香港中文大學
The Chinese University of Hong Kong

6.3 Joint Typicality

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- $|\mathcal{X}|, |\mathcal{Y}| < \infty$

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- $n^{-1}N(x, y; \mathbf{x}, \mathbf{y})$ is the relative frequency of (x, y) in $(\mathbf{x}, \mathbf{y})$.
- $\{n^{-1}N(x, y; \mathbf{x}, \mathbf{y})\}$ is the empirical distribution of $(\mathbf{x}, \mathbf{y})$.

Example

- Let

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\ \mathbf{y} &= \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}\end{aligned}$$

- Then

$$\begin{aligned}N(0,0;\mathbf{x},\mathbf{y}) &= 1 \\ N(0,1;\mathbf{x},\mathbf{y}) &= 2 \\ N(1,0;\mathbf{x},\mathbf{y}) &= 2 \\ N(1,1;\mathbf{x},\mathbf{y}) &= 1\end{aligned}$$

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Definition 6.6 The strongly jointly typical set $T^n_{[XY]\delta}$ with respect to $p(x, y)$ is the set of $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$ such that

$$N(x, y; \mathbf{x}, \mathbf{y}) = 0 \quad \text{for} \quad (x, y) \notin \mathcal{S}_{XY},$$

and

$$\sum_x \sum_y \left| \frac{1}{n} N(x, y; \mathbf{x}, \mathbf{y}) - p(x, y) \right| \leq \delta,$$

where δ is an arbitrarily small positive real number. A pair of sequences $(\mathbf{x}, \mathbf{y})$ is called strongly jointly δ-typical if it is in $T^n_{[XY]\delta}$.

Theorem 6.7 (Consistency) If $(\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}$, then $\mathbf{x} \in T^n_{[X]\delta}$ and $\mathbf{y} \in T^n_{[Y]\delta}$.

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Theorem 6.8 (Preservation) Let $Y = f(X)$. If

$$\mathbf{x} = (x_1, x_2, \cdots, x_n) \in T^n_{[X]\delta},$$

then

$$f(\mathbf{x}) = (y_1, y_2, \cdots, y_n) \in T^n_{[Y]\delta},$$

where $y_i = f(x_i)$ for $1 \leq i \leq n$.

Theorem 6.9 (Strong JAEP) Let

$$(\mathbf{X}, \mathbf{Y}) = ((X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)),$$

where (X_i, Y_i) are i.i.d. with generic pair of random variables (X, Y). Then there exists $\lambda > 0$ such that $\lambda \to 0$ as $\delta \to 0$, and the following hold:

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1) If $(\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}$, then

$$2^{-n(H(X,Y)+\lambda)} \leq p(\mathbf{x}, \mathbf{y}) \leq 2^{-n(H(X,Y)-\lambda)}.$$

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$$\Pr\{(\mathbf{X}, \mathbf{Y}) \in T^n_{[XY]\delta}\} > 1 - \delta.$$

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$$(1 - \delta)2^{n(H(X,Y)-\lambda)} \leq |T^n_{[XY]\delta}| \leq 2^{n(H(X,Y)+\lambda)}.$$

Stirling's Approximation

Lemma 6.11 (simplified) $\ln n! \sim n \ln n$.

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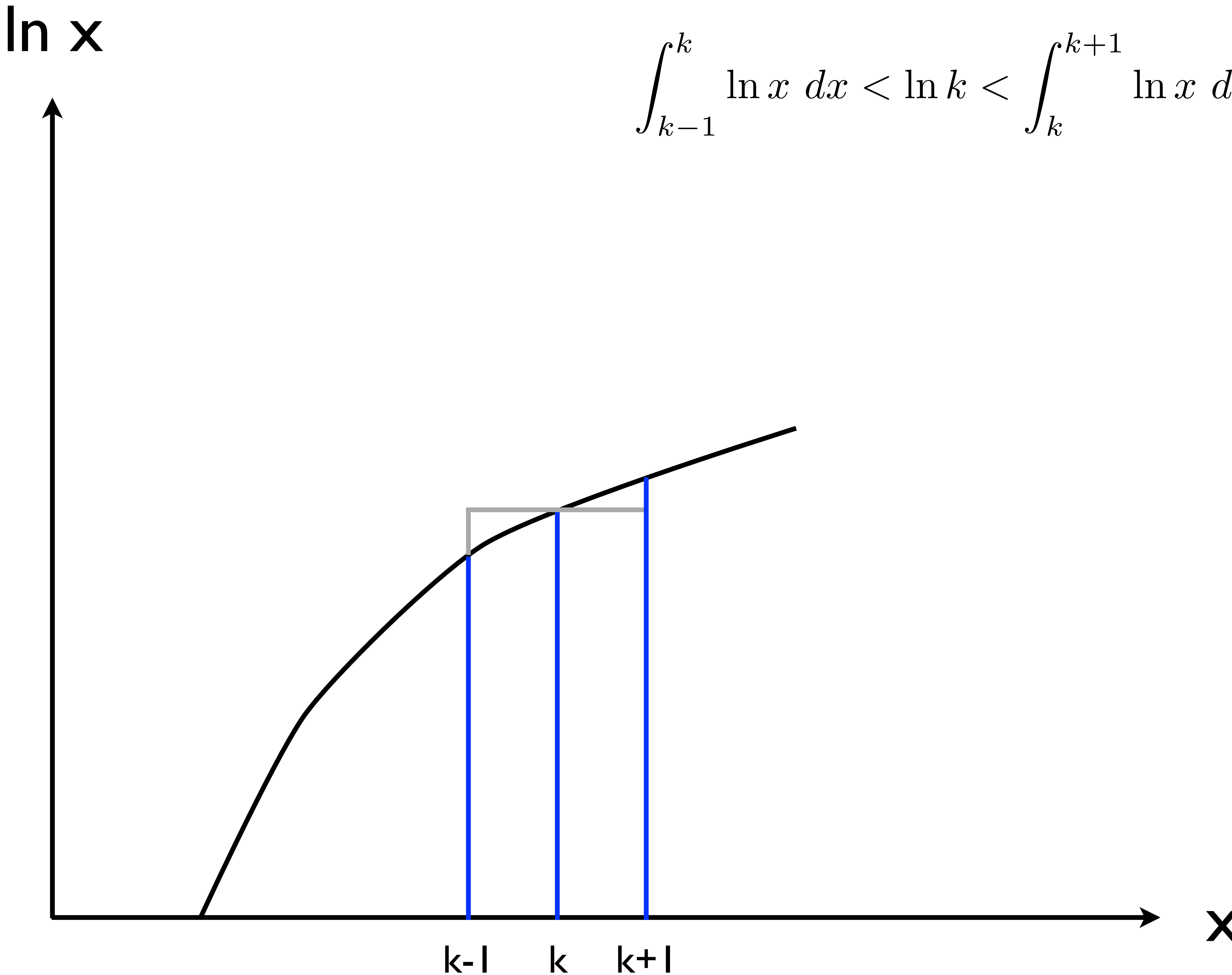

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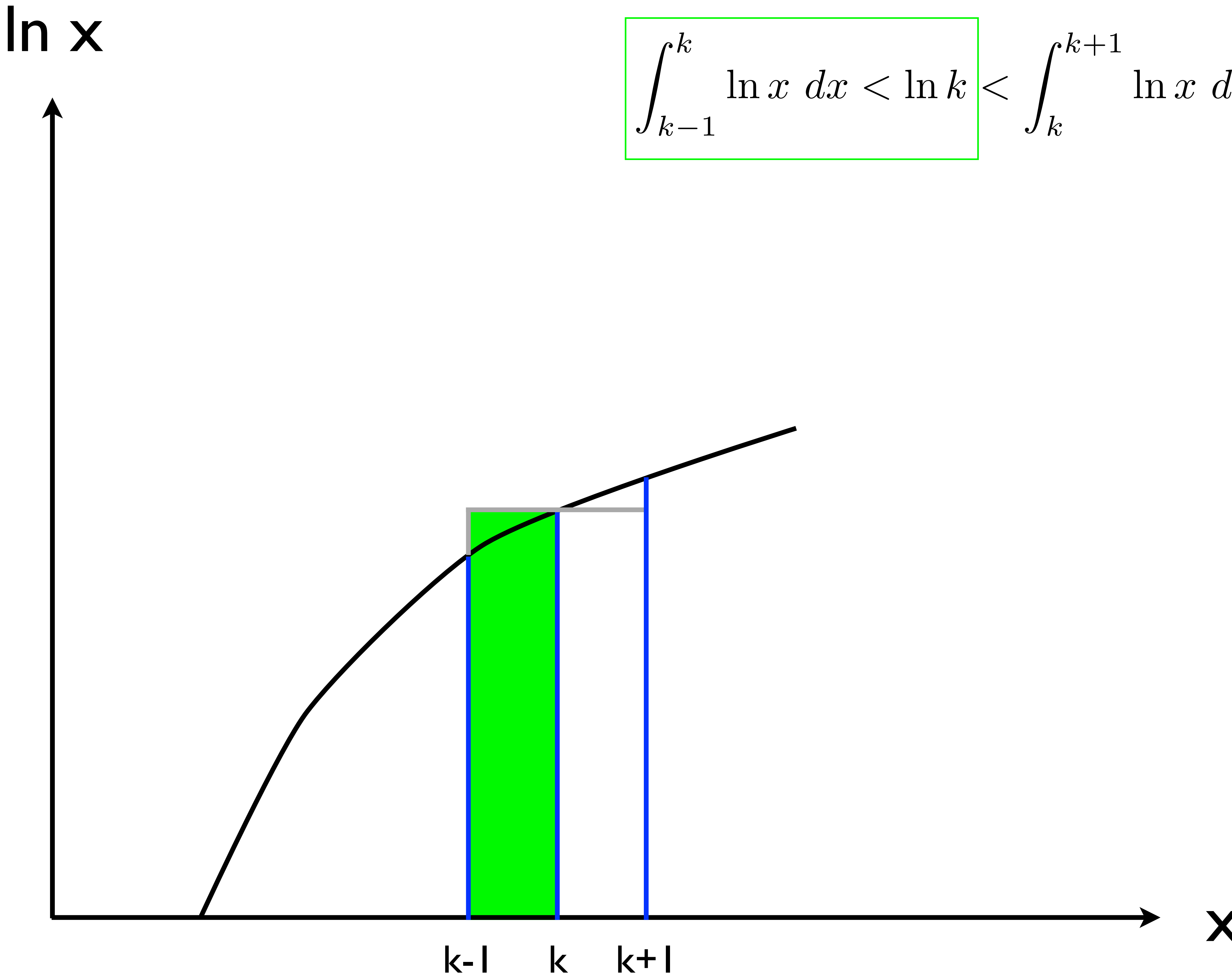

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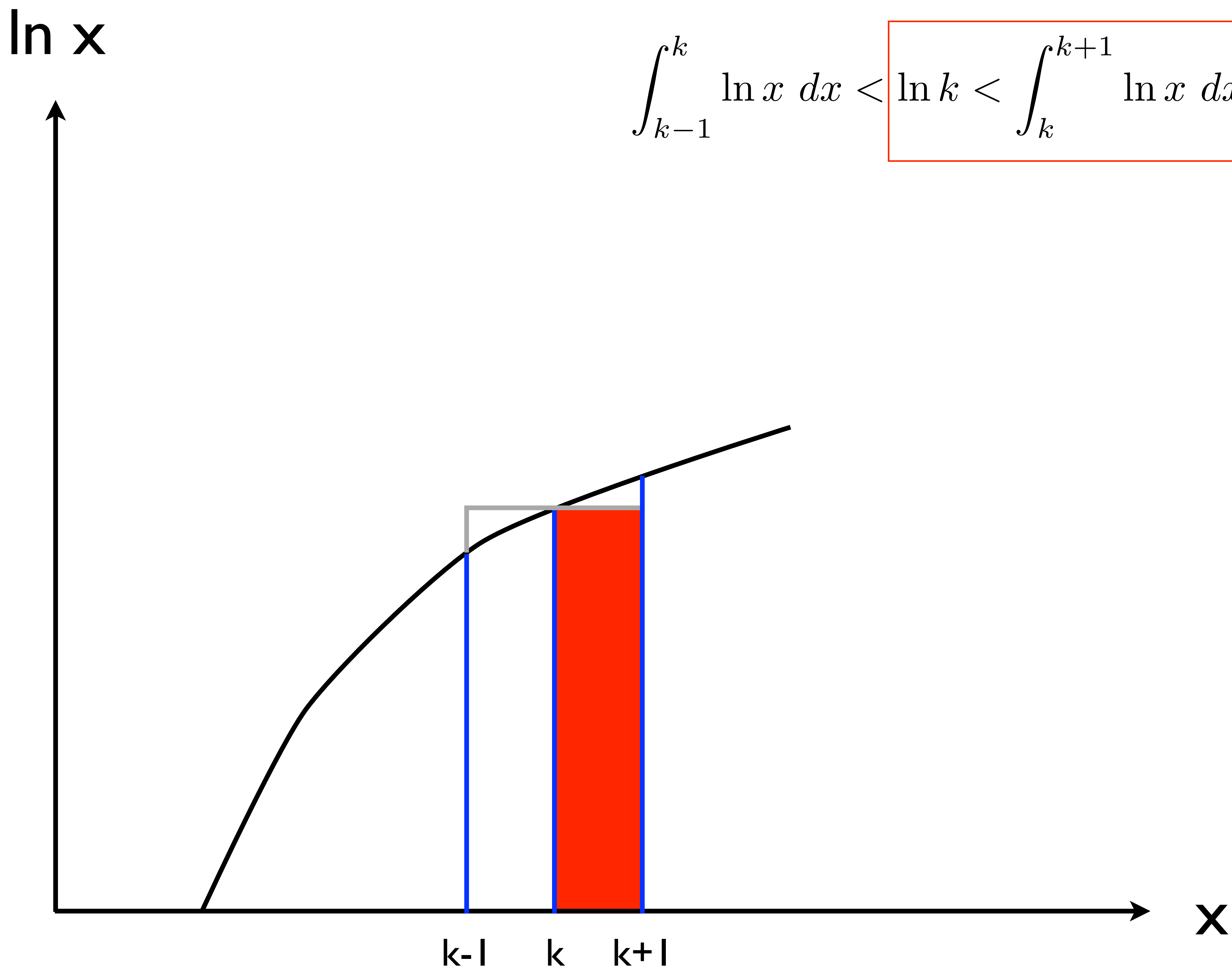

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$$\binom{n}{np,\, n(1-p)} \approx 2^{nH_2(\{p,1-p\})}.$$

Lemma 6.11 (Stirling's Approximation)

$$\ln n! \sim n \ln n.$$

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Lemma (Multinomial Coefficient) Let

$$\{p_1, p_2, \cdots, p_m\}$$

be a probability mass function. For large n,

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Theorem 6.10 (Conditional Strong AEP) For any $\mathbf{x} \in T^n_{[X]\delta}$, define

$$T^n_{[Y|X]\delta}(\mathbf{x}) = \{\mathbf{y} \in T^n_{[Y]\delta} : (\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}\}.$$

If $|T^n_{[Y|X]\delta}(\mathbf{x})| \geq 1$, then

$$2^{n(H(Y|X)-\nu)} \leq |T^n_{[Y|X]\delta}(\mathbf{x})| \leq 2^{n(H(Y|X)+\nu)},$$

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- Since
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- Theorem 6.10 guarantees that this is so for each typical $\mathbf{x}$ as long as there exists at least one $\mathbf{y}$ that is jointly typical with $\mathbf{x}$.

