香港中文大學
The Chinese University of Hong Kong

Chapter 6
Strong Typicality

The Chinese University of Hong Kong

香港中文大學
The Chinese University of Hong Kong

6.1 Strong AEP

Setup

- $\{X_k, k \geq 1\}$, X_k i.i.d. $\sim p(x)$.

Setup

- $\{X_k, k \geq 1\}$, X_k i.i.d. $\sim p(x)$.
- X denotes generic r.v. with entropy $H(X) < \infty$.

Setup

- $\{X_k, k \geq 1\}$, X_k i.i.d. $\sim p(x)$.
- X denotes generic r.v. with entropy $H(X) < \infty$.
- $\mathbf{X} = (X_1, X_2, \cdots, X_n)$. Then

$$p(\mathbf{X}) = p(X_1)p(X_2)\cdots p(X_n).$$

Setup

- $\{X_k, k \geq 1\}$, X_k i.i.d. $\sim p(x)$.
- X denotes generic r.v. with entropy $H(X) < \infty$.
- $\mathbf{X} = (X_1, X_2, \cdots, X_n)$. Then

$$p(\mathbf{X}) = p(X_1)p(X_2)\cdots p(X_n).$$

- Assume $|\mathcal{X}| < \infty$.

Setup

- $\{X_k, k \geq 1\}$, X_k i.i.d. $\sim p(x)$.
- X denotes generic r.v. with entropy $H(X) < \infty$.
- $\mathbf{X} = (X_1, X_2, \cdots, X_n)$. Then

$$p(\mathbf{X}) = p(X_1)p(X_2)\cdots p(X_n).$$

- Assume $|\mathcal{X}| < \infty$.
- Let the base of the logarithm be 2, i.e., $H(X)$ is in bits.

Notation

- Consider $\mathbf{x} \in \mathcal{X}^n$.

Notation

- Consider $\mathbf{x} \in \mathcal{X}^n$.
- Let $N(x; \mathbf{x})$ be the number of occurrences of x in the sequence $\mathbf{x}$.

Notation

- Consider $\mathbf{x} \in \mathcal{X}^n$.
- Let $N(x;\mathbf{x})$ be the number of occurrences of x in the sequence $\mathbf{x}$.
- $n^{-1}N(x;\mathbf{x})$ is the relative frequency of x in $\mathbf{x}$.

Notation

- Consider $\mathbf{x} \in \mathcal{X}^n$.
- Let $N(x; \mathbf{x})$ be the number of occurrences of x in the sequence $\mathbf{x}$.
- $n^{-1}N(x; \mathbf{x})$ is the relative frequency of x in $\mathbf{x}$.
- $\{n^{-1}N(x; \mathbf{x}) : x \in \mathcal{X}\}$ is the empirical distribution of $\mathbf{x}$.

Notation

- Consider $\mathbf{x} \in \mathcal{X}^n$.
- Let $N(x; \mathbf{x})$ be the number of occurrences of x in the sequence $\mathbf{x}$.
- $n^{-1}N(x; \mathbf{x})$ is the relative frequency of x in $\mathbf{x}$.
- $\{n^{-1}N(x; \mathbf{x}) : x \in \mathcal{X}\}$ is the empirical distribution of $\mathbf{x}$.

Example Let $\mathbf{x} = (1, 3, 2, 1, 1)$.

- $N(1; \mathbf{x}) = 3,\ N(2; \mathbf{x}) = N(3; \mathbf{x}) = 1$
- The empirical distribution of $\mathbf{x}$ is $\left\{\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right\}$.

Definition 6.1 The strongly typical set $T^n_{[X]\delta}$ with respect to $p(x)$ is the set of sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ such that

$$N(x;\mathbf{x}) = 0 \quad \text{for } x \notin \mathcal{S}_X \tag{1}$$

and

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta,$$

where δ is an arbitrarily small positive real number. The sequences in $T^n_{[X]\delta}$ are called strongly δ-typical sequences.

