香港中文大學
The Chinese University of Hong Kong

Chapter 5
Weak Typicality

The Chinese University of Hong Kong

In this chapter

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- For $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ where X_i are i.i.d. $\sim p(x)$, what would be a "typical" outcome of $\mathbf{X}$?

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- How are typical sequences related to data compression?
- A second look at data compression: Shannon's source coding theorem

5.1 The Weak AEP

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- Two common such measures in information theory: weak typicality and strong typicality.
- The main theorems are weak and strong *Asymptotic Equipartition Properties* (AEP), which are consequences of WLLN.

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- $\mathcal{X}$ may be countably infinite.
- Let the base of the logarithm be 2, i.e., $H(X)$ is in bits.

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For i.i.d. random variables $Y_1, Y_2, \cdots$ with generic random variable Y,

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3. The random variables $\log p(X_k)$ are also i.i.d. Then by WLLN, (1) tends to

$$-E\log p(X) = H(X),$$

in probability, proving the theorem.

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or equivalently,

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where ϵ is an arbitrarily small positive real number. The sequences in $W^n_{[X]\epsilon}$ are called weakly ϵ-typical sequences.

Empirical Entropy

- The empirical entropy of a sequence $\mathbf{x} = (x_1, , x_2, \cdots , x_n)$ is defined as

$$-\frac{1}{n} \log p(\mathbf{x}) = -\frac{1}{n} \log \prod_{k=1}^{n} p(x_k) = -\frac{1}{n} \sum_{k=1}^{n} \log p(x_k) = \frac{1}{n} \sum_{k=1}^{n} \left[-\log p(x_k) \right].$$

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- The empirical entropy of a weakly typical sequence is close to the true entropy $H(X)$.

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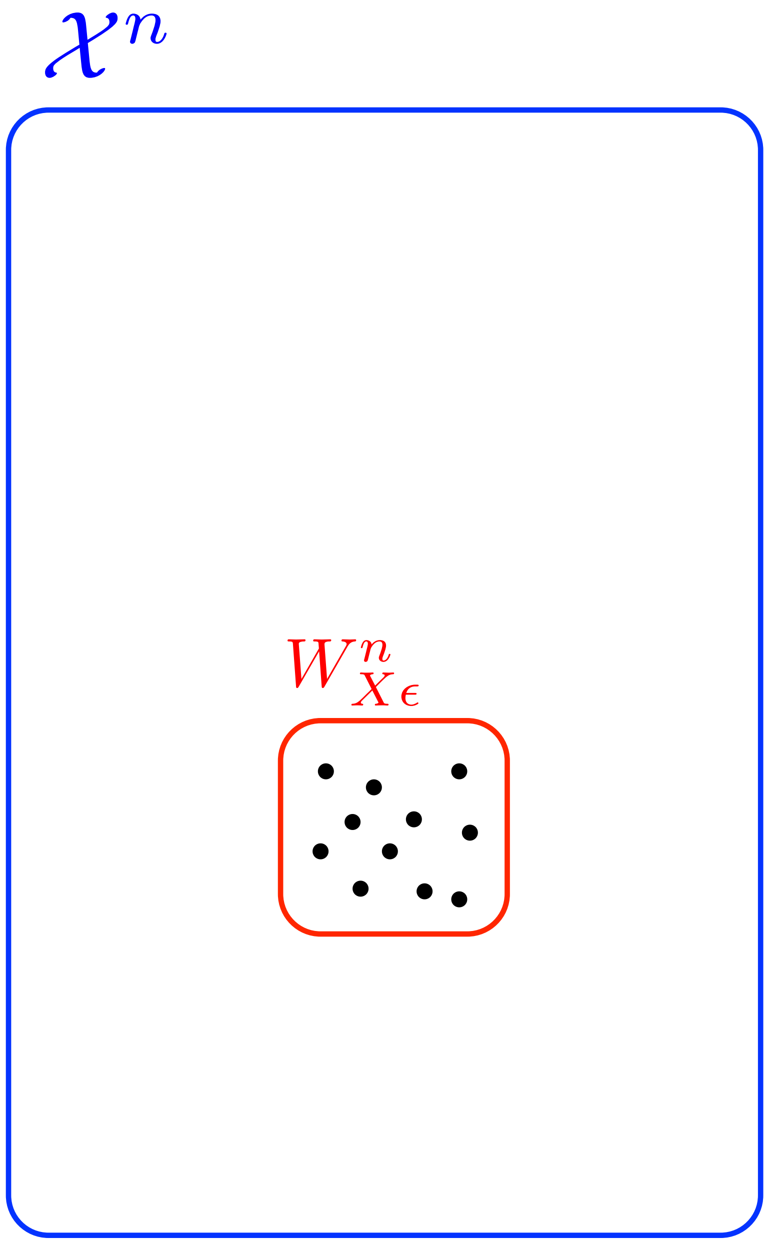
When n is large, one can almost think of the sequence $\mathbf{X}$ as being obtained by choosing a sequence from the weakly typical set according to the uniform distribution.