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If $|T^n_{[Y|X]\delta}(\mathbf{x})| \geq 1$, then

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where $\nu \to 0$ as $n \to \infty$ and $\delta \to 0$.

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$$\begin{aligned}
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\end{aligned}$$

2. The remaining steps are similar to the proof of the upper bound on $|T^n_{[X]\delta}|$ in Theorem 6.2 (SAEP):

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$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Strong JAEP

$$2^{-n(H(X,Y)+\lambda)} \leq p(\mathbf{x}, \mathbf{y}) \leq 2^{-n(H(X,Y)-\lambda)}.$$

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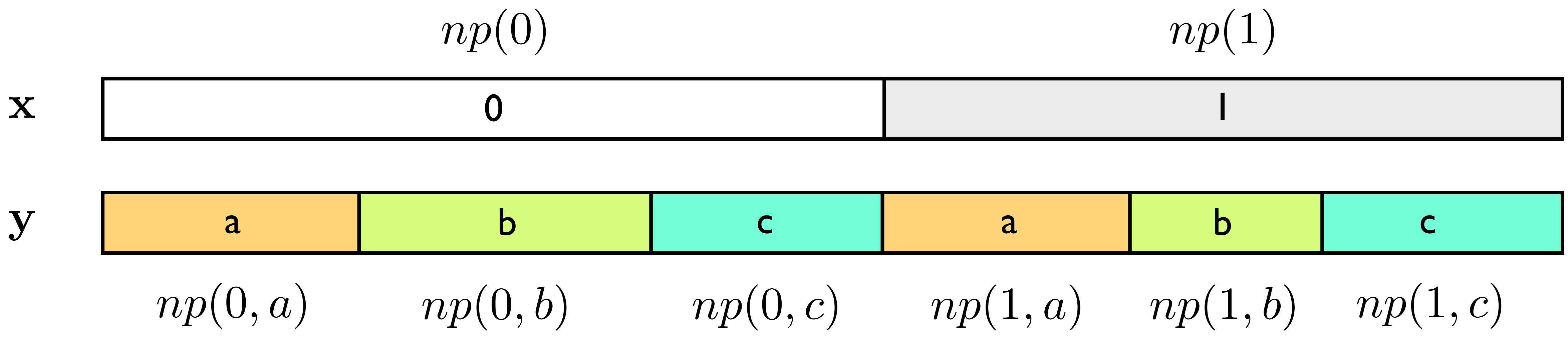
Lower Bound in Theorem 6.10

$\mathcal{X} = \{0, 1\}, \mathcal{Y} = \{a, b, c\}$



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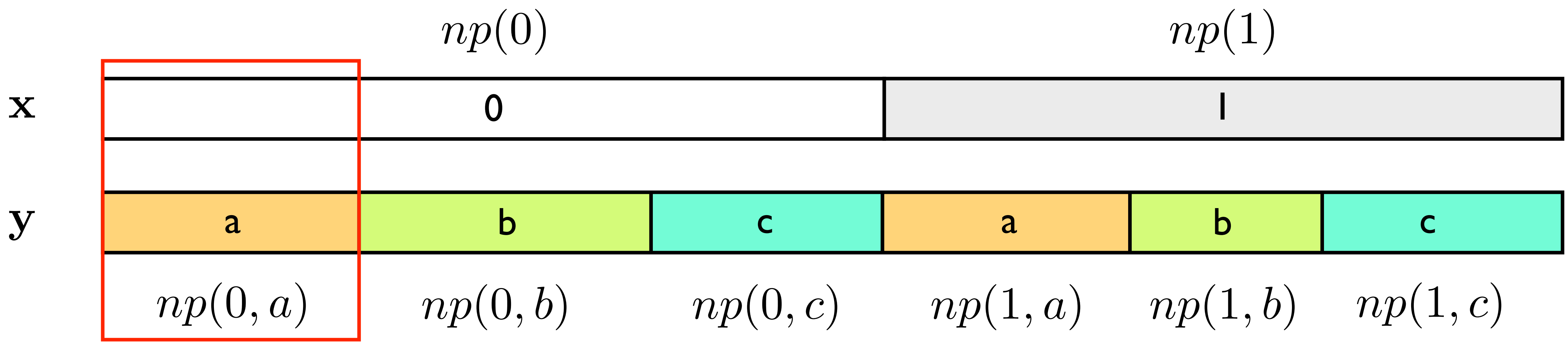

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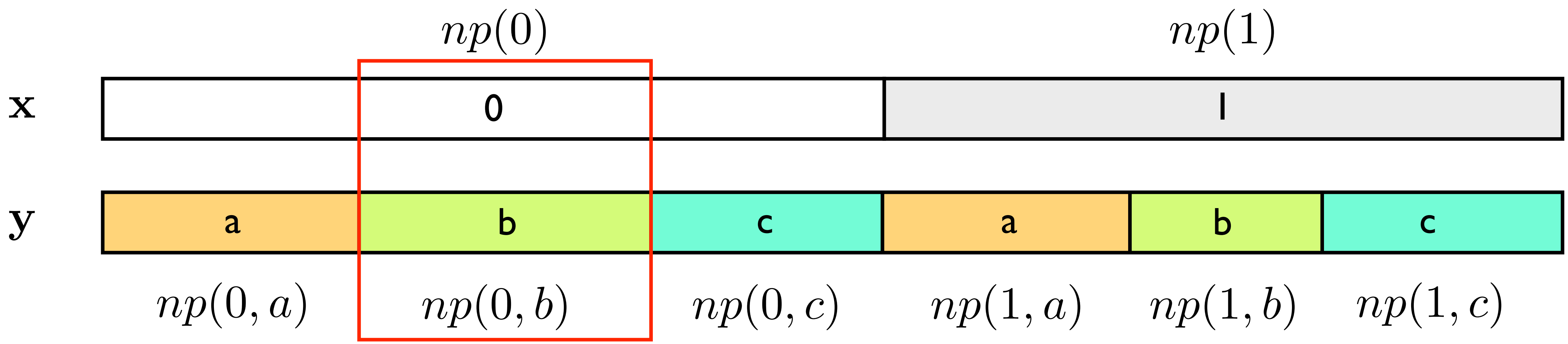

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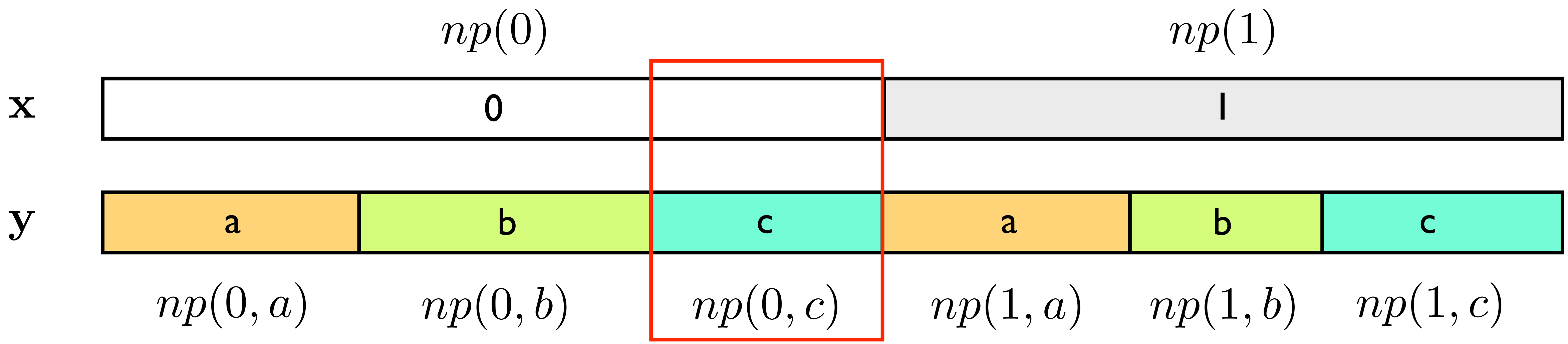

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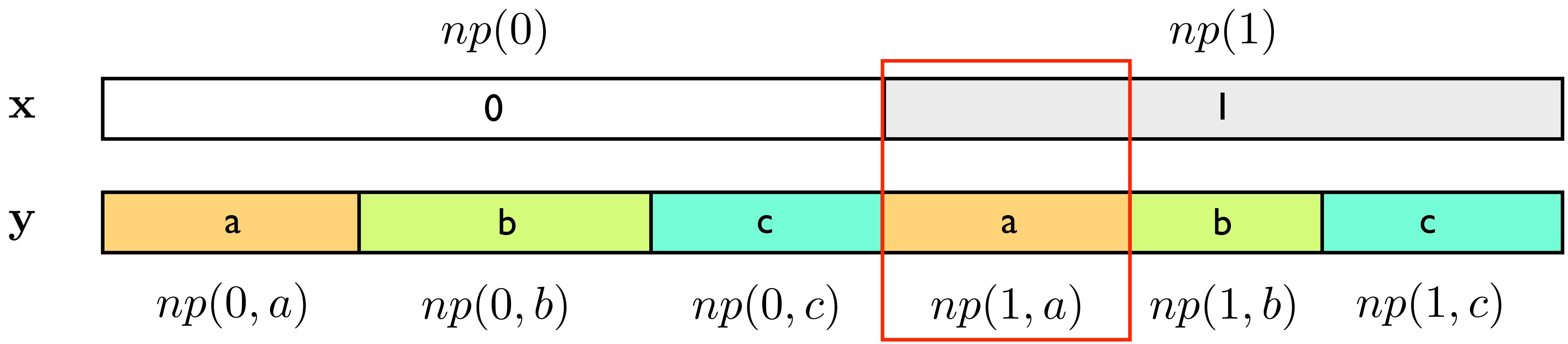

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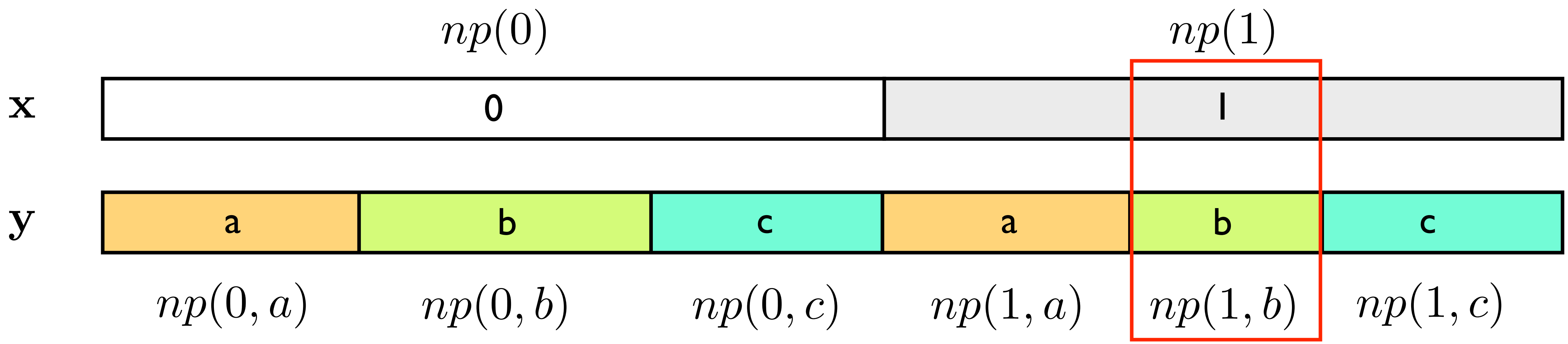

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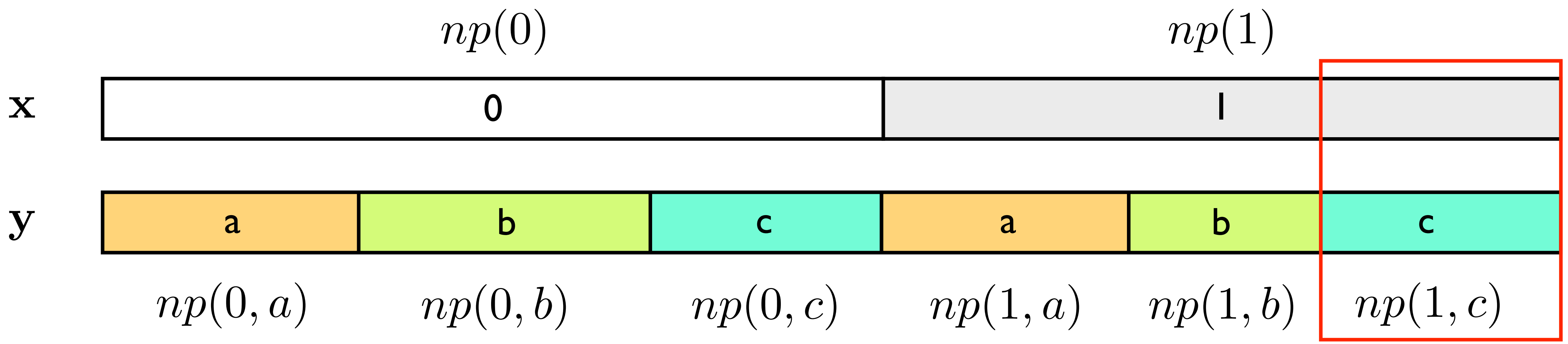