Remarks

Definition 6.1 The strongly typical set $T^n_{[X]\delta}$ with respect to $p(x)$ is the set of sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ such that

$$N(x; \mathbf{x}) = 0 \quad \text{for } x \notin \mathcal{S}_X \tag{1}$$

and

$$\sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| \leq \delta,$$

where δ is an arbitrarily small positive real number. The sequences in $T^n_{[X]\delta}$ are called strongly δ-typical sequences.

Remarks

- If $\sum_x \left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ is small, then so is $\left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ for every $x \in \mathcal{X}$.

Definition 6.1 The strongly typical set $T^n_{[X]\delta}$ with respect to $p(x)$ is the set of sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ such that

$$N(x; \mathbf{x}) = 0 \quad \text{for } x \notin \mathcal{S}_X \tag{1}$$

and

$$\sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| \leq \delta,$$

where δ is an arbitrarily small positive real number. The sequences in $T^n_{[X]\delta}$ are called strongly δ-typical sequences.

Remarks

- If $\sum_x \left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ is small, then so is $\left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ for every $x \in \mathcal{X}$.
- In other words, $n^{-1} N(x; \mathbf{x}) \approx p(x)$ for all $x \in \mathcal{X}$.

Definition 6.1 The strongly typical set $T^n_{[X]\delta}$ with respect to $p(x)$ is the set of sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ such that

$$N(x; \mathbf{x}) = 0 \quad \text{for } x \notin \mathcal{S}_X \tag{1}$$

and

$$\sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| \leq \delta,$$

where δ is an arbitrarily small positive real number. The sequences in $T^n_{[X]\delta}$ are called strongly δ-typical sequences.

Remarks

- If $\sum_x \left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ is small, then so is $\left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ for every $x \in \mathcal{X}$.
- In other words, $n^{-1} N(x; \mathbf{x}) \approx p(x)$ for all $x \in \mathcal{X}$.
- Therefore, if $\mathbf{x}$ is strongly typical, the empirical distribution of $\mathbf{x}$ is approximately equal to the generic distribution $p(x)$.

Definition 6.1 The strongly typical set $T^n_{[X]\delta}$ with respect to $p(x)$ is the set of sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathcal{X}^n$ such that

$$N(x; \mathbf{x}) = 0 \quad \text{for } x \notin \mathcal{S}_X \tag{1}$$

and

$$\sum_x \left| \frac{1}{n} N(x; \mathbf{x}) - p(x) \right| \leq \delta,$$

where δ is an arbitrarily small positive real number. The sequences in $T^n_{[X]\delta}$ are called strongly δ-typical sequences.

Remarks

- If $\sum_x \left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ is small, then so is $\left| n^{-1} N(x; \mathbf{x}) - p(x) \right|$ for every $x \in \mathcal{X}$.
- In other words, $n^{-1} N(x; \mathbf{x}) \approx p(x)$ for all $x \in \mathcal{X}$.
- Therefore, if $\mathbf{x}$ is strongly typical, the empirical distribution of $\mathbf{x}$ is approximately equal to the generic distribution $p(x)$.
- If $\mathbf{x}$ is strongly typical, then $p(x_k) > 0$ for all k because of (1).

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

3) For n sufficiently large,

$$(1-\delta)2^{n(H(X)-\eta)} \le |T^n_{[X]\delta}| \le 2^{n(H(X)+\eta)}.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof Idea

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof Idea

- If $\mathbf{x}$ is strongly typical, then the empirical distribution is "about right".

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof Idea

- If $\mathbf{x}$ is strongly typical, then the empirical distribution is "about right".
- If the empirical distribution is about right, then everything else, including the empirical entropy, would be about right, i.e.,

$$-\frac{1}{n} \log p(\mathbf{x}) \approx H(X).$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof Idea

- If $\mathbf{x}$ is strongly typical, then the empirical distribution is "about right".
- If the empirical distribution is about right, then everything else, including the empirical entropy, would be about right, i.e.,

$$-\frac{1}{n}\log p(\mathbf{x}) \approx H(X).$$

- This is equivalent to $p(\mathbf{x}) \approx 2^{-nH(X)}$.

