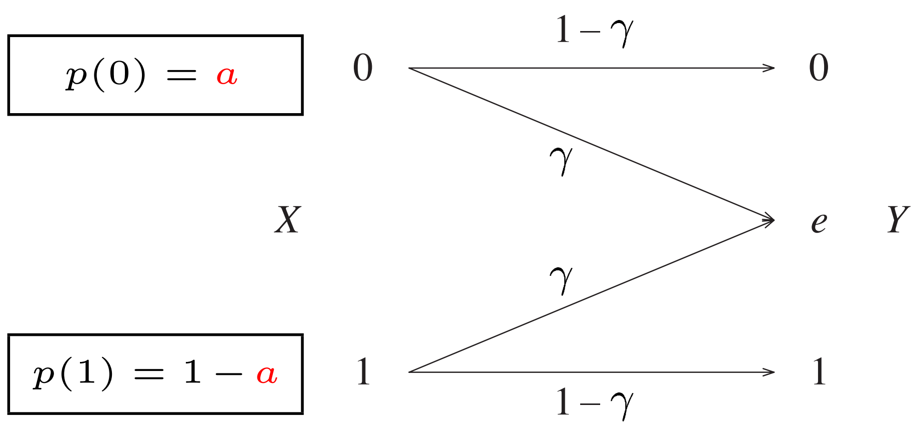

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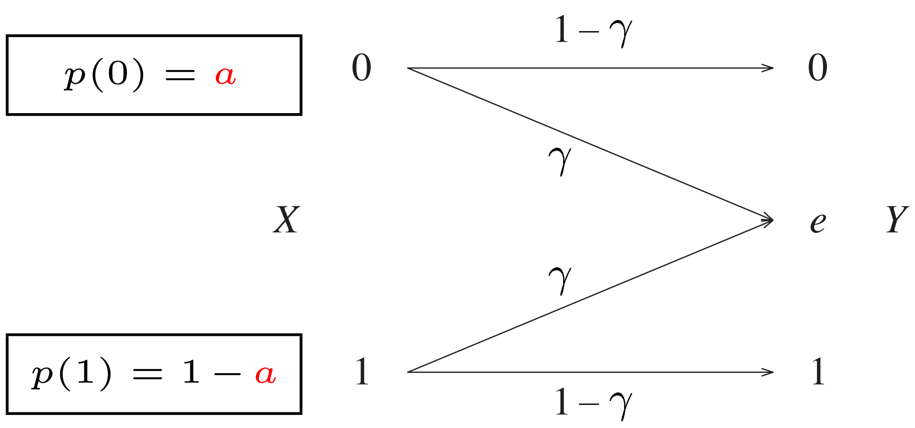

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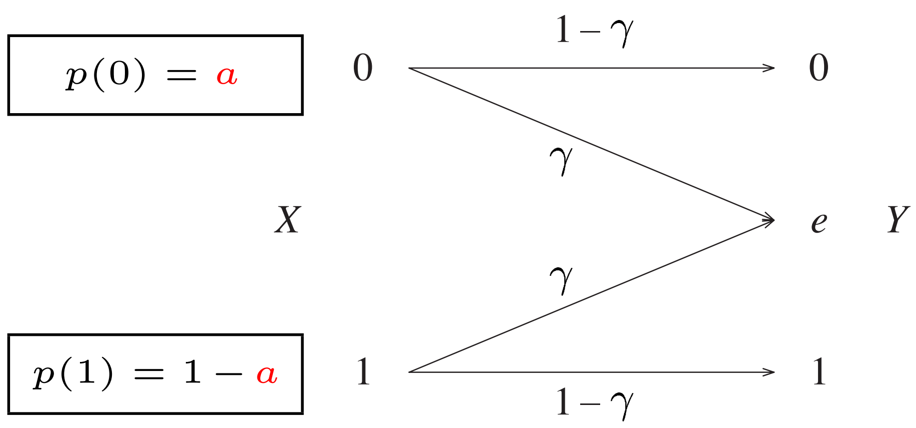