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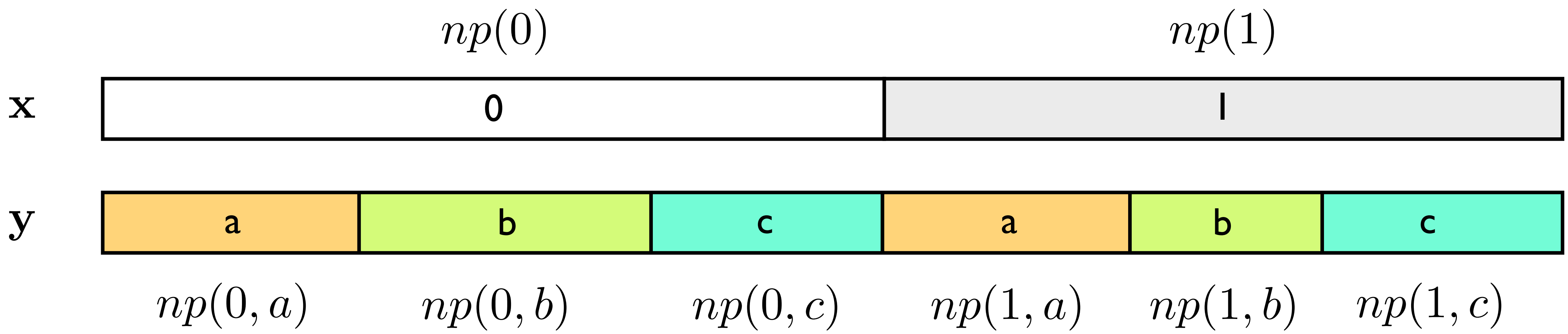

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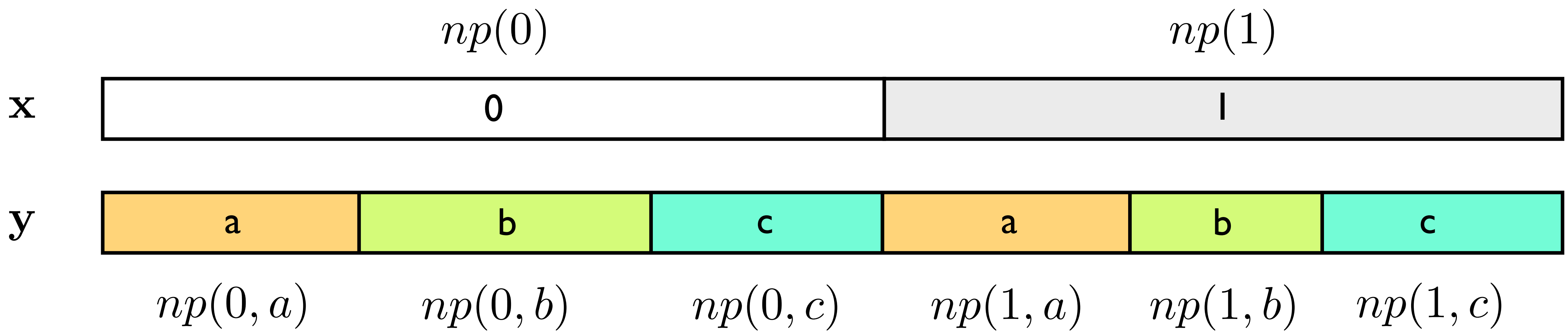

Rearrange the components of $\mathbf{y}$ corresponding to $x_k = 0$ and rearrange the components of $\mathbf{y}$ corresponding to $x_k = 1$. This preserves joint typicality.

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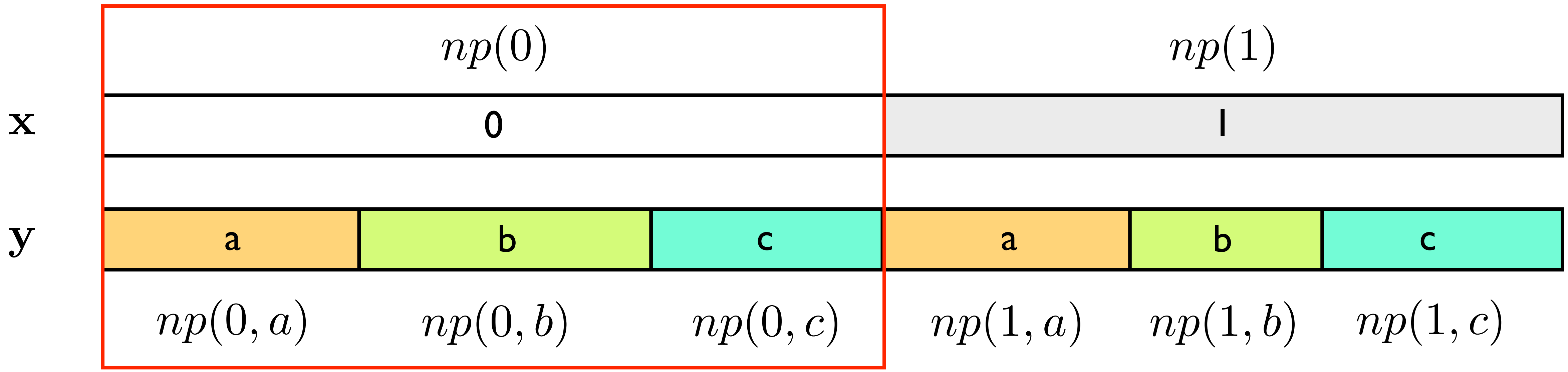

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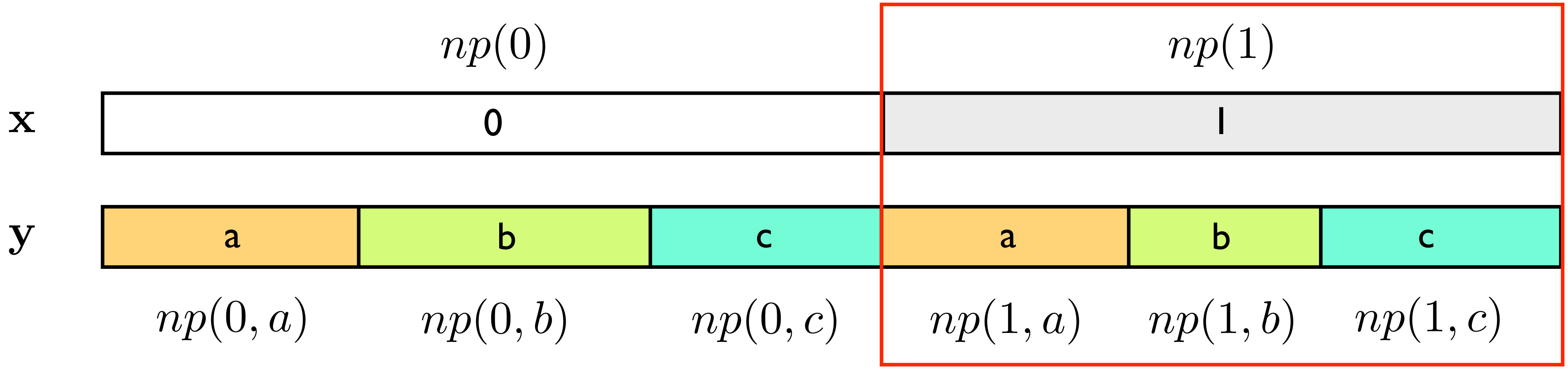

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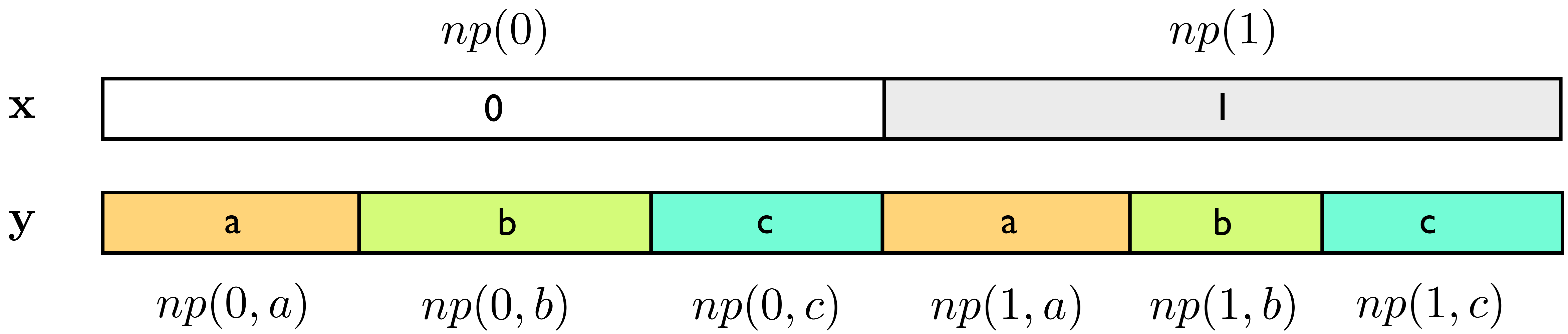

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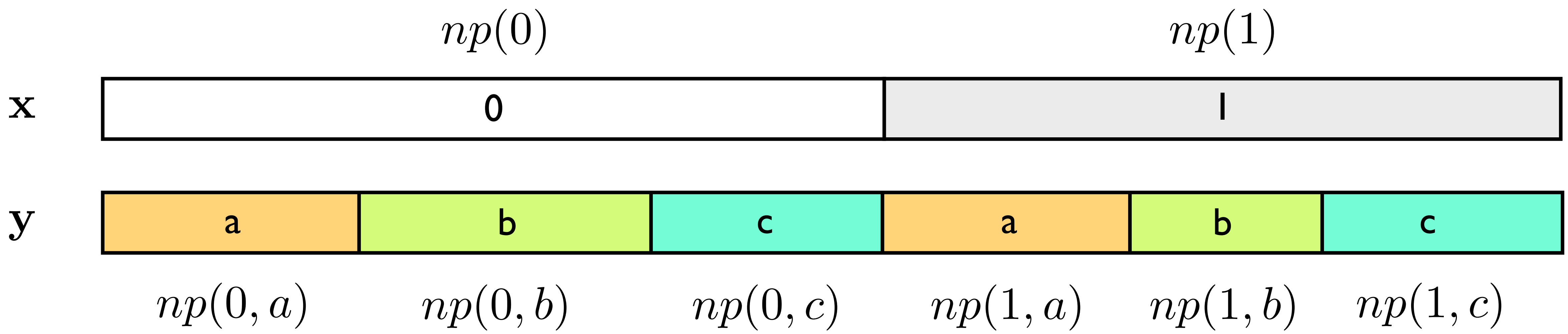

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$$\#arrangements \approx \binom{np(0)}{np(0,a)\ np(0,b)\ np(0,c)} \binom{np(1)}{np(1,a)\ np(1,b)\ np(1,c)}$$

Lower Bound in Theorem 6.10

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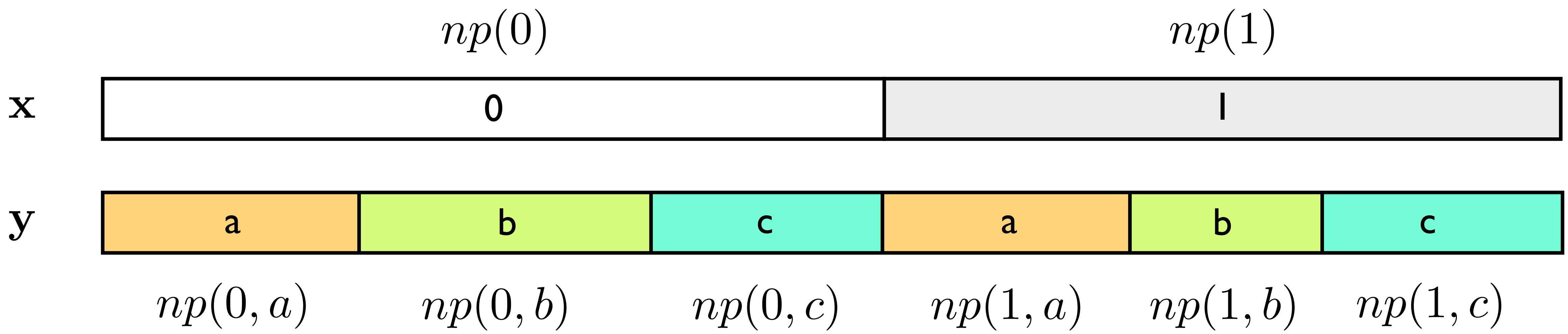

Rearrange the components of $\mathbf{y}$ corresponding to $x_k = 0$ and rearrange the components of $\mathbf{y}$ corresponding to $x_k = 1$. This preserves joint typicality.

$$\begin{aligned}\#arrangements &\approx \binom{np(0)}{np(0,a)\ np(0,b)\ np(0,c)} \binom{np(1)}{np(1,a)\ np(1,b)\ np(1,c)} \\ &\approx 2^{np(0)H(\{p(\cdot|0)\})} 2^{np(1)H(\{p(\cdot|1)\})}\end{aligned}$$

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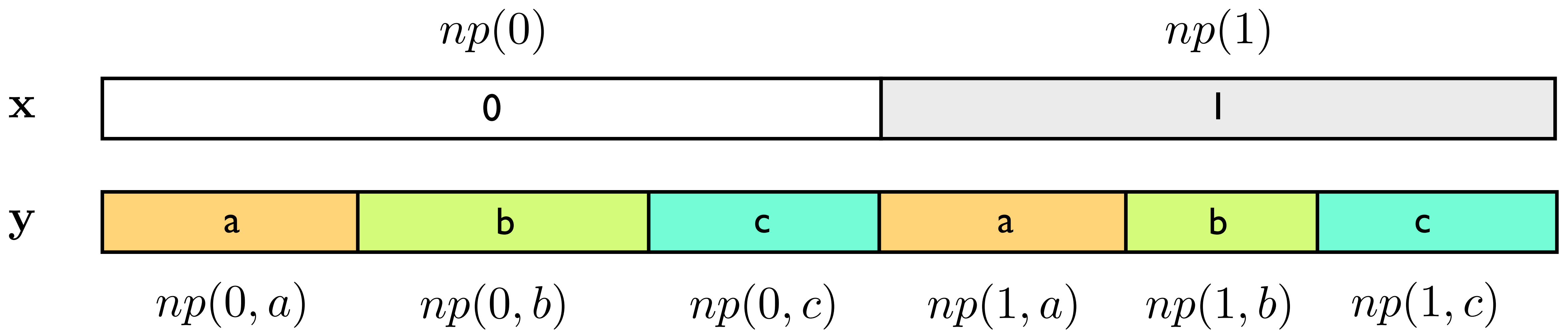