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$\log p(\mathbf{x})$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$\log p(\mathbf{x})$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned} &\log p(\mathbf{x}) \\ &\quad = \sum_x N(x;\mathbf{x}) \log p(x) \end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

log

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\log p(\mathbf{x})$$
$$= \sum_x N(x;\mathbf{x}) \log p(x)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

log

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\log p(\mathbf{x})$$
$$= \sum_x N(x;\mathbf{x}) \log p(x)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) \underline{- np(x)} + np(x)) \log p(x)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) \underline{-\, np(x)}\ \underline{+\, np(x)}) \log p(x)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned} &\log p(\mathbf{x}) \\ &\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\ &\quad = \sum_x (N(x;\mathbf{x}) - np(x) \underline{+ np(x)}) \log p(x) \end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned} &\log p(\mathbf{x}) \\ &= \sum_x N(x;\mathbf{x}) \log p(x) \\ &= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) \underline{+ np(x)}) \underline{\log p(x)} \\
&= \underline{n \sum_x p(x) \log p(x)}
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) & \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (\underline{N(x;\mathbf{x}) - np(x)} + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x \left(\underline{N(x;\mathbf{x}) - np(x)} + np(x)\right) \underline{\log p(x)} \\
&\quad = n \sum_x p(x) \log p(x)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) & \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (\underline{N(x;\mathbf{x}) - np(x)} + np(x)) \underline{\log p(x)} \\
&= n \sum_x p(x) \log p(x) \\
&\quad \underline{-n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x))}
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&\quad = n \sum_x p(x) \log p(x) \\
&\qquad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x))
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x))
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x\in\mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (\underline{N(x;\mathbf{x}) - np(x)} + np(x)) \underline{\log p(x)} \\
&= n\sum_x p(x)\log p(x) \\
&\quad -n\sum_x \left(\frac{1}{n}N(x;\mathbf{x}) - p(x)\right)(\underline{-}\log p(x))
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (\underline{N(x;\mathbf{x}) - np(x)} + np(x)) \underline{\log p(x)} \\
&\quad = n \sum_x p(x) \log p(x) \\
&\qquad \underline{-} n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (\underline{-} \log p(x))
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&\quad = n \sum_x p(x) \log p(x) \\
&\qquad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \underline{\sum_x p(x) \log p(x)} \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \\
&= -n \left[\underline{H(X)} + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right].
\end{aligned} \tag{1}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \underline{\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))} \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right].
\end{aligned} \tag{1}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \underline{\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x))} \\
&= -n \left[H(X) + \underline{\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x))} \right].
\end{aligned} \quad (1)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n\left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&\quad = n \sum_x p(x) \log p(x) \\
&\qquad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&\quad = -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&\quad = \sum_x N(x;\mathbf{x}) \log p(x) \\
&\quad = \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&\quad = n \sum_x p(x) \log p(x) \\
&\qquad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&\quad = -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n}N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n\left[H(X) + \sum_x \left(\frac{1}{n}N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| \leq \delta.$$

Now consider

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right|$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n\left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right| \\
&\leq \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right|(-\log p(x))
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \right| \\
&\leq \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \underline{(-\log p(x))}
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x))\right| \\
&\le \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \underline{(-\log p(x))} \\
&\le \underline{-\log \left(\min_x p(x)\right)} \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right|
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n}N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n}N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n}N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right| \\
&\leq \sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right|(-\log p(x)) \\
&\leq -\log\left(\min_x p(x)\right) \sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right|
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right| \\
&\le \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right|(-\log p(x)) \\
&\le -\log\left(\min_x p(x)\right) \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \\
&\le -\delta \log\left(\min_x p(x)\right)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right| \\
&\leq \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
&\leq -\log \left(\min_x p(x) \right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
&\leq -\delta \log \left(\min_x p(x) \right)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right| \\
&\leq \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right|(-\log p(x)) \\
&\leq -\log\left(\min_x p(x)\right) \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \\
&\leq -\delta \log\left(\min_x p(x)\right)
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \right].
\end{aligned} \quad (1)$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \right| \\
&\leq \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
&\leq -\log \left(\min_x p(x)\right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
&\leq -\delta \log \left(\min_x p(x)\right) \\
&= \eta,
\end{aligned}$$

