4.3 Redundancy of Prefix Codes

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- We will present an alternative proof specifically for prefix codes which offers much insight into the redundancy of such codes.

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- Let $\mathcal{I}$ be the index set of all the internal nodes (including the root) in the code tree.

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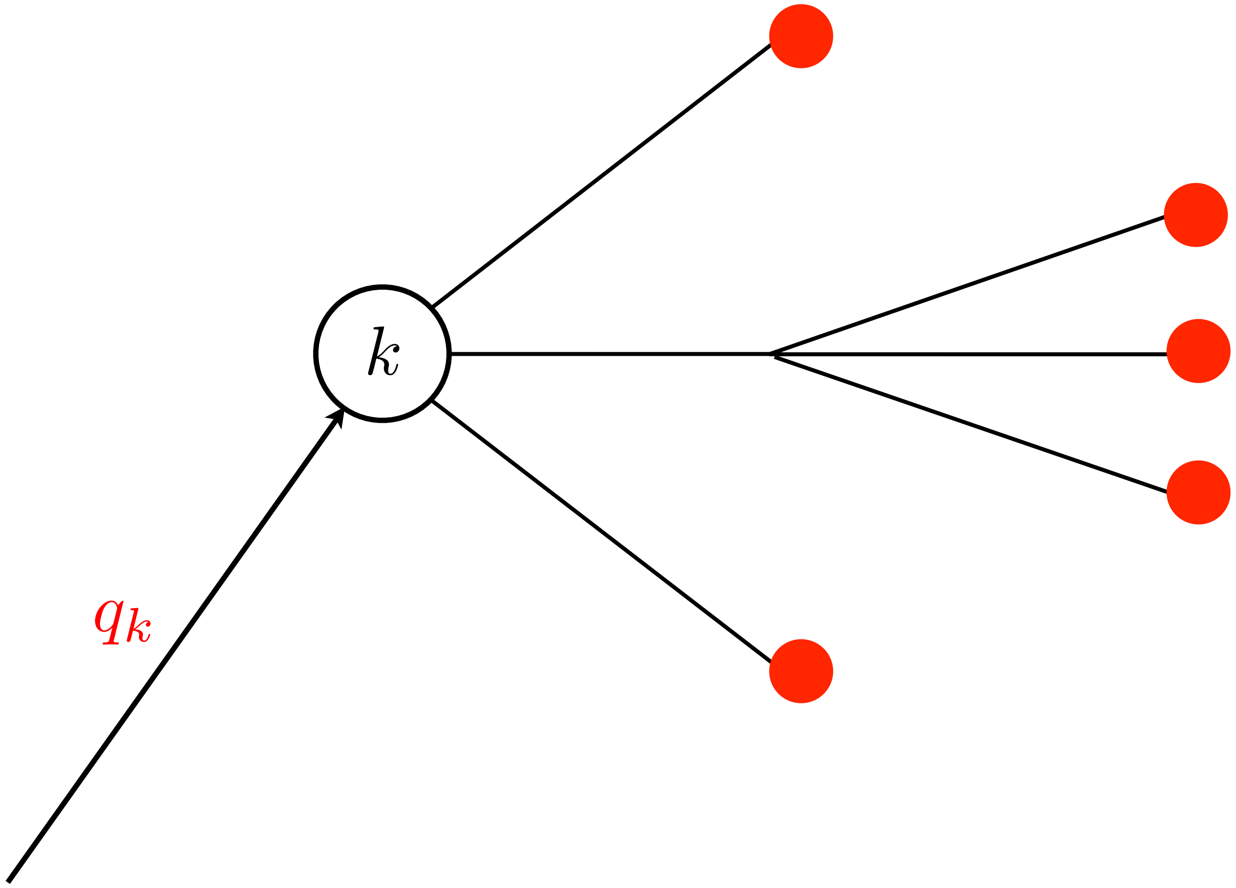

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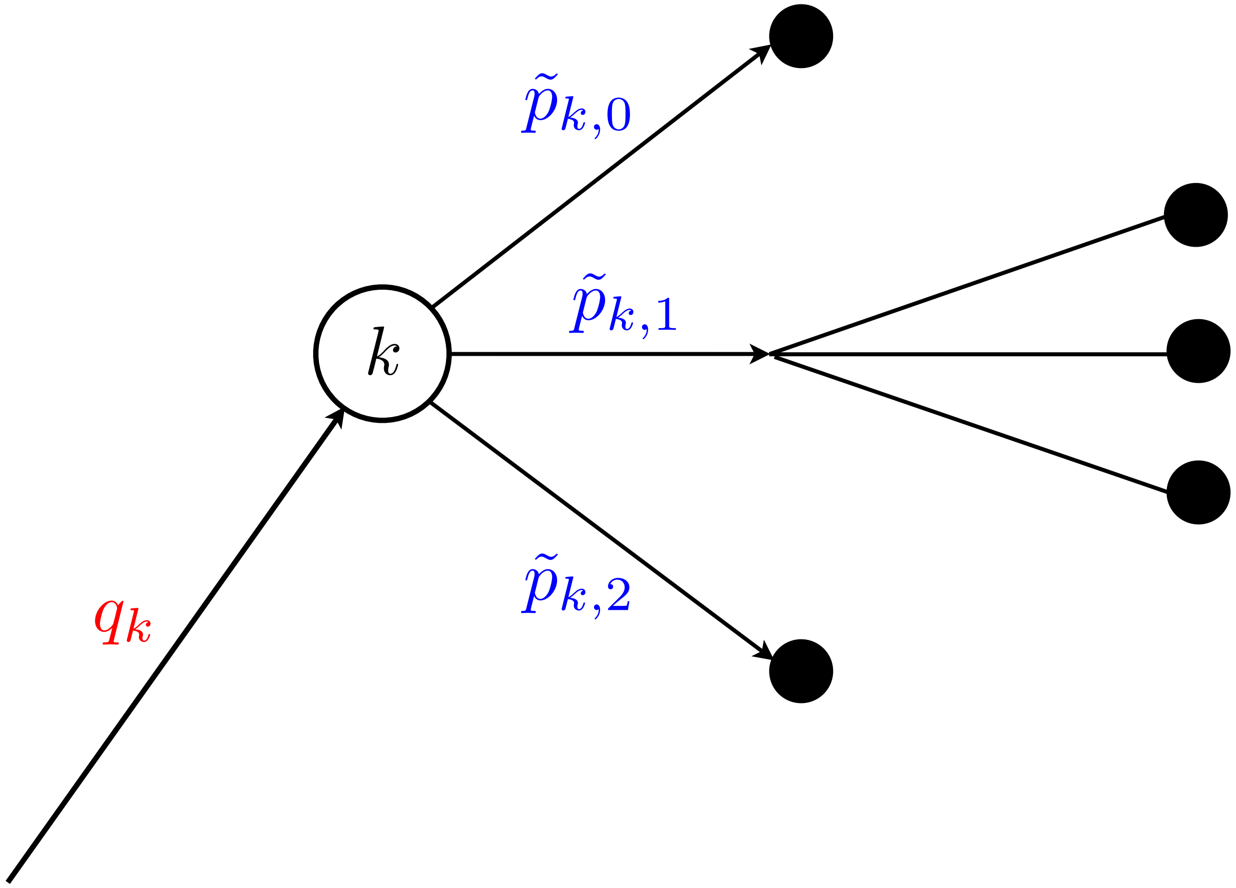
- Then define the conditional entropy of node k by