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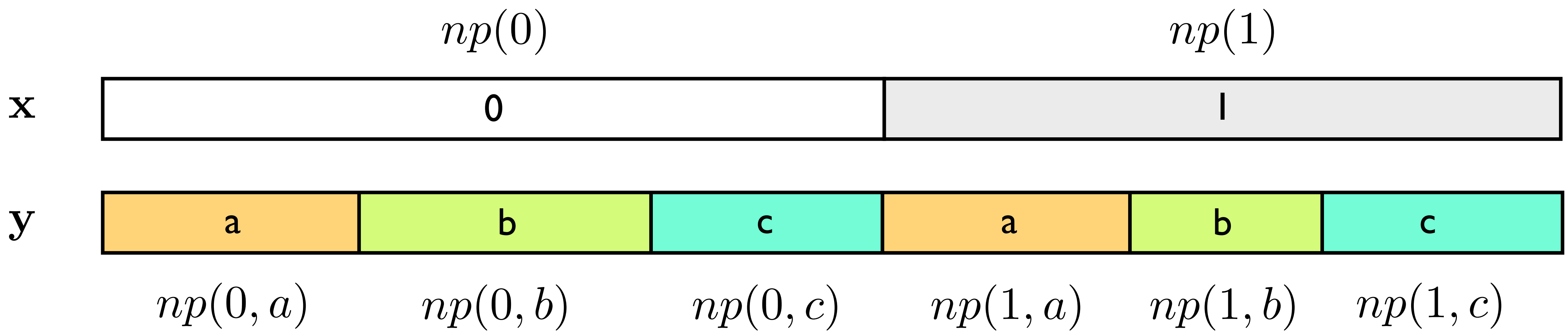

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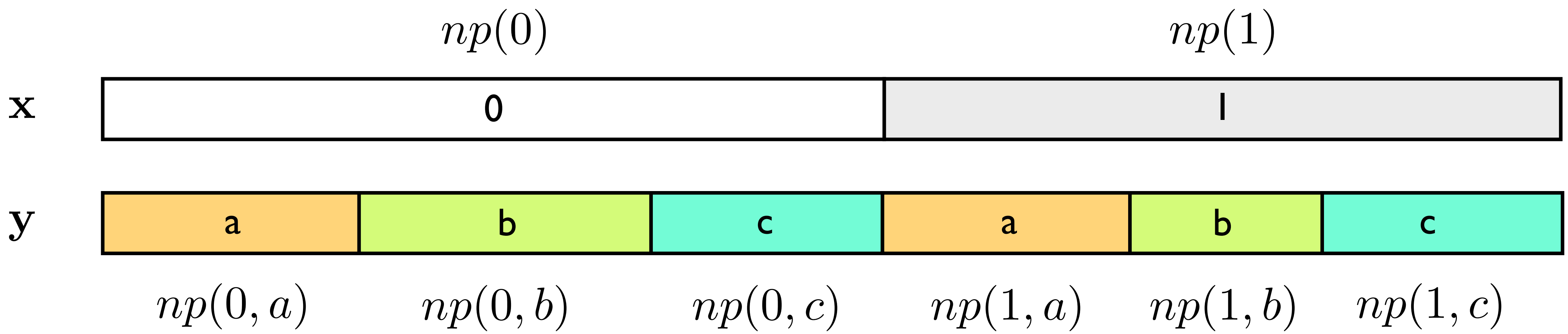

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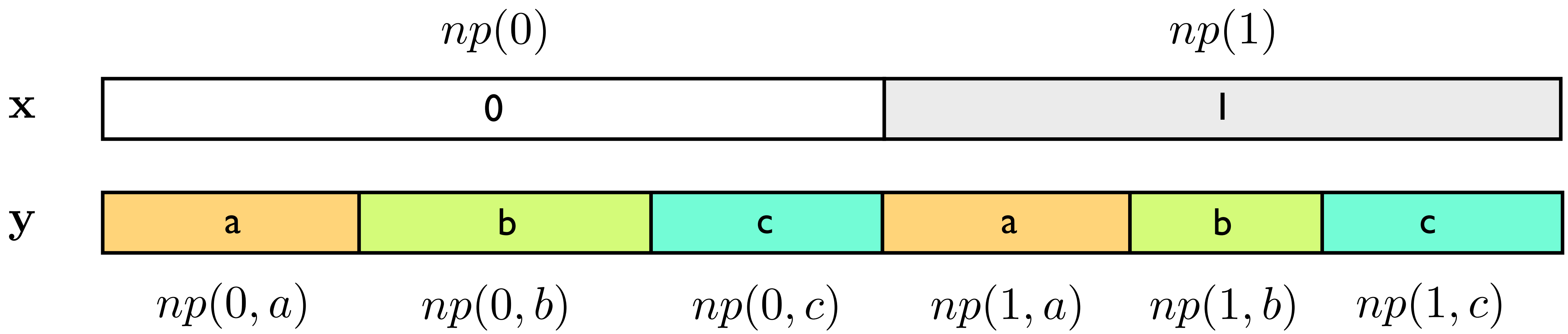

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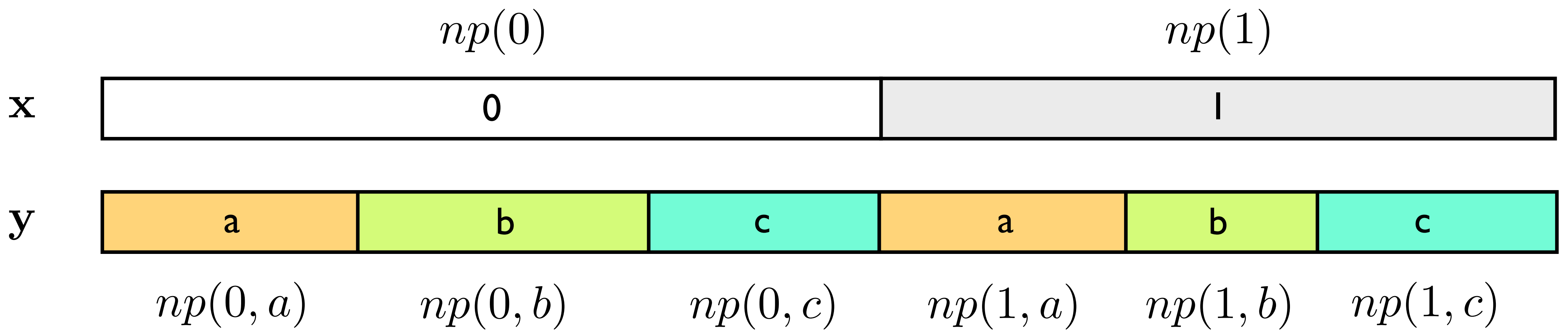

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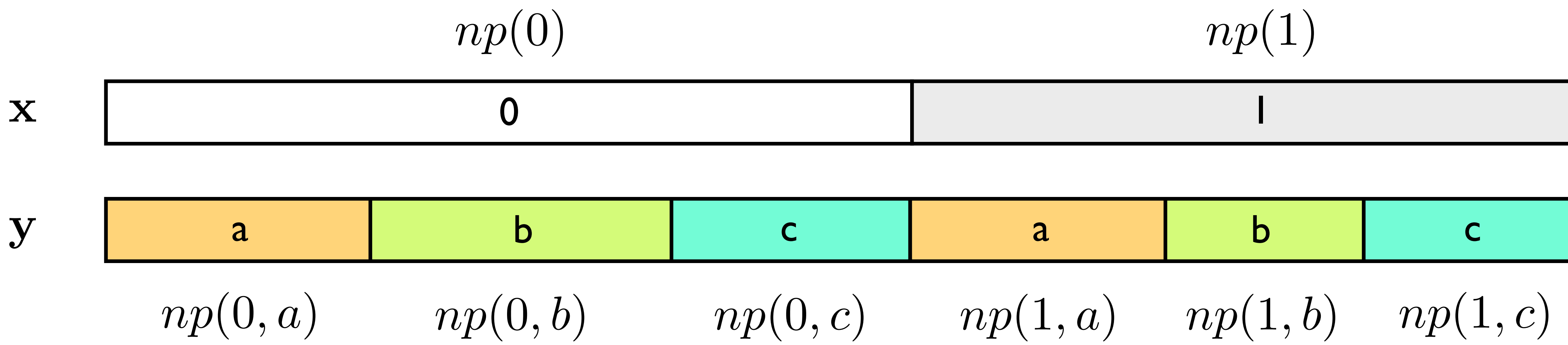

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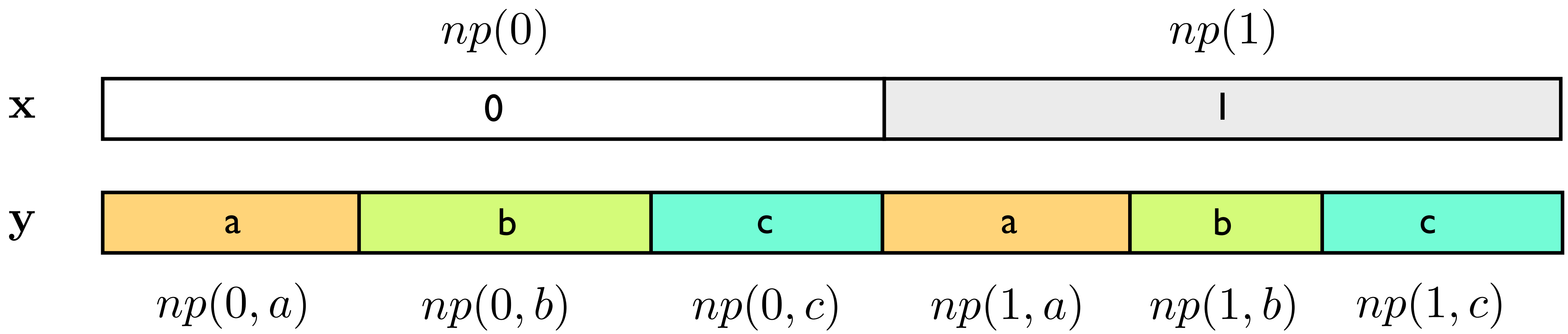

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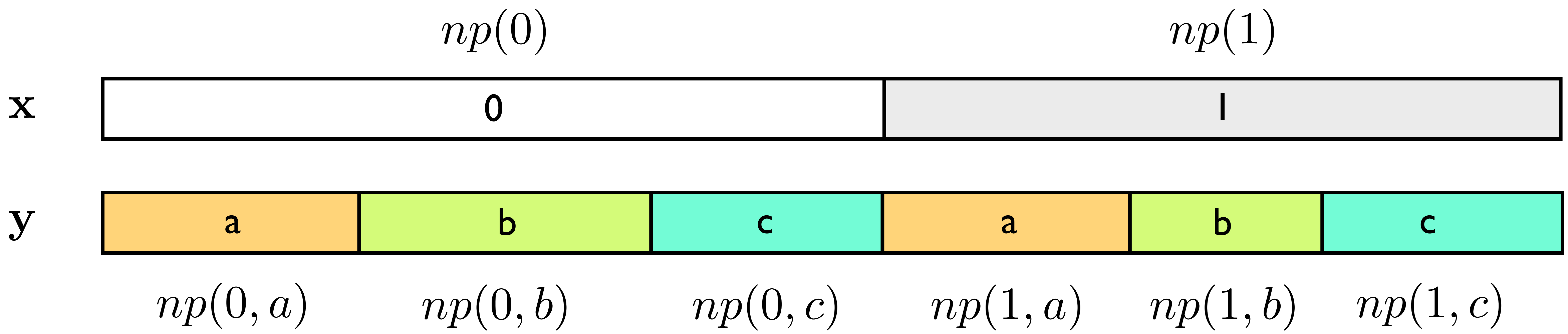

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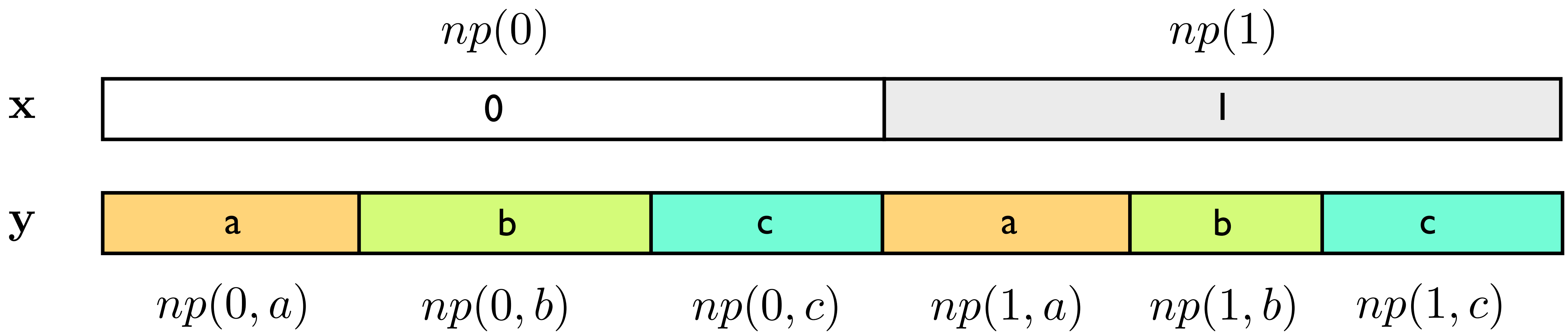

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\end{aligned}
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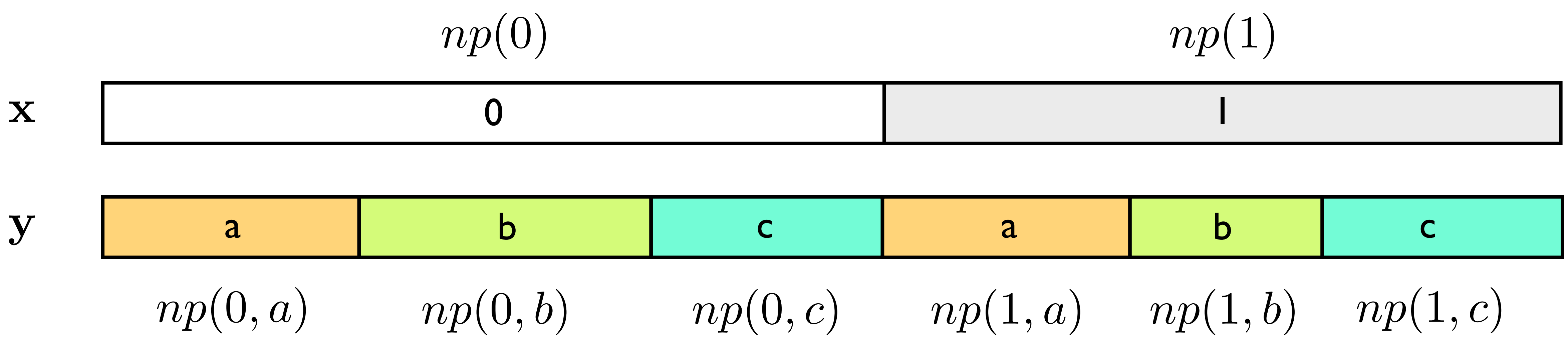

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&= 2^{n(p(0)H(Y|X=0)+p(1)H(Y|X=1))} \\
&= 2^{nH(Y|X)}
\end{aligned}
$$