where

$$\eta = -\delta \log \left(\min_x p(x)\right) > 0.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right| \\
&\le \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right|(-\log p(x)) \\
&\le -\log\left(\min_x p(x)\right) \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \\
&\le -\delta \log\left(\min_x p(x)\right) \\
&= \eta,
\end{aligned}$$

where

$$\eta = -\delta \log\left(\min_x p(x)\right) > 0.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n\left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left|\sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x))\right| \\
&\le \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right|(-\log p(x)) \\
&\le -\log\left(\min_x p(x)\right) \sum_x \left|\frac{1}{n} N(x;\mathbf{x}) - p(x)\right| \\
&\le -\delta \log\left(\min_x p(x)\right) \\
&= \eta,
\end{aligned}$$

where

$$\eta = -\delta \log\left(\min_x p(x)\right) > 0.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right| \\
\le &\quad \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
\le &\quad -\log \left(\min_x p(x) \right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
\le &\quad -\delta \log \left(\min_x p(x) \right) \\
= &\quad \eta,
\end{aligned}$$

where

$$\eta = -\delta \log \left(\min_x p(x) \right) > 0.$$

3. Therefore,

$$-\eta \le \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \le \eta.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \right| \\
\le\ & \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
\le\ & -\log \left(\min_x p(x)\right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
\le\ & -\delta \log \left(\min_x p(x)\right) \\
=\ & \eta,
\end{aligned}$$

where

$$\eta = -\delta \log \left(\min_x p(x)\right) > 0.$$

3. Therefore,

$$-\eta \le \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right) (-\log p(x)) \le \eta.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right]. \qquad (1)
\end{aligned}$$

$-\eta \le \quad \le \eta$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right| \\
&\le \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
&\le -\log\left(\min_x p(x)\right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
&\le -\delta \log\left(\min_x p(x)\right) \\
&= \eta,
\end{aligned}$$

where

$$\eta = -\delta \log\left(\min_x p(x)\right) > 0.$$

3. Therefore,

$$-\eta \le \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \le \eta.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) &= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right]. \qquad (1)
\end{aligned}$$

$-\eta \le$ $\le \eta$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right| \\
&\le \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
&\le -\log\left(\min_x p(x)\right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
&\le -\delta \log\left(\min_x p(x)\right) \\
&= \eta,
\end{aligned}$$

where

$$\eta = -\delta \log\left(\min_x p(x)\right) > 0.$$

3. Therefore,

$$-\eta \le \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \le \eta.$$

4. It then follows from (1) that

$$-n(H(X)+\eta) \le \log p(\mathbf{x}) \le -n(H(X)-\eta),$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad -n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right]. \qquad (1)
\end{aligned}$$

$-\eta \le$ $\le \eta$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right| \\
\le &\quad \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
\le &\quad -\log\left(\min_x p(x)\right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
\le &\quad -\delta \log\left(\min_x p(x)\right) \\
= &\quad \eta,
\end{aligned}$$

where

$$\eta = -\delta \log\left(\min_x p(x)\right) > 0.$$

3. Therefore,

$$-\eta \le \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \le \eta.$$

4. It then follows from (1) that

$$-n(H(X)+\eta) \le \log p(\mathbf{x}) \le -n(H(X)-\eta),$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
&\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \leq \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \right| \\
\leq\ & \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
\leq\ & -\log\left(\min_x p(x)\right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
\leq\ & -\delta \log\left(\min_x p(x)\right) \\
=\ & \eta,
\end{aligned}$$

where

$$\eta = -\delta \log\left(\min_x p(x)\right) > 0.$$

3. Therefore,

$$-\eta \leq \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x)\right)(-\log p(x)) \leq \eta.$$