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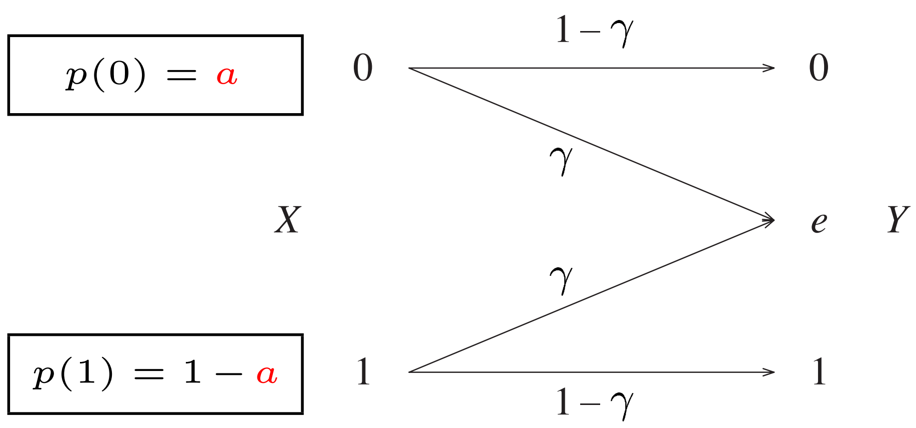

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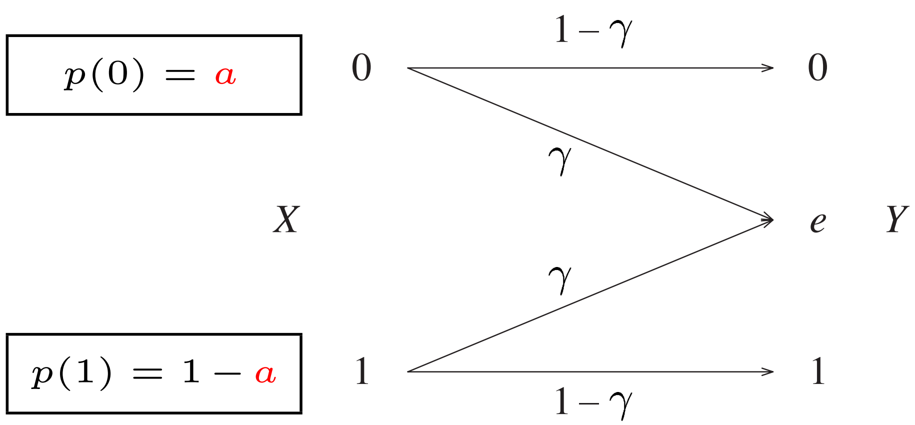

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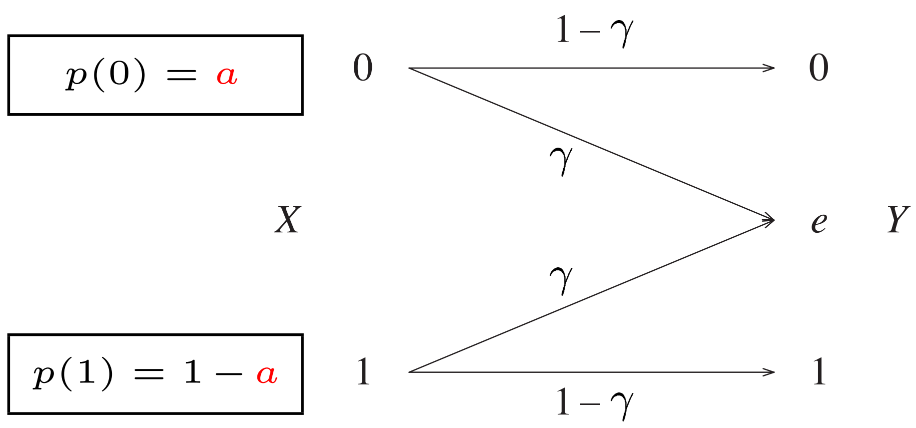