$$h_k = H_D\left(\left\{ \frac{\tilde{p}_{k,0}}{q_k}, \frac{\tilde{p}_{k,1}}{q_k}, \cdots, \frac{\tilde{p}_{k,D-1}}{q_k} \right\}\right) \le \log_D D = 1.$$

Conditional Branching Distribution



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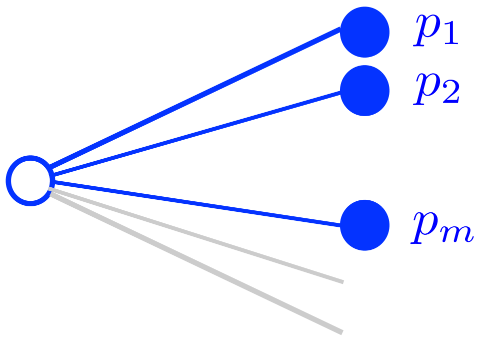

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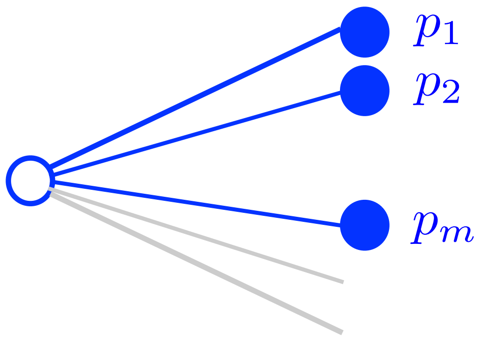

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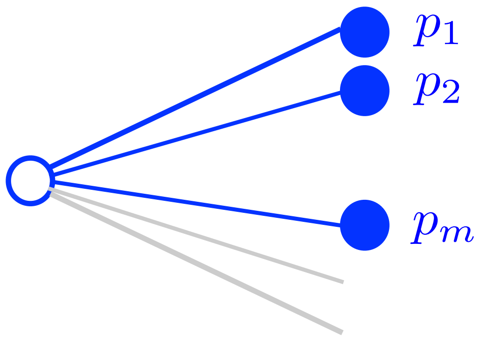

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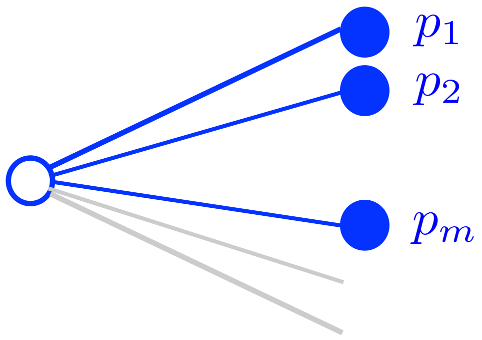

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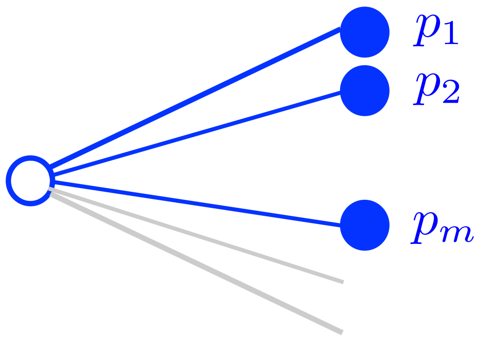