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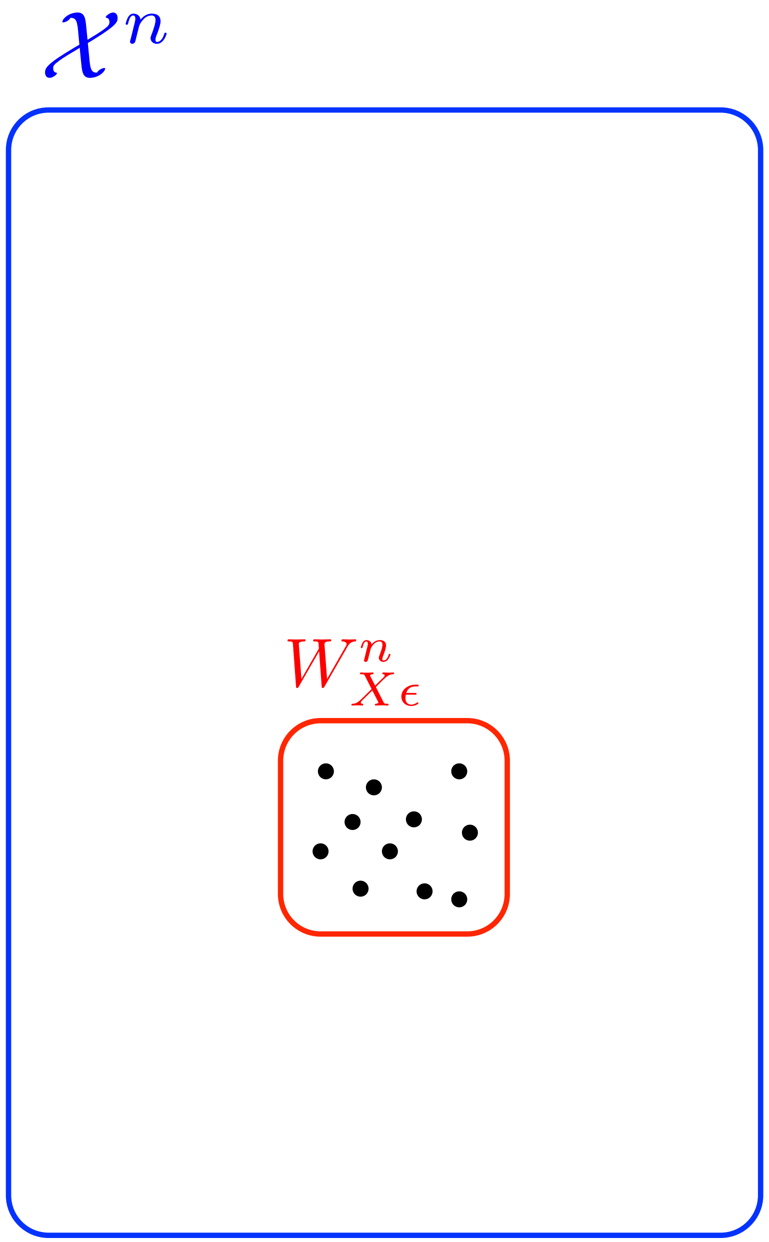