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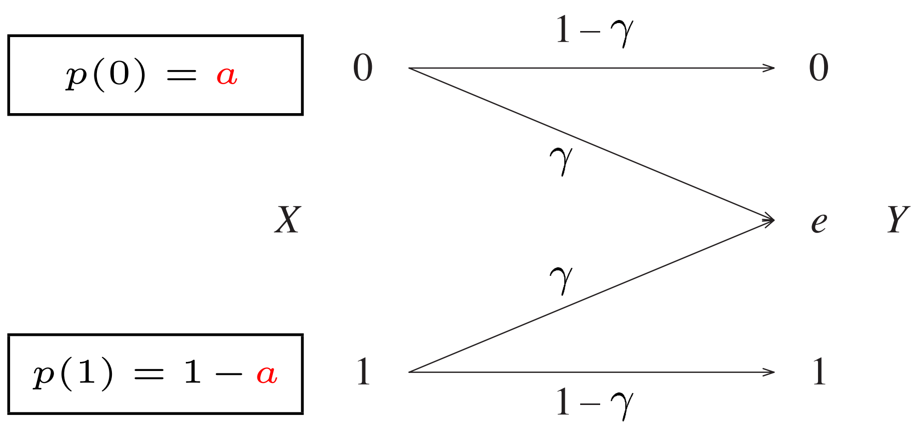

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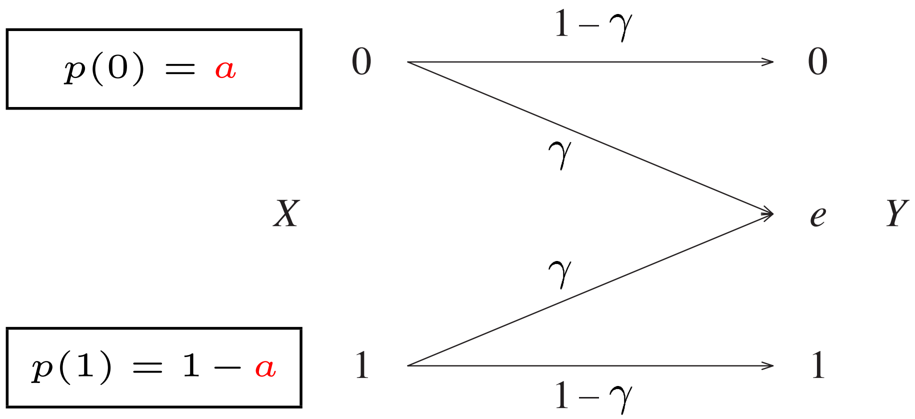

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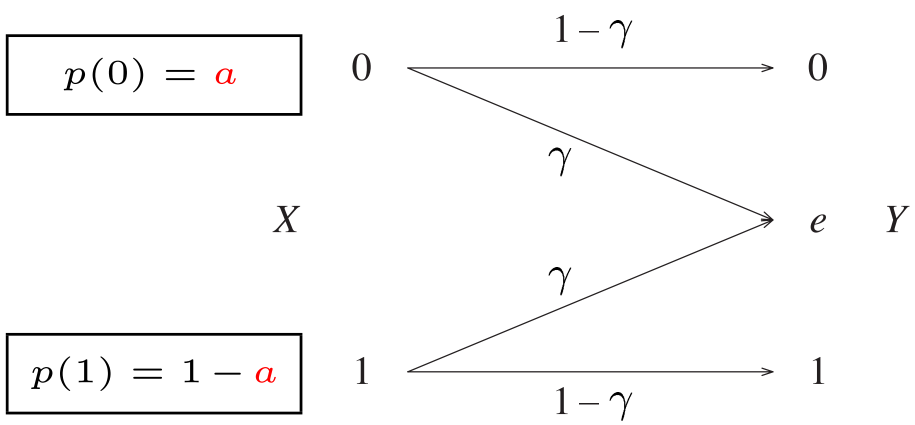