Hence,

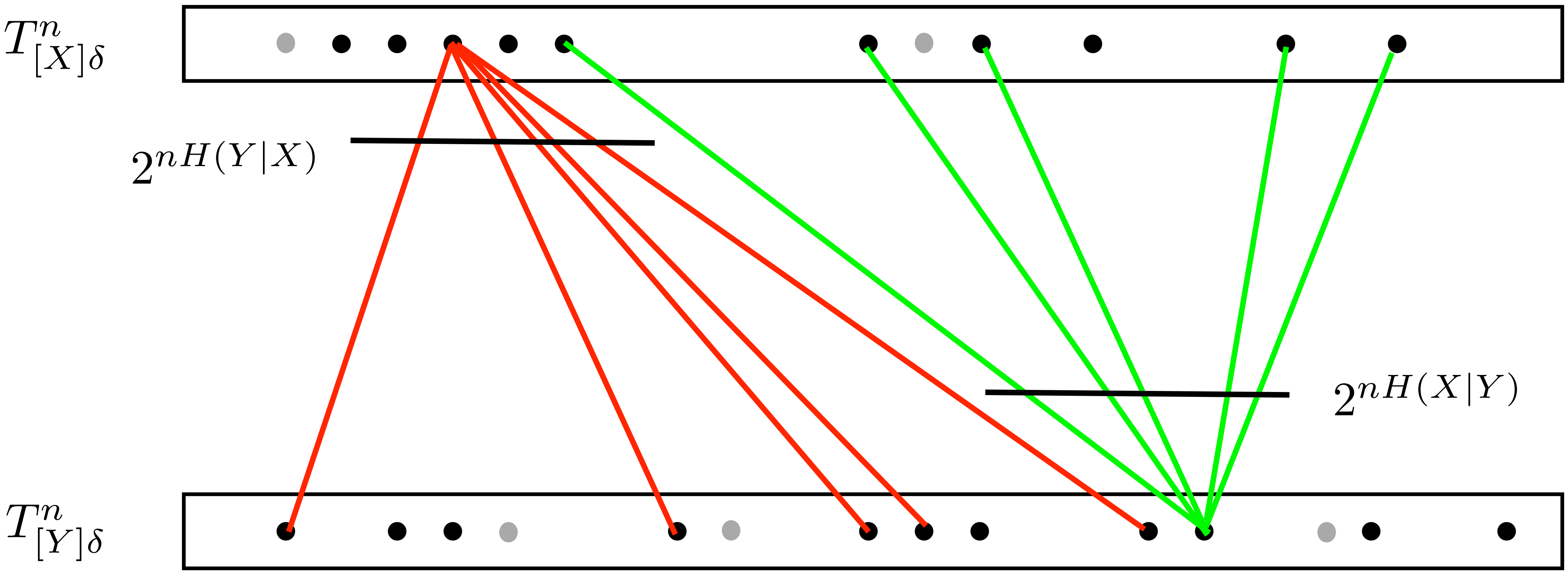
$$
|T^n_{[Y|X]\delta}(\mathbf{x})| \geq 2^{n(H(Y|X)-\nu)}.
$$

Theorem 6.10 (Conditional Strong AEP) If $|T^n_{[Y|X]\delta}(\mathbf{x})| \geq 1$, then

$$2^{n(H(Y|X)-\nu)} \leq |T^n_{[Y|X]\delta}(\mathbf{x})| \leq 2^{n(H(Y|X)+\nu)},$$

where $\nu \to 0$ as $n \to \infty$ and $\delta \to 0$.

An Illustration of Conditional SAEP



Corollary 6.12 For a joint distribution $p(x, y)$ on $\mathcal{X} \times \mathcal{Y}$, let $S^n_{[X]\delta}$ be the set of all sequences $\mathbf{x} \in T^n_{[X]\delta}$ such that $T^n_{[Y|X]\delta}(\mathbf{x})$ is nonempty. Then

$$|S^n_{[X]\delta}| \geq (1 - \delta)2^{n(H(X) - \psi)},$$

where $\psi \to 0$ as $n \to \infty$ and $\delta \to 0$.

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where $\psi \to 0$ as $n \to \infty$ and $\delta \to 0$.

Proposition 6.13 With respect to a joint distribution $p(x,y)$ on $\mathcal{X} \times \mathcal{Y}$, for any $\delta > 0$,

$$\Pr\{\mathbf{X} \in S^n_{[X]\delta}\} > 1 - \delta$$

for n sufficiently large.

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Proof

1. By the consistency of strong typicality (Theorem 6.7), if $(\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}$, then $\mathbf{x} \in T^n_{[X]\delta}$. In particular, $\mathbf{x} \in S^n_{[X]\delta}$.

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2. Then

$$T^n_{[XY]\delta} = \bigcup_{\mathbf{x} \in S^n_{[X]\delta}} \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in T^n_{[Y|X]\delta}(\mathbf{x})\}.$$

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Strong JAEP

$$(1-\delta)2^{n(H(X,Y)-\lambda)} \leq |T^n_{[XY]\delta}| \leq 2^{n(H(X,Y)+\lambda)}.$$

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Conditional Strong AEP

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Strong JAEP

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which implies

$$|S^n_{[X]\delta}| \quad \geq \quad (1-\delta)2^{n(H(X,Y)-H(Y|X)-(\lambda+\nu))}$$

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Conditional Strong AEP

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Corollary 6.12 Let $S^n_{[X]\delta}$ be the set of all sequences $\mathbf{x} \in T^n_{[X]\delta}$ such that $T^n_{[Y|X]\delta}(\mathbf{x})$ is nonempty. Then

$$|S^n_{[X]\delta}| \geq (1-\delta)2^{n(H(X)-\psi)},$$

where $\psi \to 0$ as $n \to \infty$ and $\delta \to 0$.

Proof

1. By the consistency of strong typicality (Theorem 6.7), if $(\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}$, then $\mathbf{x} \in T^n_{[X]\delta}$. In particular, $\mathbf{x} \in S^n_{[X]\delta}$.

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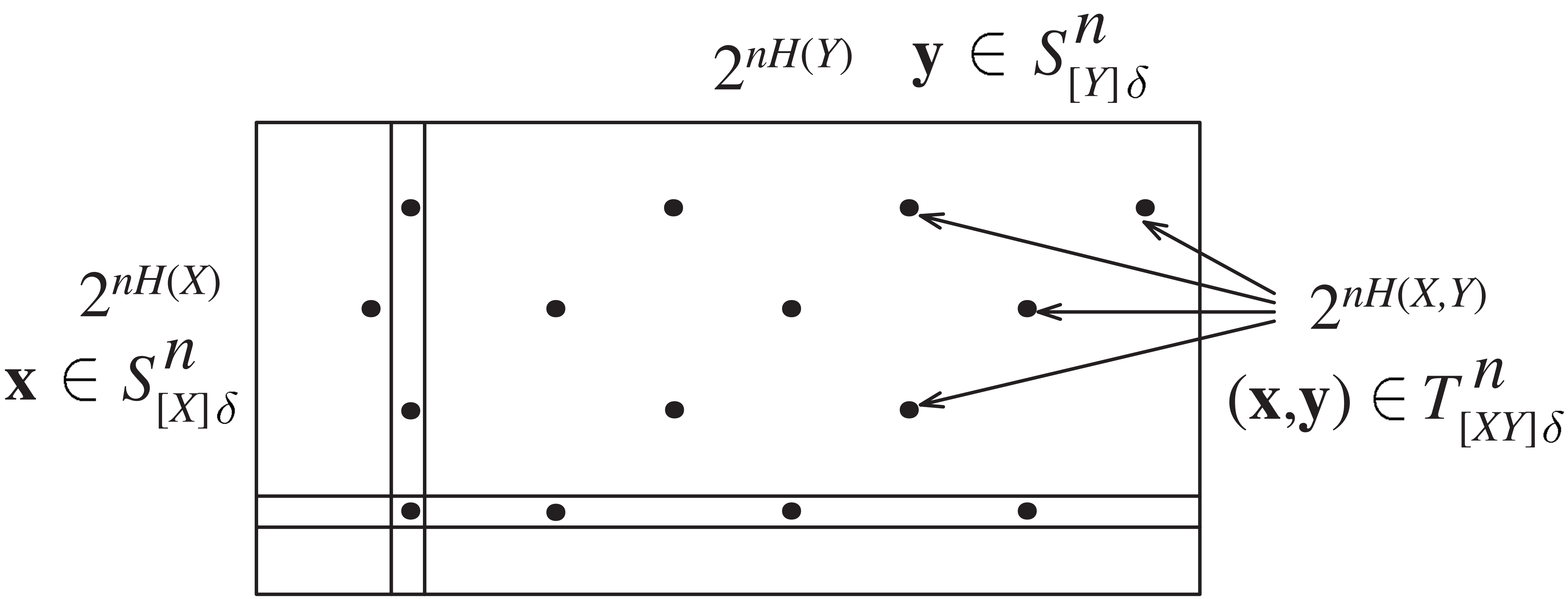
Strongly Joint Typicality Array

- Exhibits an "asymptotic quasi-uniform" structure.
- Two-Dimensional:



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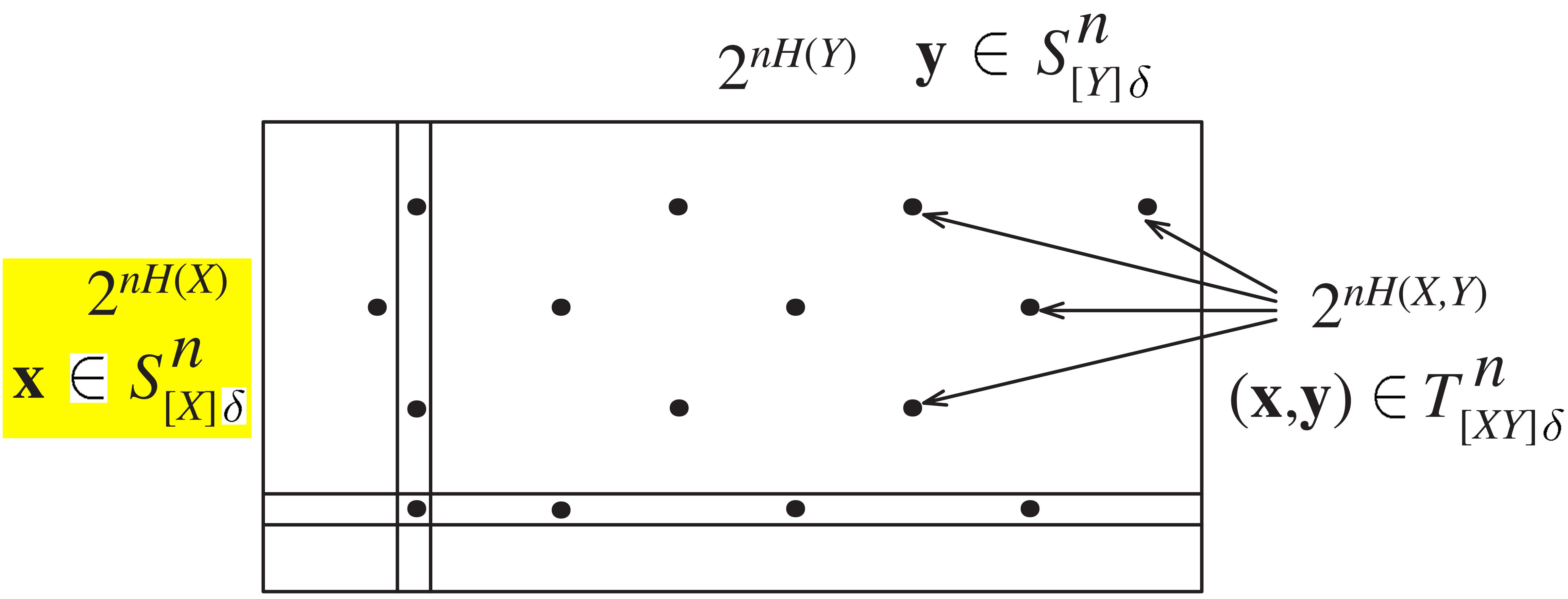

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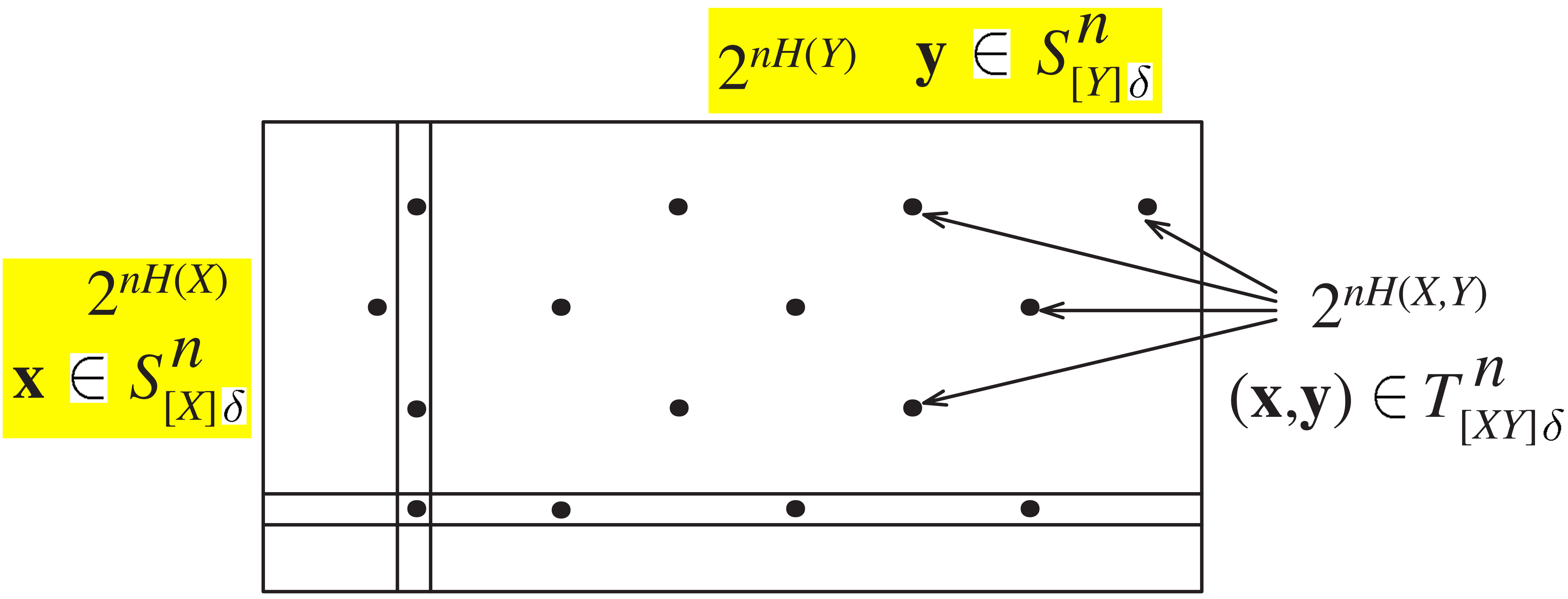