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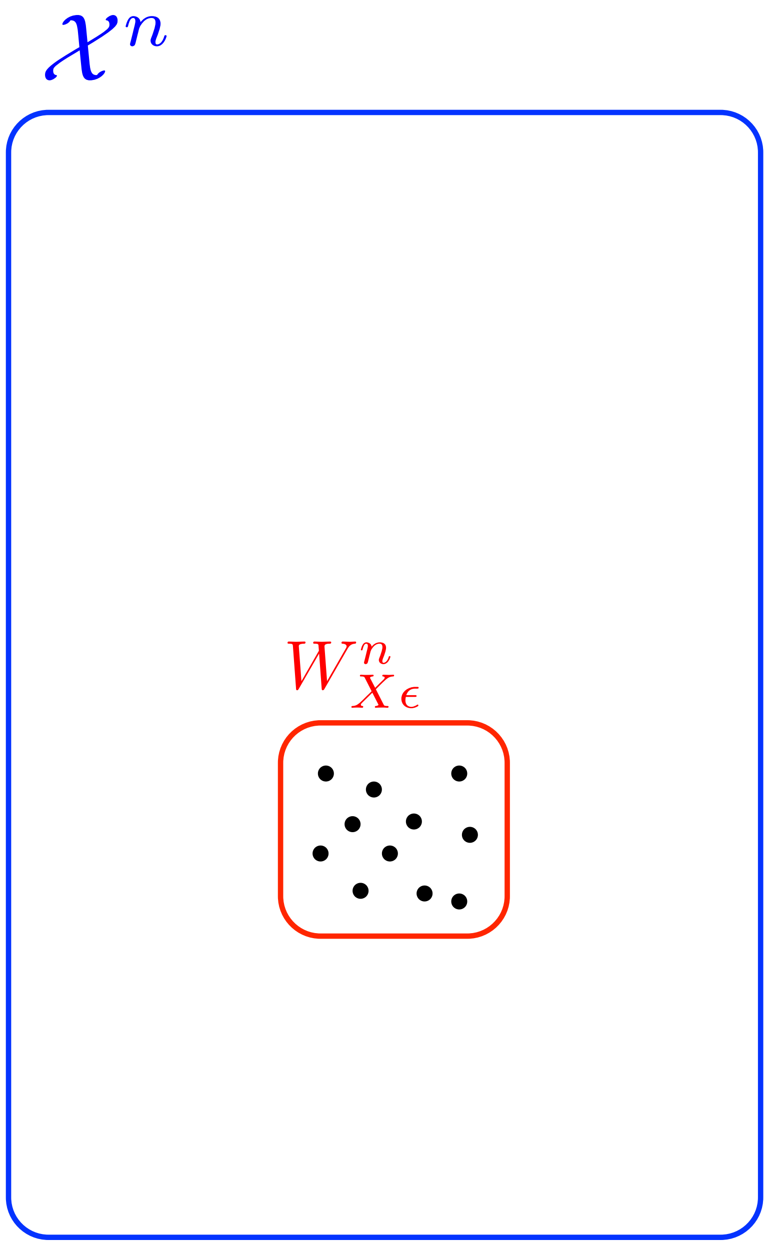
When n is large, one can almost think of the sequence $\mathbf{X}$ as being obtained by choosing a sequence from the weakly typical set according to the uniform distribution.



- $|W^n_{[X]\epsilon}| \approx 2^{nH(X)}$
- If $H(X) < \log|\mathcal{X}|$, then

$$2^{nH(X)} \ll 2^{n\log|\mathcal{X}|} = \left(2^{\log|\mathcal{X}|}\right)^n = |\mathcal{X}|^n = |\mathcal{X}^n|$$