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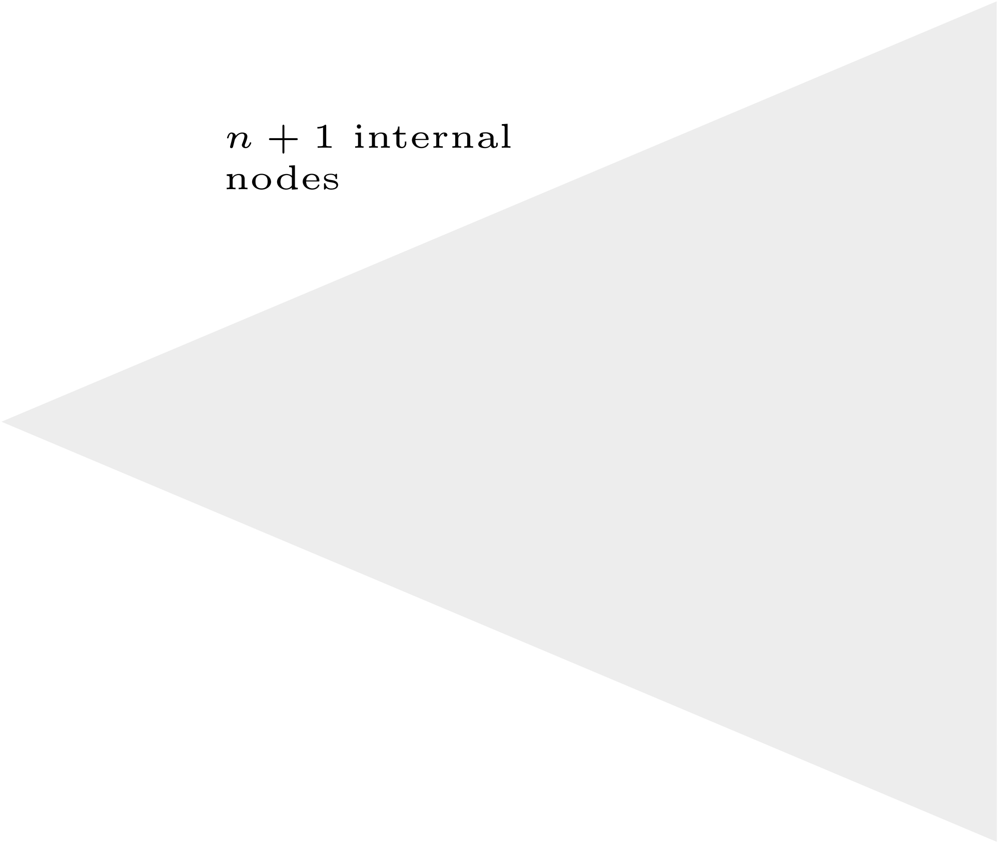

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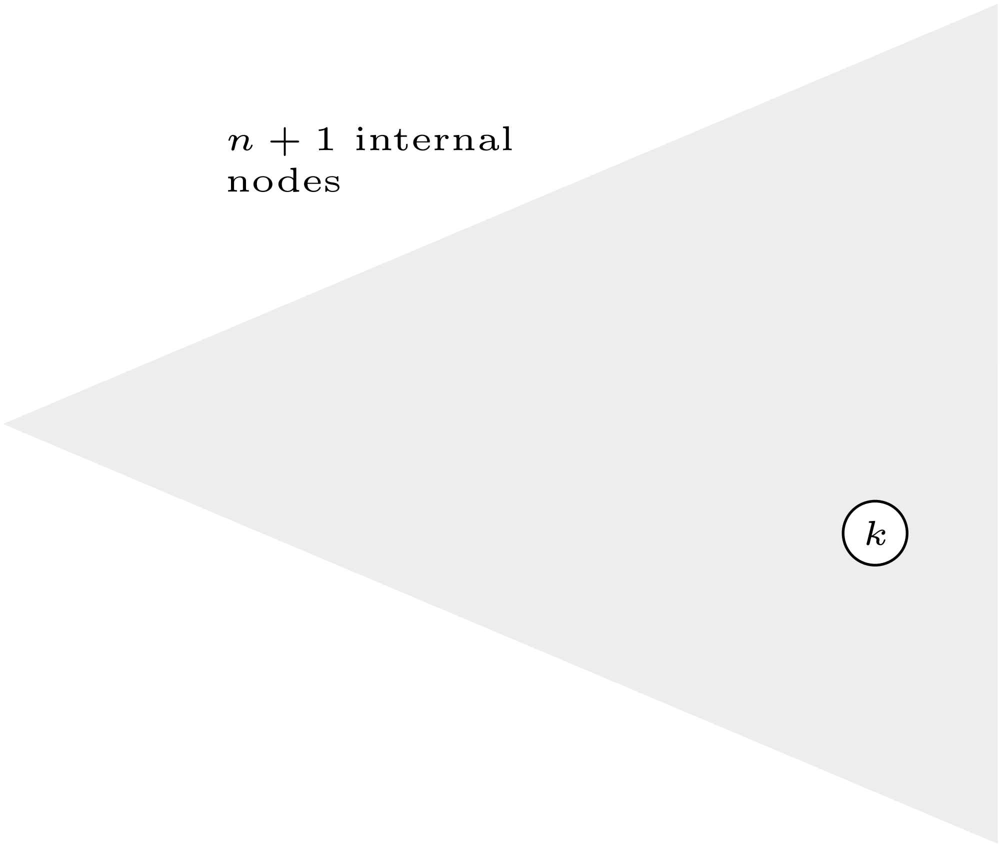