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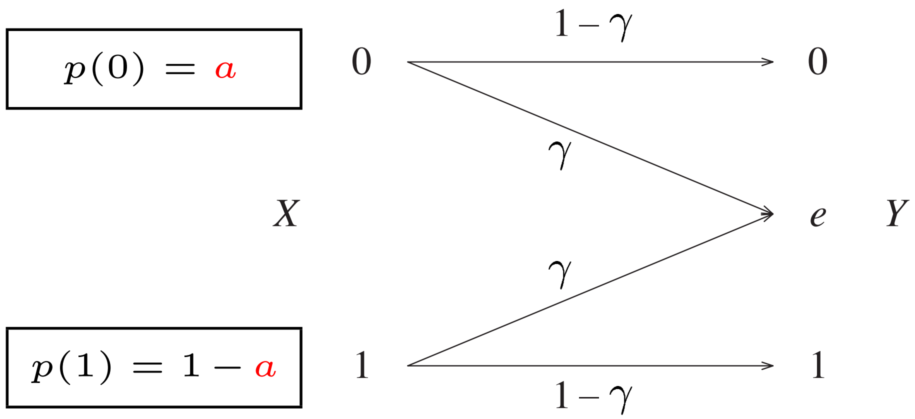

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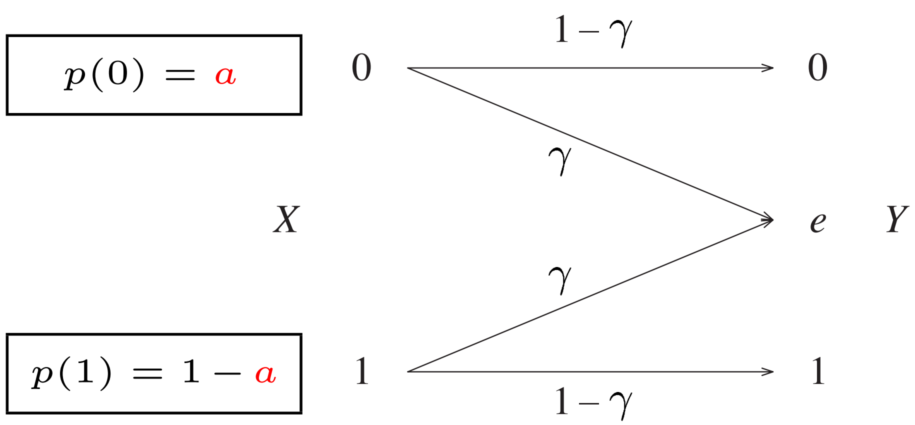

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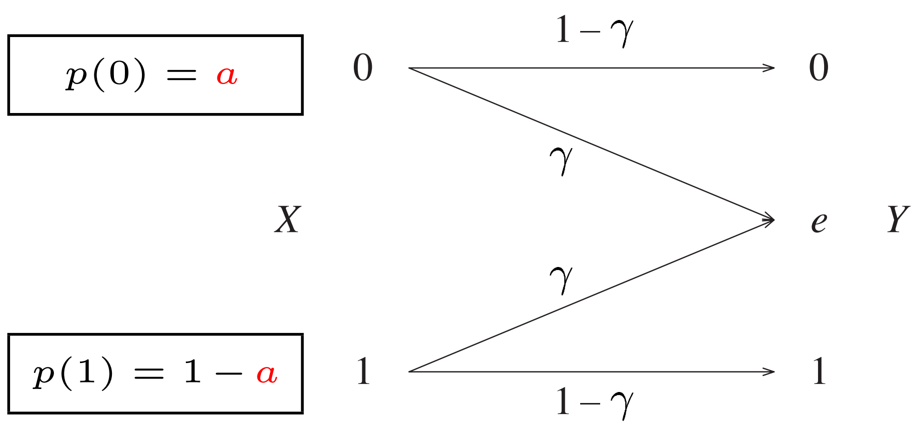