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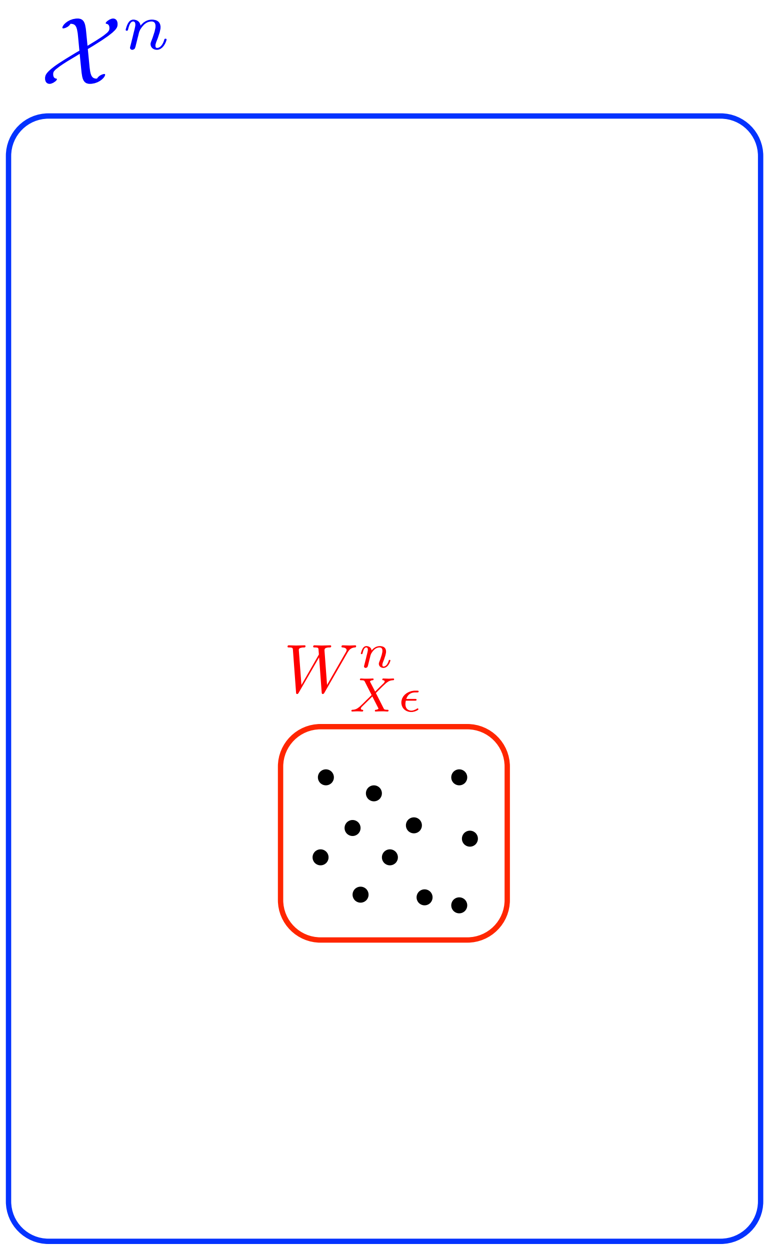

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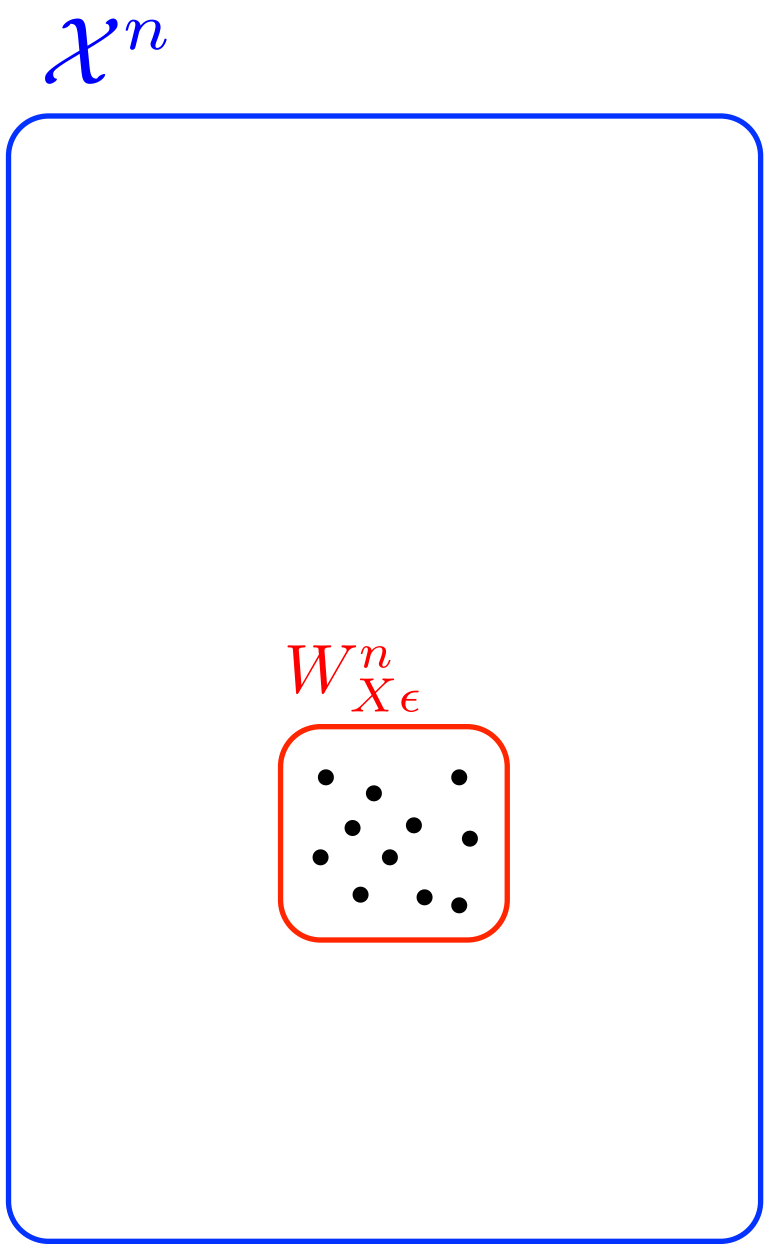