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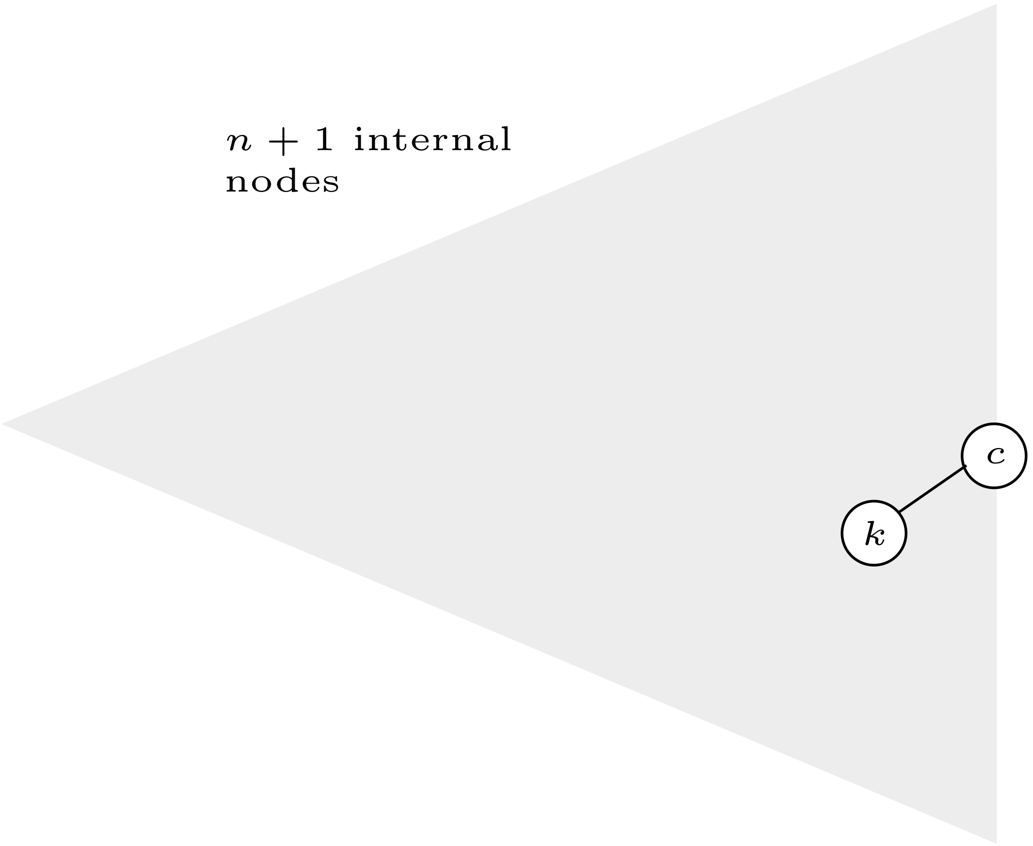

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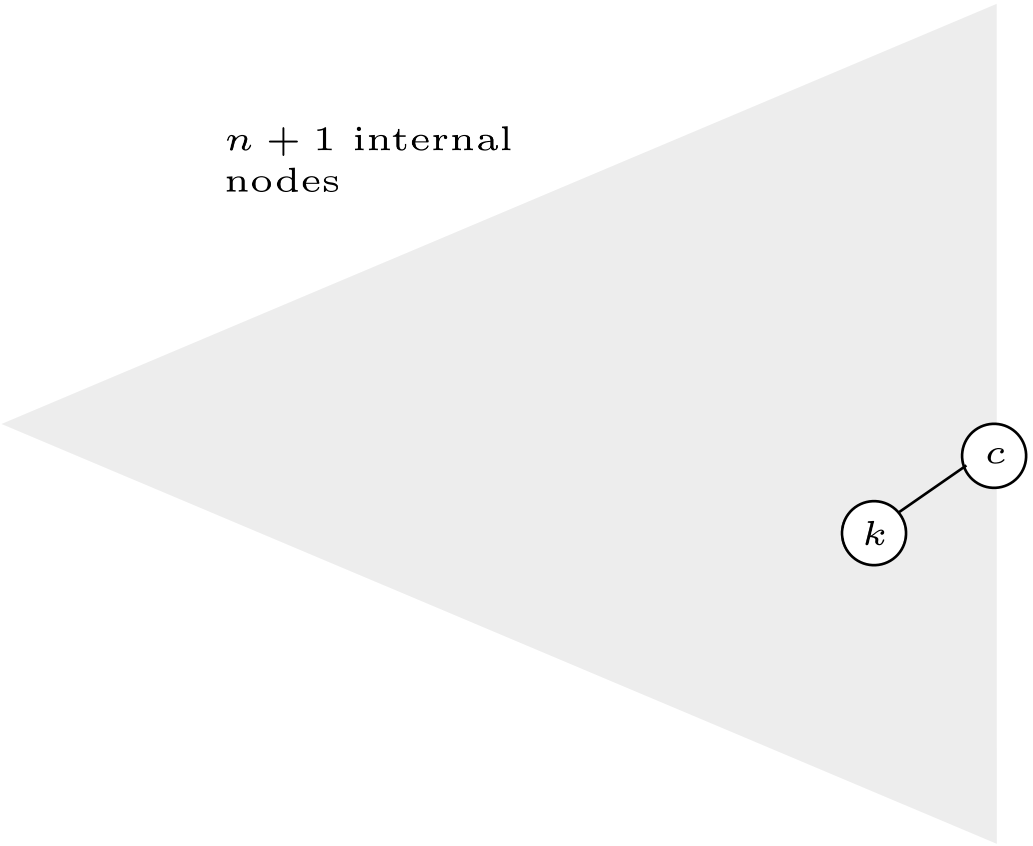