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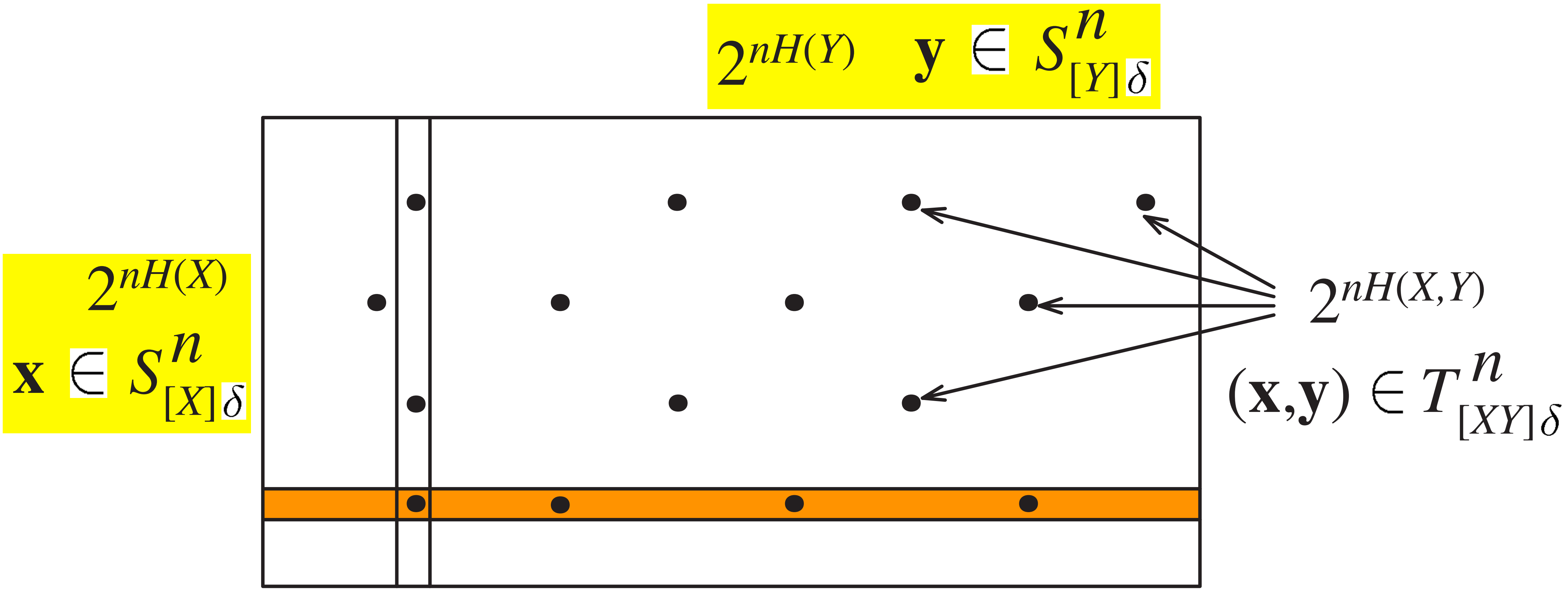

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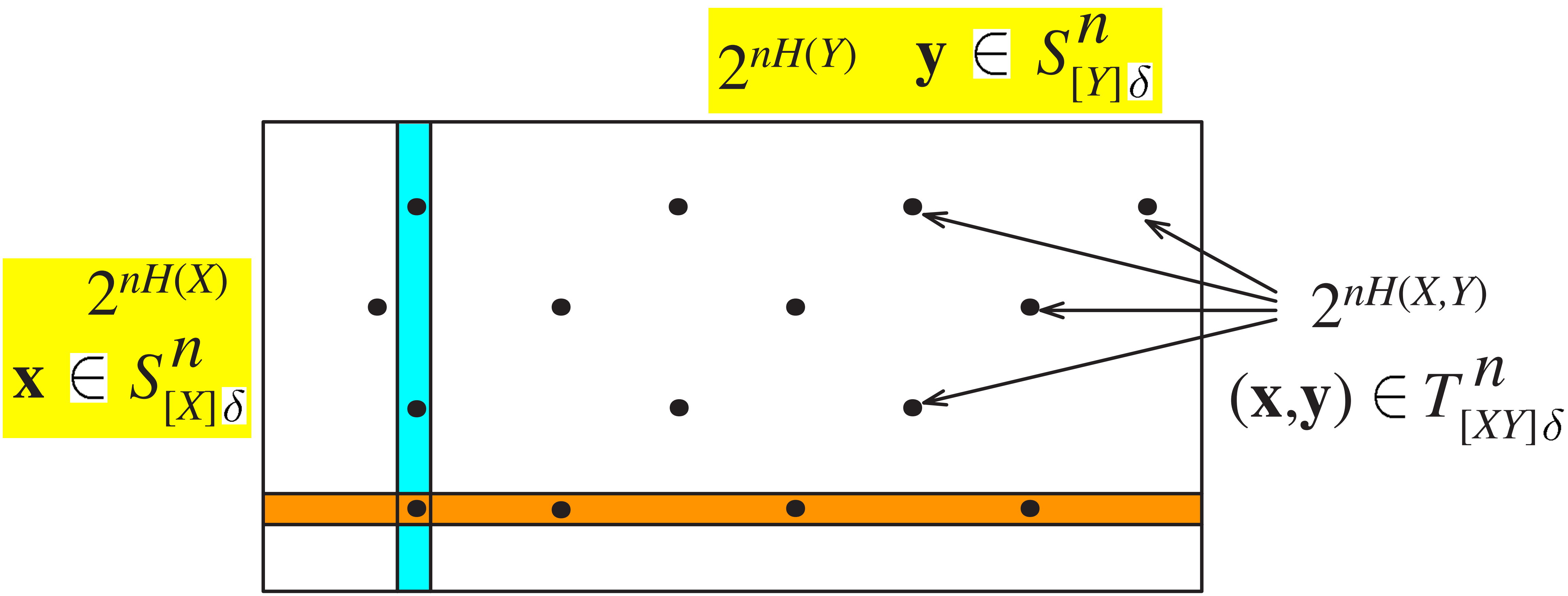

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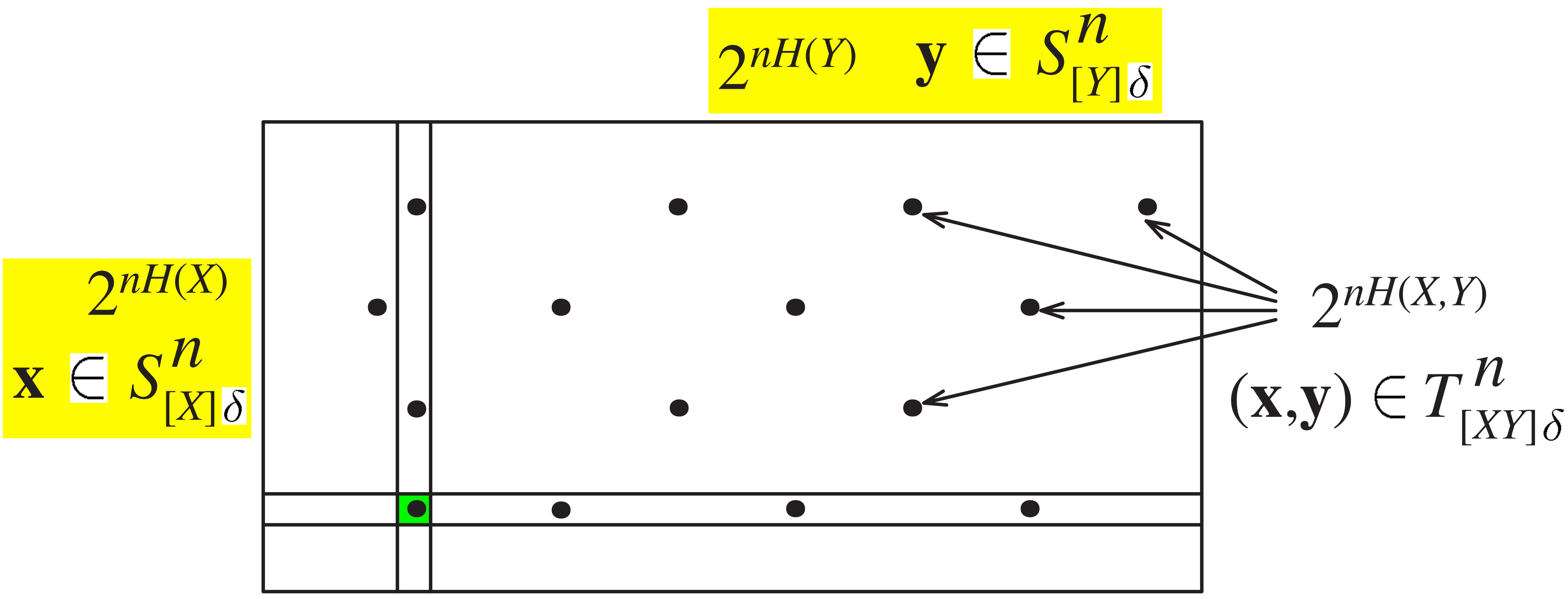

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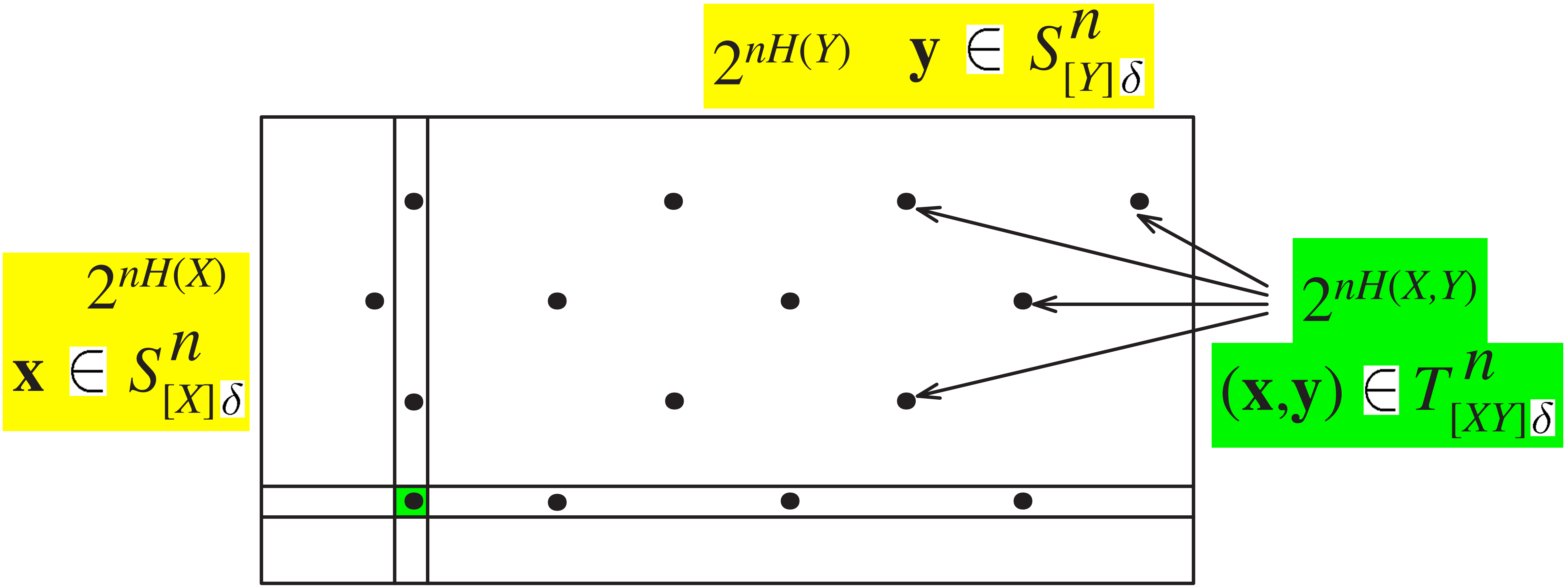

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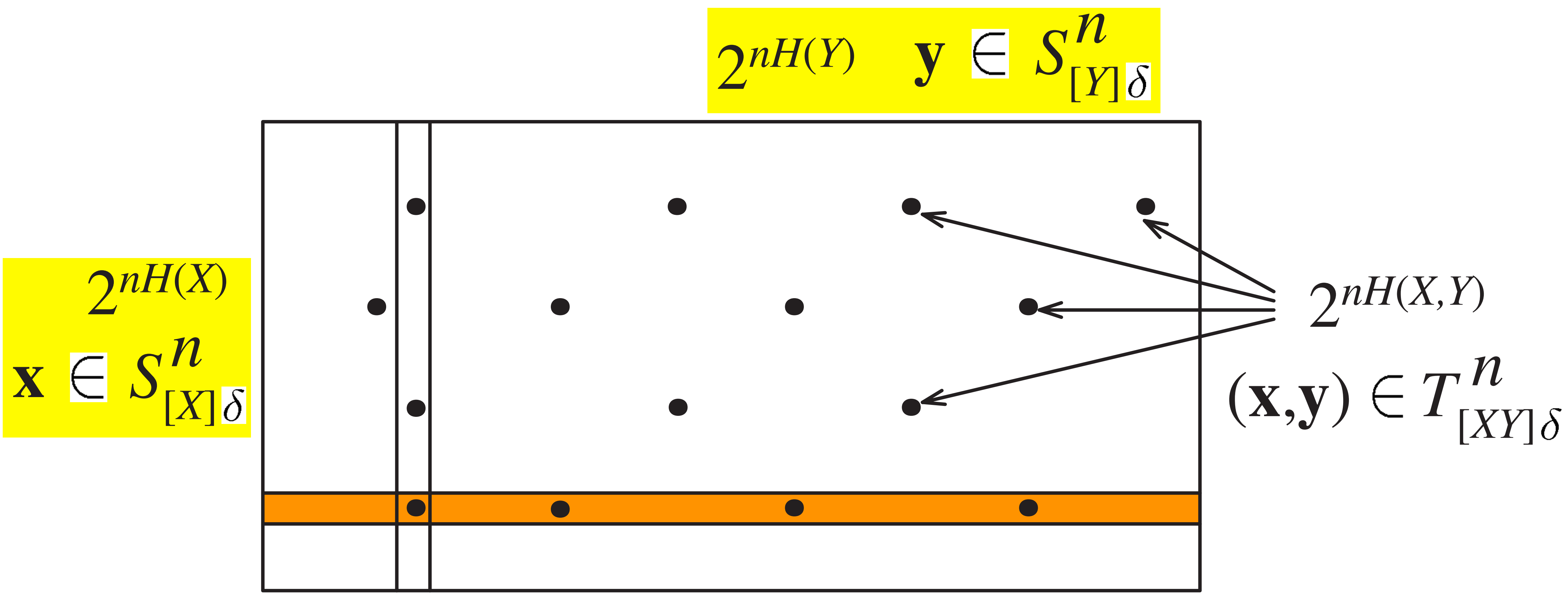