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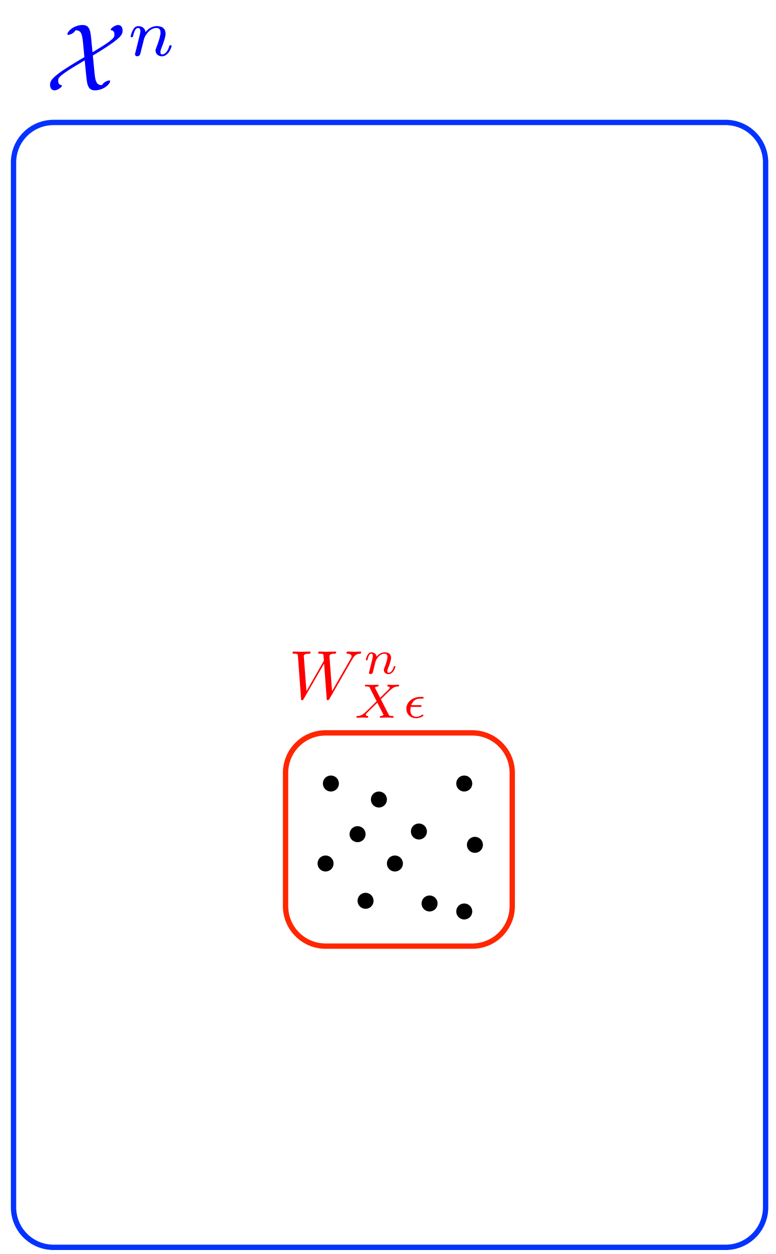