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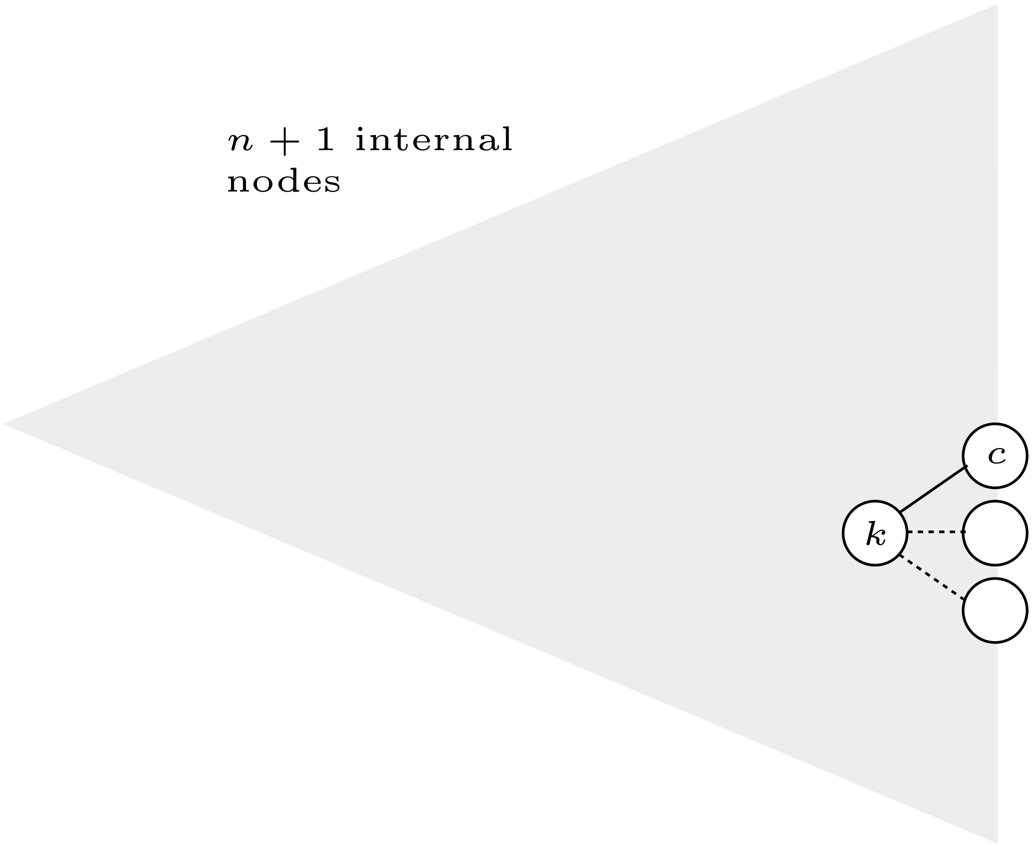

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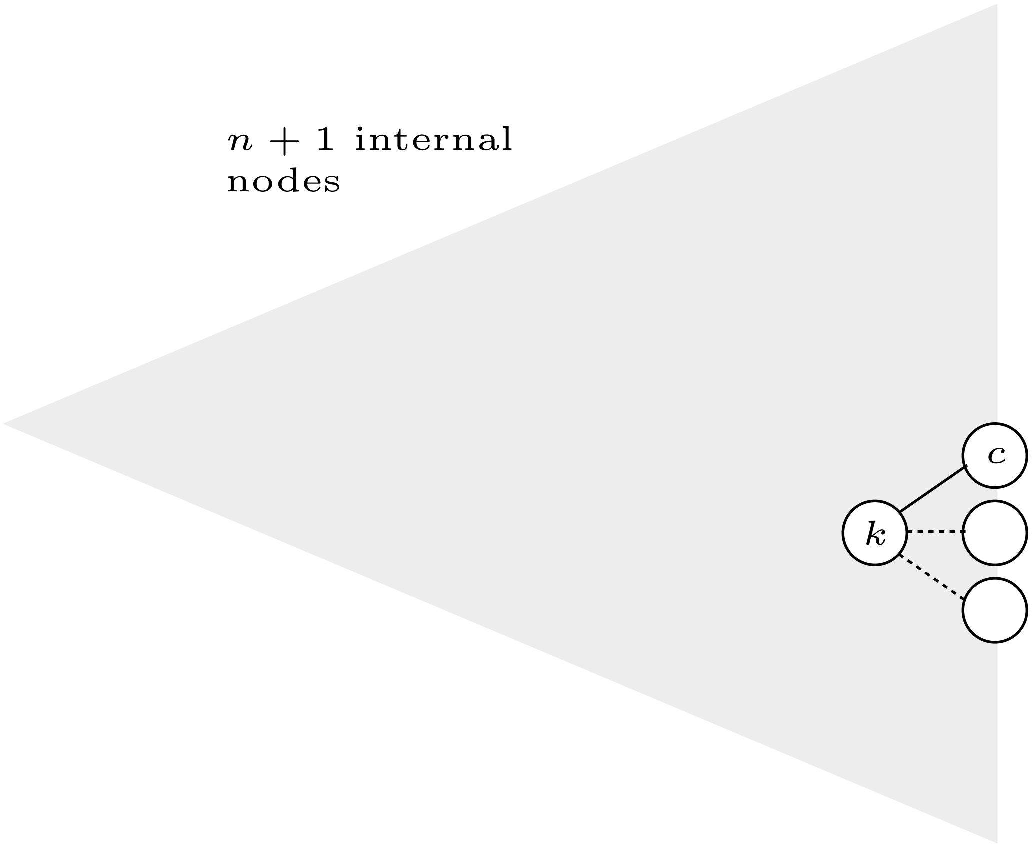