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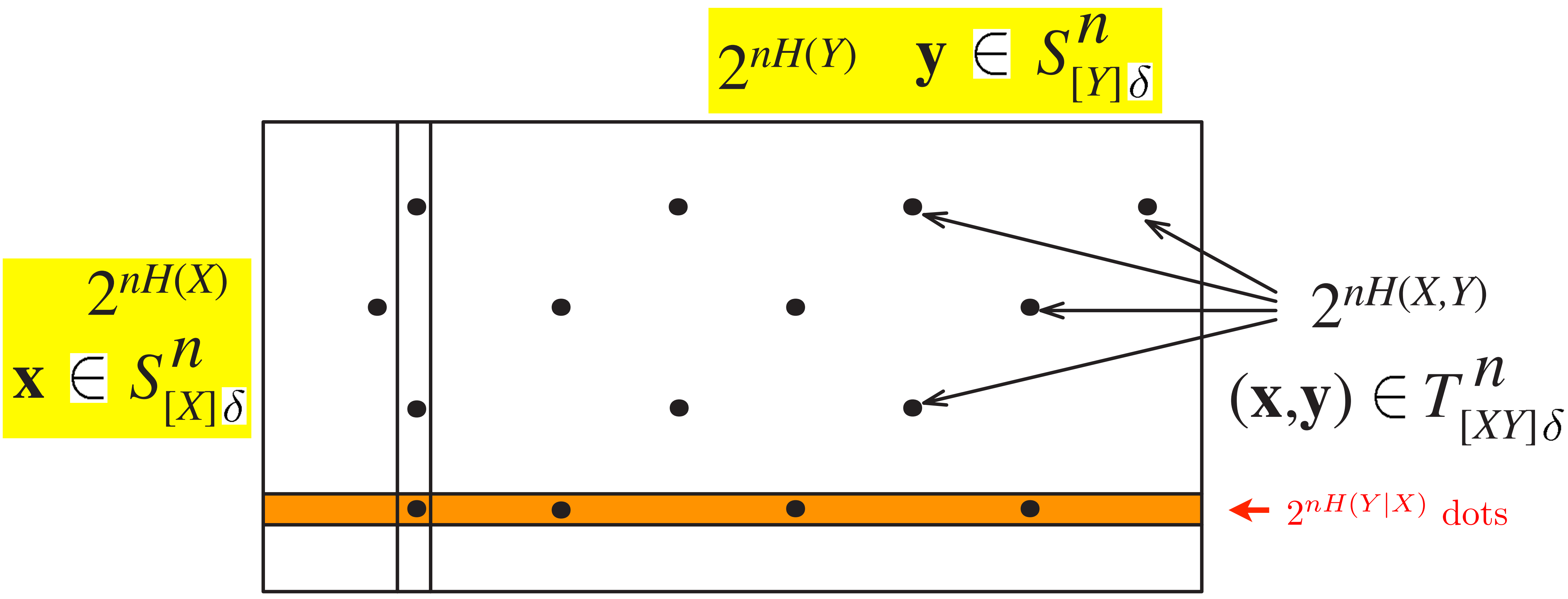

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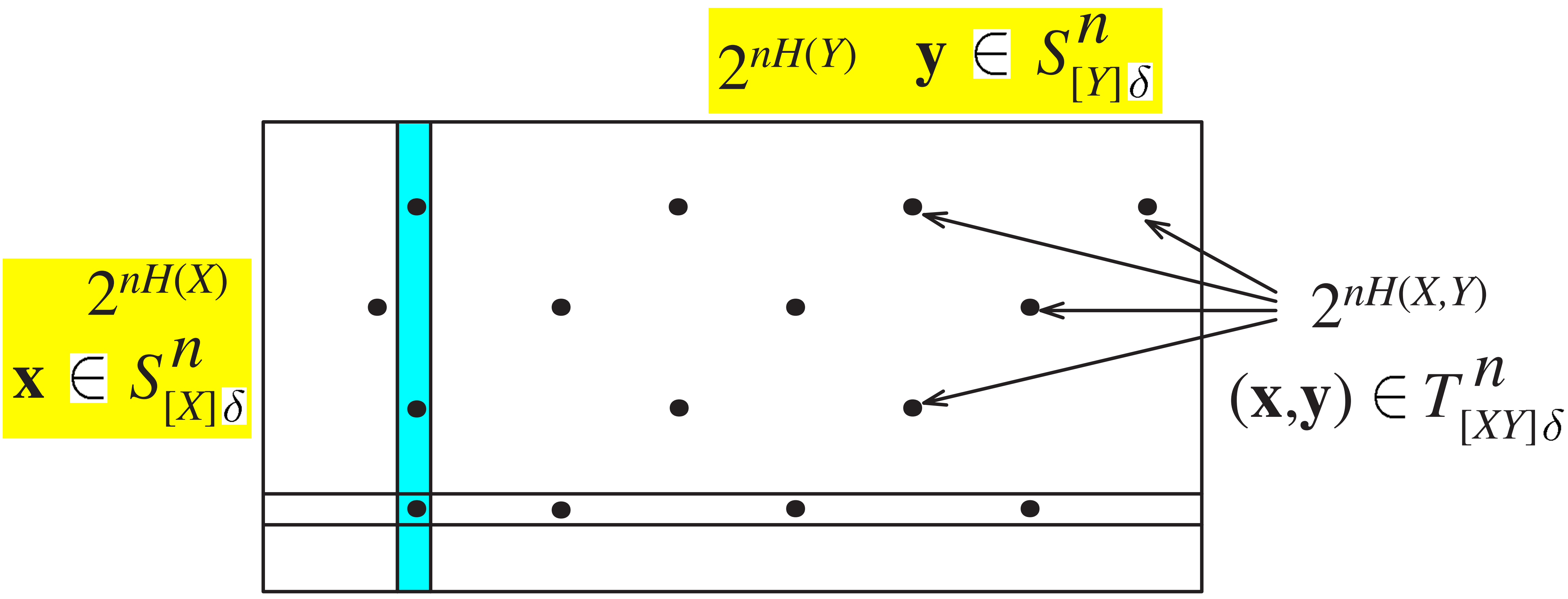

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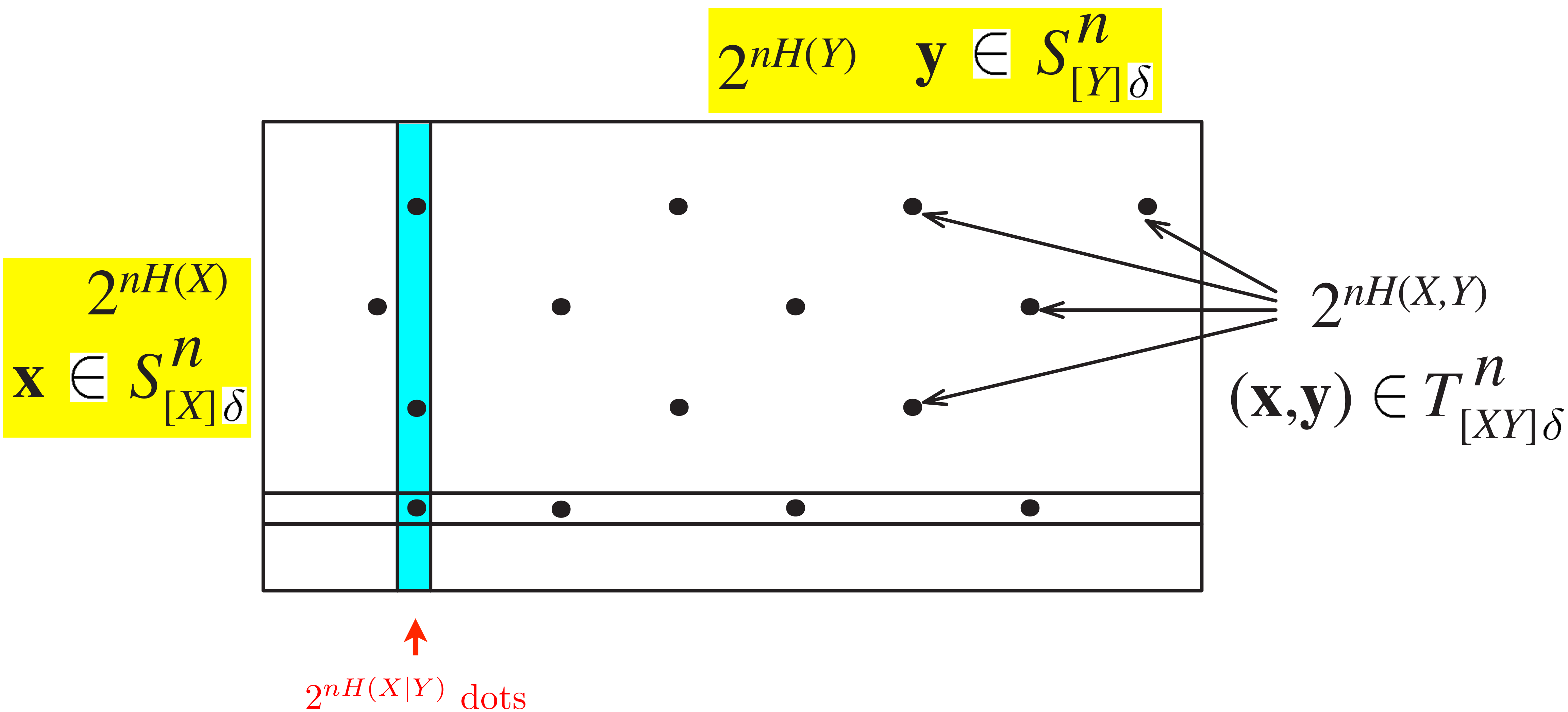

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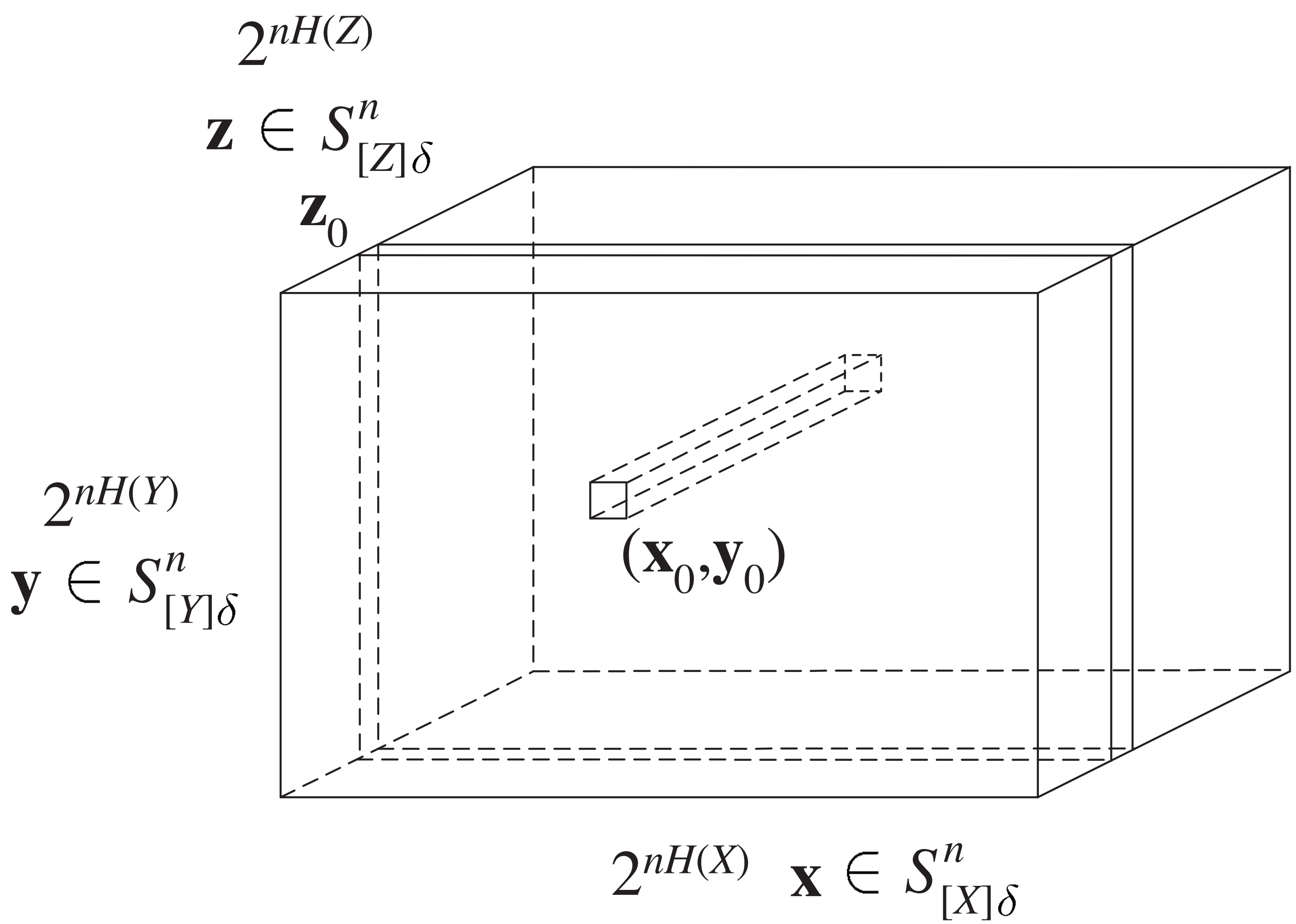