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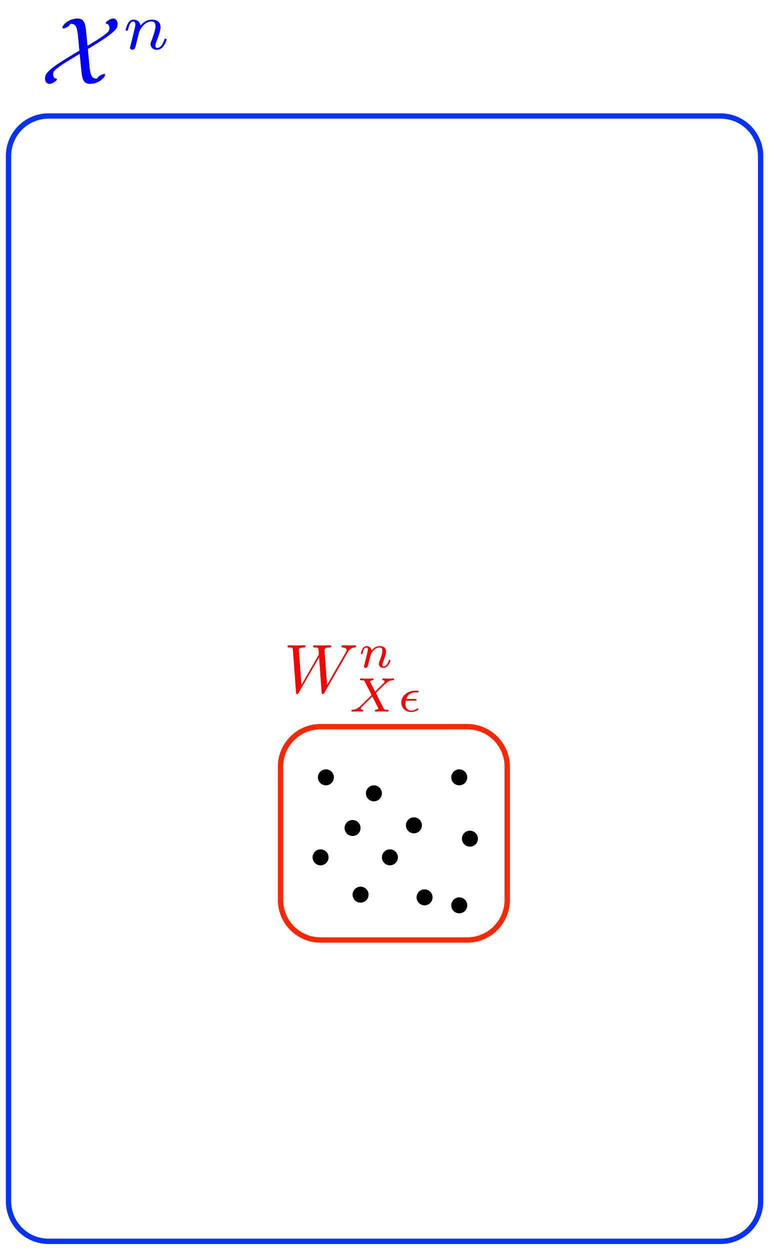

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- Therefore, $|W^n_{[X]\epsilon}| \ll |\mathcal{X}^n|$.
- $\Pr\{W^n_{[X]\epsilon}\} \approx 1$
- $p(\mathbf{x}) \approx 2^{-nH(X)}$ for $\mathbf{x} \in W^n_{[X]\epsilon}$
- The conditional distribution on $W^n_{[X]\epsilon}$ is almost uniform.