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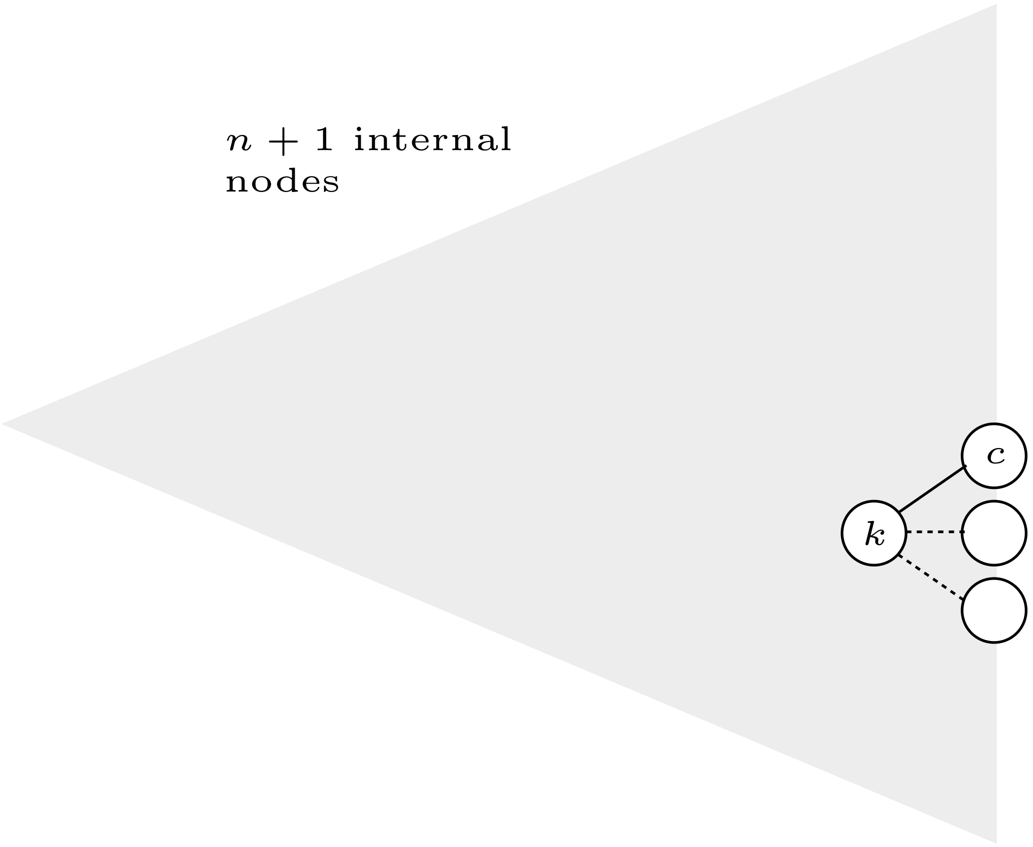

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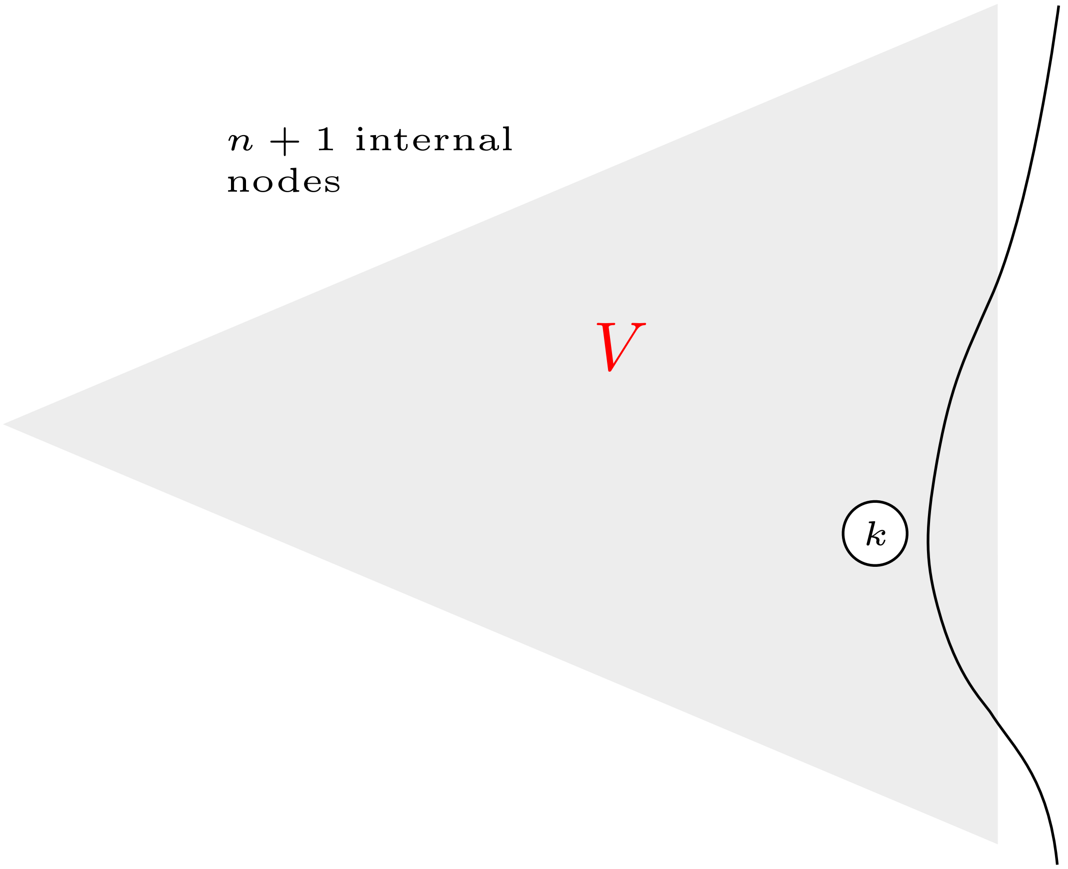