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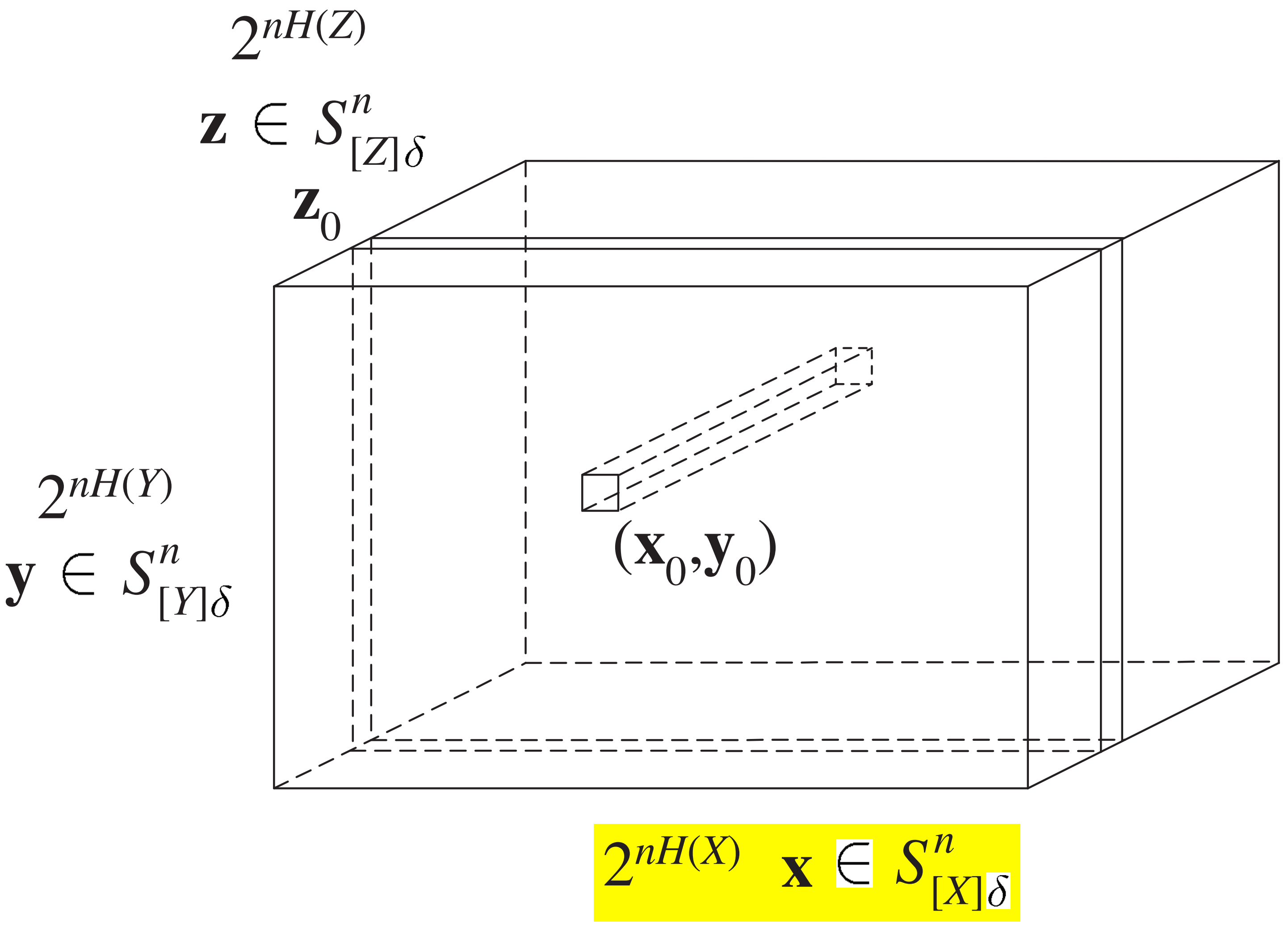

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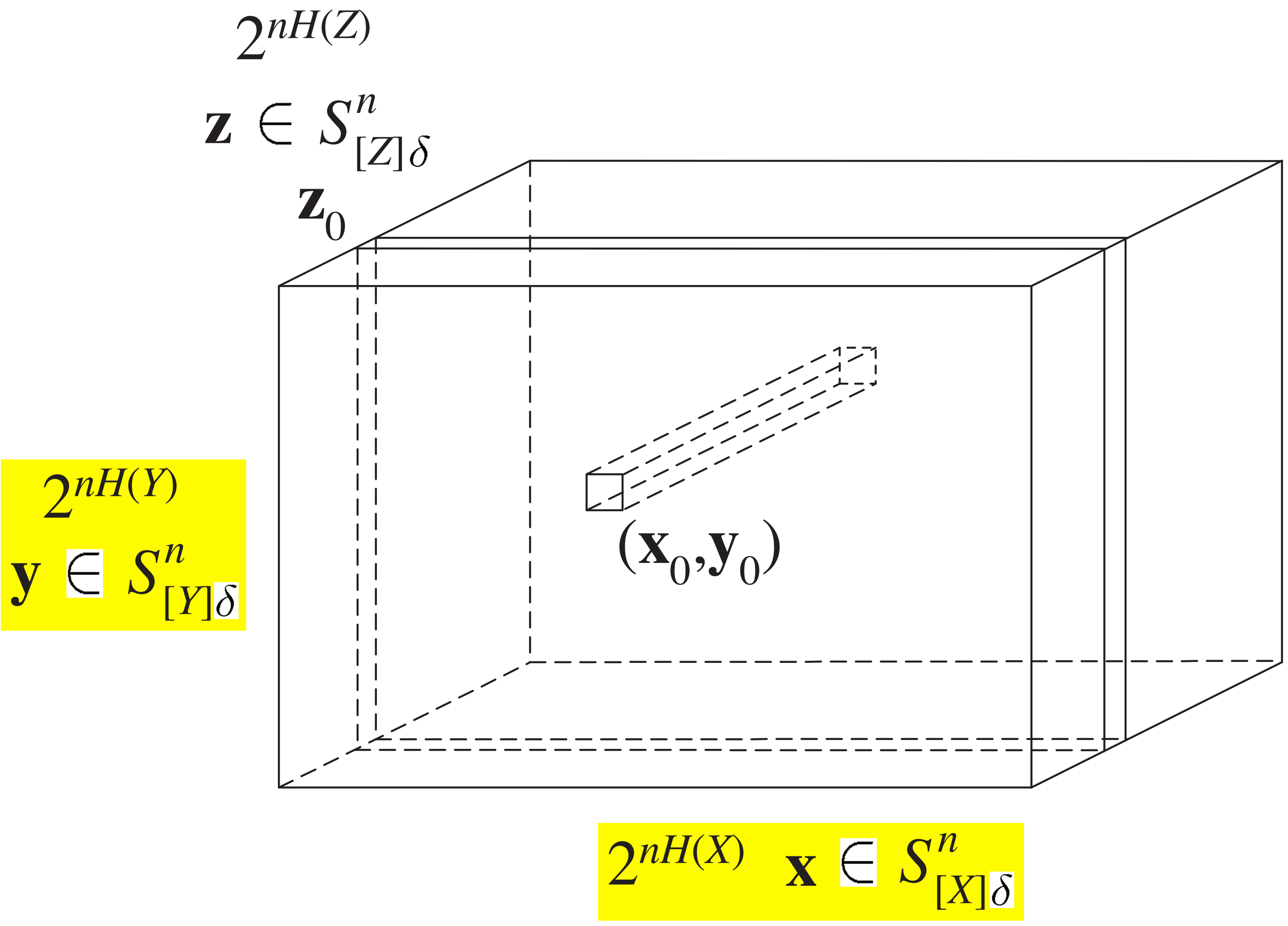

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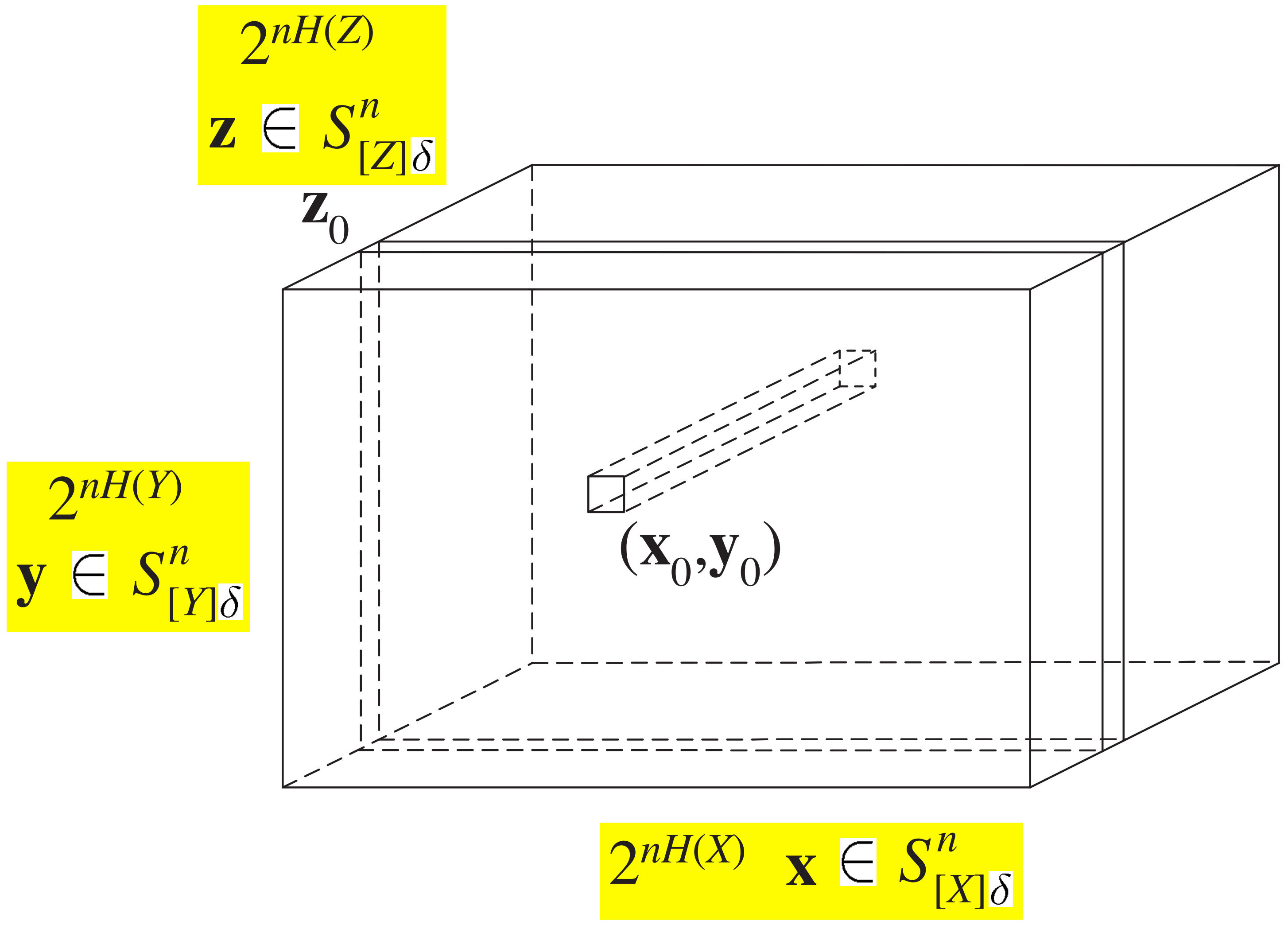

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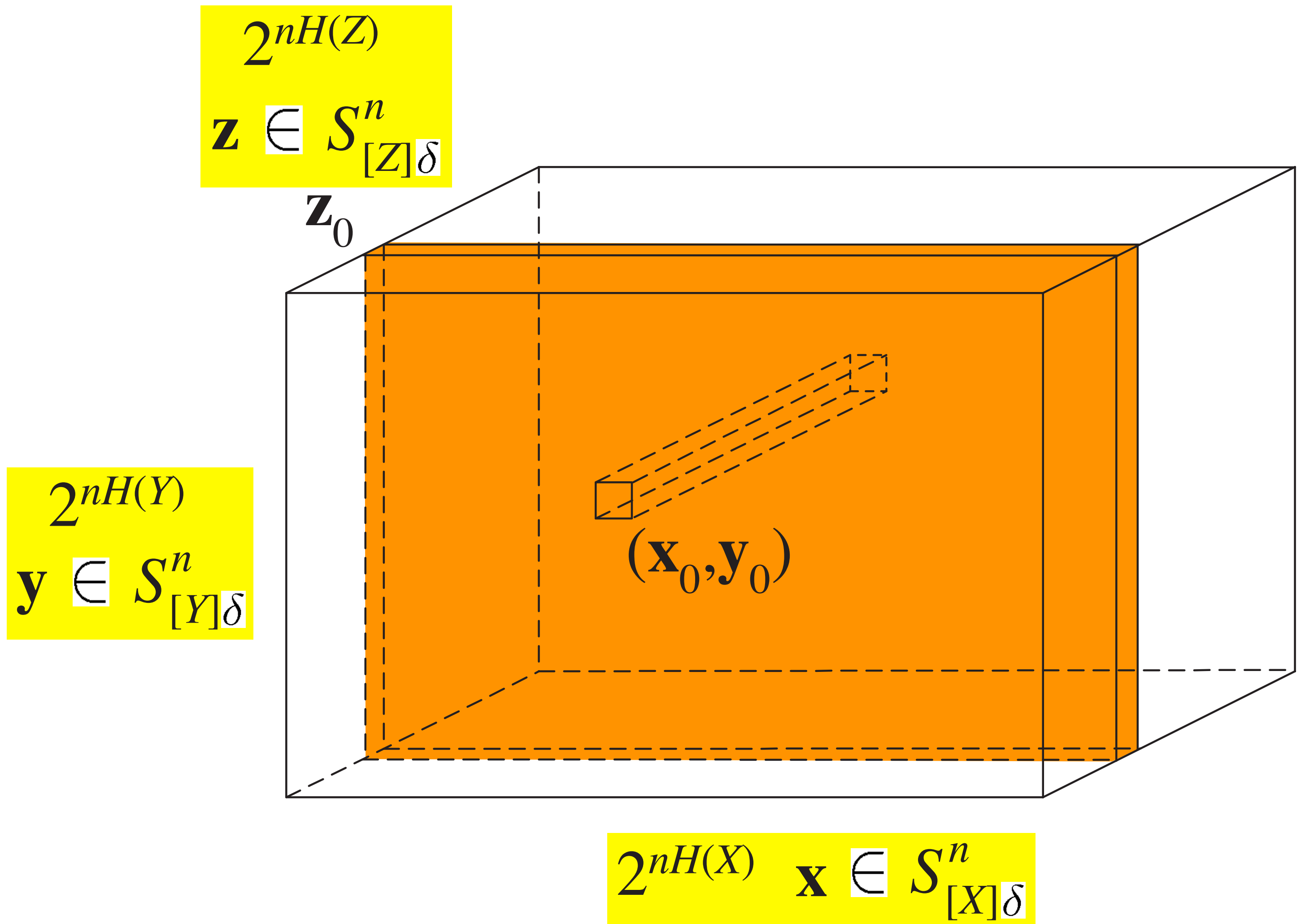

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