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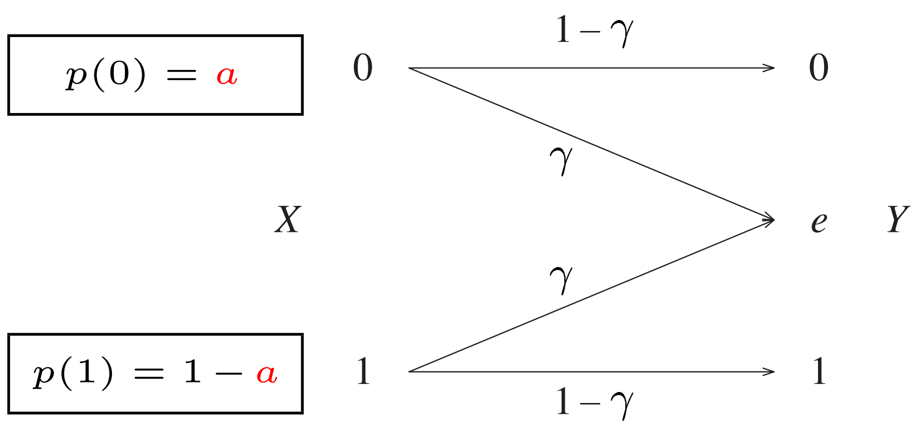

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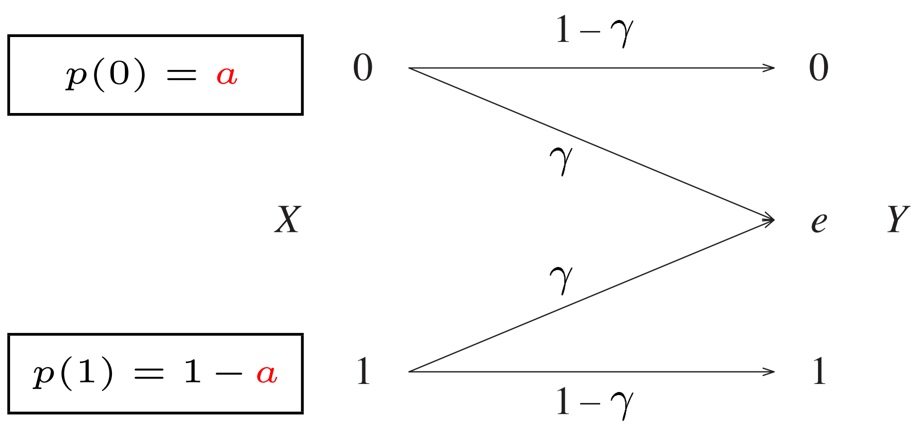

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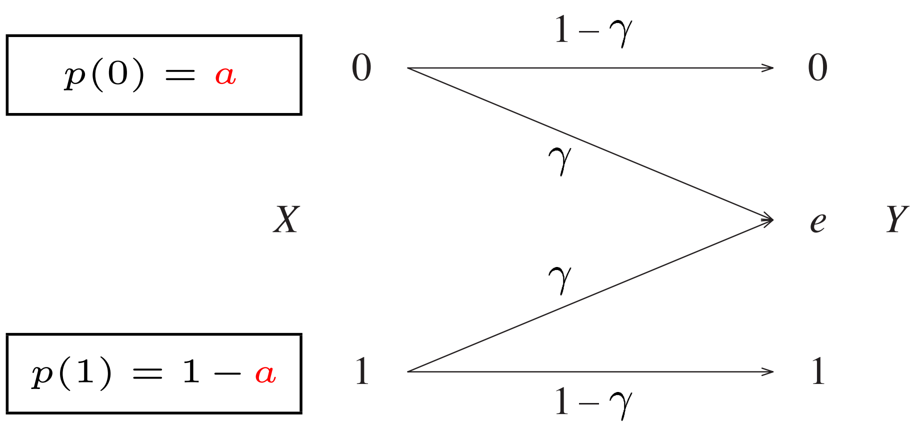

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4. Hence,

$$\begin{aligned} C &= \max_{p(x)} H(Y) - h_b(\gamma) \\ &= \max_{a} [\cancel{h_b(\gamma)} + (1-\gamma)h_b(a)] - \cancel{h_b(\gamma)} \\ &= (1-\gamma) \max_{a} h_b(a) \\ &= (1-\gamma) \text{ bit per use,} \end{aligned}$$

where the capacity is achieved by letting $a = 0.5$, i.e., the input distribution is uniform.

Exercise: Show that $H(Y|E) = (1-\gamma)h_b(a)$.