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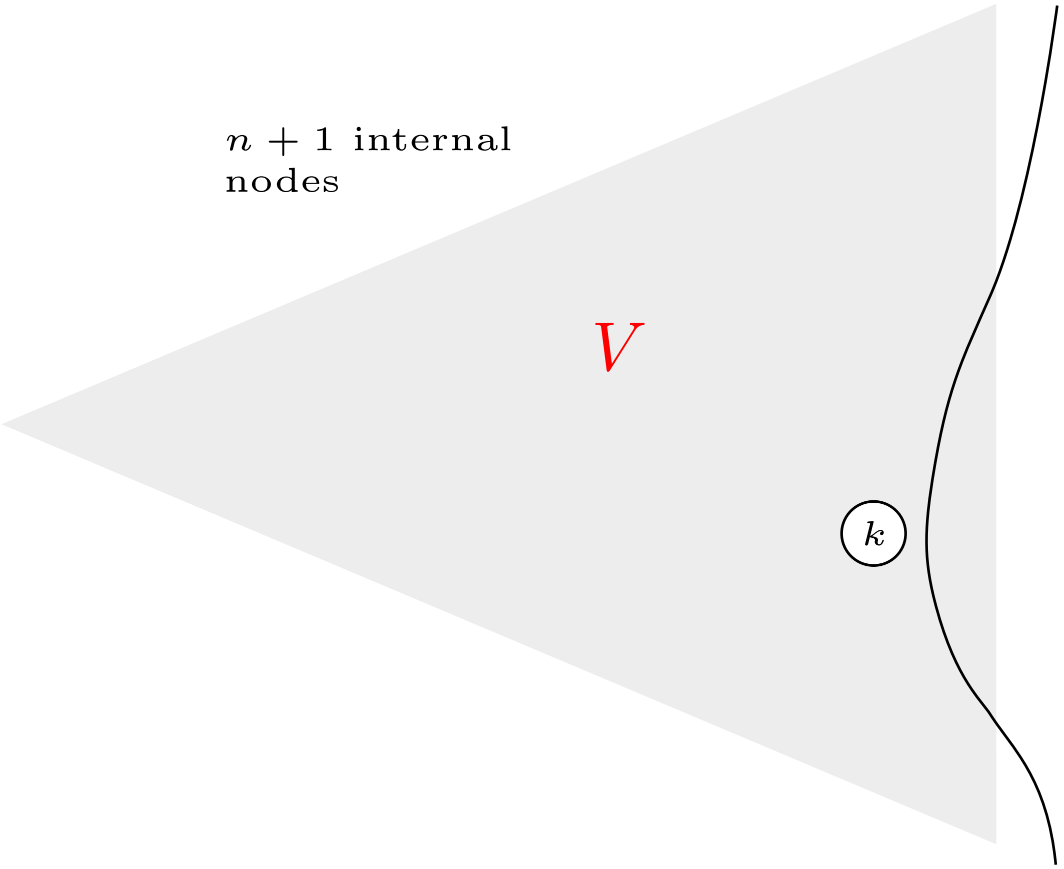

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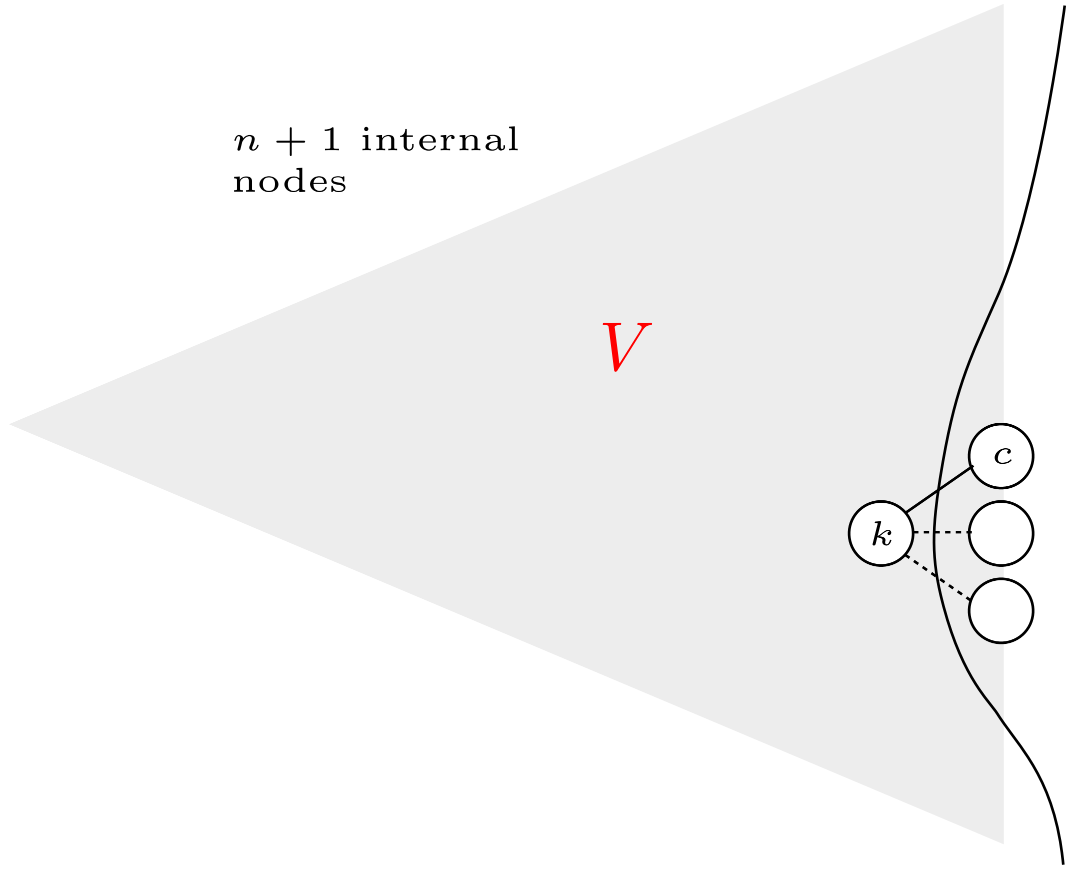