4. It then follows from (1) that

$$-n(H(X) + \eta) \leq \log p(\mathbf{x}) \leq -n(H(X) - \eta),$$

or

$$2^{-n(H(X)+\eta)} \leq p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)},$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

1) If $\mathbf{x} \in T^n_{[X]\delta}$, then

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)}.$$

Proof

1. To prove Property 1, for $\mathbf{x} \in T^n_{[X]\delta}$, we have

$$p(\mathbf{x}) = p(x_1)p(x_2)\cdots p(x_n) = \prod_{x \in \mathcal{S}_X} p(x)^{N(x;\mathbf{x})} > 0$$

because $N(x;\mathbf{x}) = 0$ for all $x \notin \mathcal{S}_X$. Then

$$\begin{aligned}
\log p(\mathbf{x}) \\
&= \sum_x N(x;\mathbf{x}) \log p(x) \\
&= \sum_x (N(x;\mathbf{x}) - np(x) + np(x)) \log p(x) \\
&= n \sum_x p(x) \log p(x) \\
&\quad - n \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \\
&= -n \left[H(X) + \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right].
\end{aligned} \tag{1}$$

2. Since $\mathbf{x} \in T^n_{[X]\delta}$,

$$\sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \le \delta.$$

Now consider

$$\begin{aligned}
&\left| \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \right| \\
&\le \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| (-\log p(x)) \\
&\le -\log \left(\min_x p(x) \right) \sum_x \left| \frac{1}{n} N(x;\mathbf{x}) - p(x) \right| \\
&\le -\delta \log \left(\min_x p(x) \right) \\
&= \eta,
\end{aligned}$$

where

$$\eta = -\delta \log \left(\min_x p(x) \right) > 0.$$

3. Therefore,

$$-\eta \le \sum_x \left(\frac{1}{n} N(x;\mathbf{x}) - p(x) \right) (-\log p(x)) \le \eta.$$

4. It then follows from (1) that

$$-n(H(X)+\eta) \le \log p(\mathbf{x}) \le -n(H(X)-\eta),$$

or

$$2^{-n(H(X)+\eta)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\eta)},$$

where $\eta \to 0$ as $\delta \to 0$, proving Property 1.

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof Idea

- By WLLN, w.p. $\to 1$ (with probability tends to 1), the empirical distribution of $\mathbf{X}$ is close to $p(x)$, and so by definition $\mathbf{X}$ is strongly typical.

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \rightarrow 0$ as $\delta \rightarrow 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{ \left| \frac{1}{n} \sum_{k=1}^{n} B_k(x) - p(x) \right| > \frac{\delta}{|\mathcal{X}|} \right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\Pr\left\{\left|\frac{1}{n}\underline{N(x; \mathbf{X})} - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x; \mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x; \mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$
$$= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned} &\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\ &= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned} &\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\ &= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \underline{\text{ for some } x}\right\} \\ &= \Pr\left\{\underline{\bigcup_x}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned} &\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\ &= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\ &= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \end{aligned}$$

Union Bound

$$\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x; \mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n} N(x; \mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}
\end{aligned}$$

Union Bound

$$\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned} &\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\ &= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\ &= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\ &\le \underline{\sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}} \end{aligned}$$

Union Bound

$$\Pr(A \cup B) \le \Pr(A) + \Pr(B)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}
\end{aligned}$$

Union Bound

$$\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|}
\end{aligned}$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in T^n_{[X]\delta}\} > 1 - \delta.$$

Proof

1. To prove Property 2, we write

$$N(x;\mathbf{X}) = \sum_{k=1}^{n} B_k(x),$$

where

$$B_k(x) = \begin{cases} 1 & \text{if } X_k = x \\ 0 & \text{if } X_k \neq x. \end{cases}$$

2. Then $B_k(x), k = 1, 2, \cdots, n$ are i.i.d. random variables with

$$\Pr\{B_k(x) = 1\} = p(x)$$

and

$$\Pr\{B_k(x) = 0\} = 1 - p(x).$$

Note that

$$EB_k(x) = (1 - p(x)) \cdot 0 + p(x) \cdot 1 = p(x).$$

3. By WLLN, for any $\delta > 0$ and for any $x \in \mathcal{X}$,

$$\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} < \frac{\delta}{|\mathcal{X}|}$$

for n sufficiently large.

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

4. Then

$$
\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\le \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}
$$

4. Then

$$
\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_{x}\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_{x}\Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_{x}\frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}
$$

4. Then

$$
\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}
$$

5. Now

$$
\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta
$$

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

4. Then

$$
\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\right\}\\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\right\}\\
&= \Pr\left\{\bigcup_x\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\right\}\\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\\
&< \sum_x \frac{\delta}{|\mathcal{X}|}\\
&= \delta.
\end{aligned}
$$

5. Now

$$\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\in\mathcal{X}.$$

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

$$\begin{aligned}
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\le \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

Then we have

$$\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\}$$

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

$$\begin{aligned}
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\le \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

Then we have

$$\begin{aligned}
&\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\} \\
&\le \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}.
\end{aligned}$$

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$
$$= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$
$$= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\}$$
$$\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}$$
$$< \sum_x \frac{\delta}{|\mathcal{X}|}$$
$$= \delta.$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

Then we have

$$\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\}$$
$$\leq \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}.$$

Then

$$\Pr\left\{\mathbf{X} \in T^n_{[X]\delta}\right\}$$

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\right\}\\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\right\}\\
&= \Pr\left\{\bigcup_x\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\right\}\\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\\
&< \sum_x \frac{\delta}{|\mathcal{X}|}\\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\in\mathcal{X}.$$

Then we have

$$\begin{aligned}
&\Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta\right\}\\
&\leq \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\in\mathcal{X}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Pr\left\{\mathbf{X}\in T^n_{[X]\delta}\right\}\\
&= \Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|\leq\delta\right\}
\end{aligned}$$

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\le \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

Then we have

$$\begin{aligned}
&\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\} \\
&\le \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Pr\left\{\mathbf{X} \in T^n_{[X]\delta}\right\} \\
&= \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| \le \delta\right\} \\
&= 1 - \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \delta\right\}
\end{aligned}$$

4. Then

$$
\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\le \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}
$$

5. Now

$$
\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta
$$

implies

$$
\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.
$$

Then we have

$$
\begin{aligned}
&\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\} \\
&\le \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Pr\left\{\mathbf{X} \in T^n_{[X]\delta}\right\} \\
&= \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| \le \delta\right\} \\
&= 1 - \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \delta\right\}
\end{aligned}
$$

4. Then

$$
\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}
$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

Then we have

$$
\begin{aligned}
&\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\} \\
&\leq \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Pr\left\{\mathbf{X} \in T^n_{[X]\delta}\right\} \\
&= \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| \leq \delta\right\} \\
&= 1 - \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \delta\right\} \\
&\geq 1 - \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}
\end{aligned}
$$

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\}$$

$$\begin{aligned}
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x\right\} \\
&= \Pr\left\{\bigcup_x \left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\}\right\} \\
&\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n} B_k(x) - p(x)\right| > \frac{\delta}{|\mathcal{X}|}\right\} \\
&< \sum_x \frac{\delta}{|\mathcal{X}|} \\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}.$$

Then we have

$$\begin{aligned}
&\Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \delta\right\} \\
&\quad\leq \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Pr\left\{\mathbf{X} \in T^n_{[X]\delta}\right\} \\
&\quad= \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| \leq \delta\right\} \\
&\quad= 1 - \Pr\left\{\sum_x \left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \delta\right\} \\
&\quad\geq 1 - \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X}) - p(x)\right| > \frac{\delta}{|\mathcal{X}|} \text{ for some } x \in \mathcal{X}\right\}
\end{aligned}$$

4. Then

$$\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\right\}$$

$$= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\right\}$$

$$= \Pr\left\{\bigcup_x\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\right\}$$

$$\leq \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}$$

$$< \sum_x \frac{\delta}{|\mathcal{X}|}$$

$$= \delta.$$

5. Now

$$\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\in\mathcal{X}.$$

Then we have

$$\Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta\right\}$$

$$\leq \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\in\mathcal{X}\right\}.$$

Then

$$\Pr\left\{\mathbf{X}\in T^n_{[X]\delta}\right\}$$

$$= \Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|\leq\delta\right\}$$

$$= 1-\Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\delta\right\}$$

$$\geq 1-\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some }x\in\mathcal{X}\right\}$$

$$> 1-\delta,$$

4. Then

$$\begin{aligned}
&\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some } x\right\}\\
&= \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some } x\right\}\\
&= \Pr\left\{\bigcup_x\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\right\}\\
&\le \sum_x \Pr\left\{\left|\frac{1}{n}\sum_{k=1}^{n}B_k(x)-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\right\}\\
&< \sum_x \frac{\delta}{|\mathcal{X}|}\\
&= \delta.
\end{aligned}$$

5. Now

$$\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta$$

implies

$$\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some } x\in\mathcal{X}.$$

Then we have

$$\begin{aligned}
&\Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\delta\right\}\\
&\le \Pr\left\{\left|\frac{1}{n}N(x;\mathbf{x})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some } x\in\mathcal{X}\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Pr\left\{\mathbf{X}\in T^n_{[X]\delta}\right\}\\
&= \Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|\le\delta\right\}\\
&= 1-\Pr\left\{\sum_x\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\delta\right\}\\
&\ge 1-\Pr\left\{\left|\frac{1}{n}N(x;\mathbf{X})-p(x)\right|>\frac{\delta}{|\mathcal{X}|}\text{ for some } x\in\mathcal{X}\right\}\\
&> 1-\delta,
\end{aligned}$$

proving Property 2.

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

3) For n sufficiently large,

$$(1-\delta)2^{n(H(X)-\eta)} \leq |T^n_{[X]\delta}| \leq 2^{n(H(X)+\eta)}.$$

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

3) For n sufficiently large,

$$(1-\delta)2^{n(H(X)-\eta)} \leq |T^n_{[X]\delta}| \leq 2^{n(H(X)+\eta)}.$$

Proof Follows from Property 1 and Property 2 in exactly the same way as in Theorem 5.3. (Exercise)

Theorem 6.2 (Strong AEP) There exists $\eta > 0$ such that $\eta \to 0$ as $\delta \to 0$, and the following hold:

3) For n sufficiently large,

$$(1-\delta)2^{n(H(X)-\eta)} \le |T^n_{[X]\delta}| \le 2^{n(H(X)+\eta)}.$$

Proof Follows from Property 1 and Property 2 in exactly the same way as in Theorem 5.3. (Exercise)

Theorem 5.3 (Weak AEP II)

1) If $\mathbf{x} \in W^n_{[X]\epsilon}$, then

$$2^{-n(H(X)+\epsilon)} \le p(\mathbf{x}) \le 2^{-n(H(X)-\epsilon)}.$$

2) For n sufficiently large,

$$\Pr\{\mathbf{X} \in W^n_{[X]\epsilon}\} > 1 - \epsilon.$$

3) For n sufficiently large,

$$(1-\epsilon)2^{n(H(X)-\epsilon)} \le |W^n_{[X]\epsilon}| \le 2^{n(H(X)+\epsilon)}.$$

Theorem 6.3 For sufficiently large n, there exists $\varphi(\delta) > 0$ such that

$$\Pr\{\mathbf{X} \notin T^n_{[X]\delta}\} < 2^{-n\varphi(\delta)}.$$

Proof Chernoff bound.