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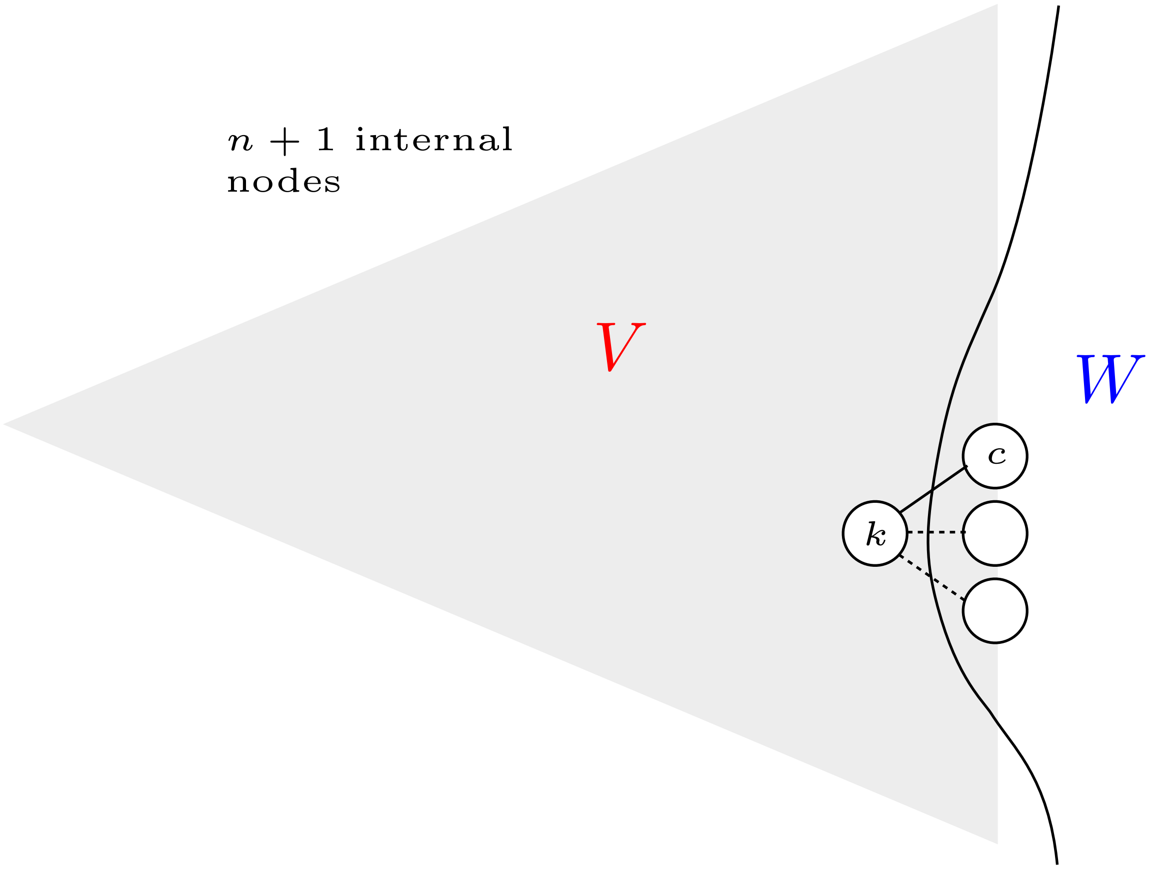
5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n + 1$ internal nodes.

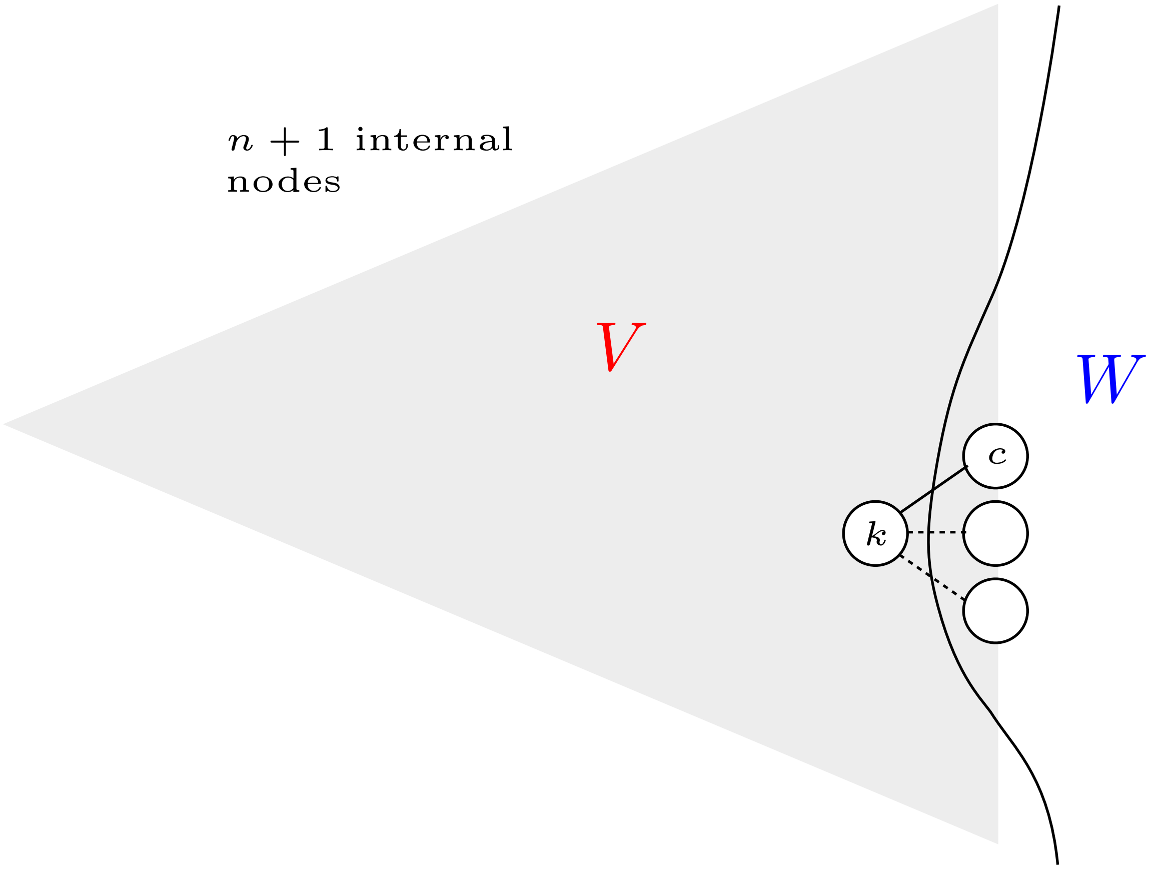
4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n + 1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

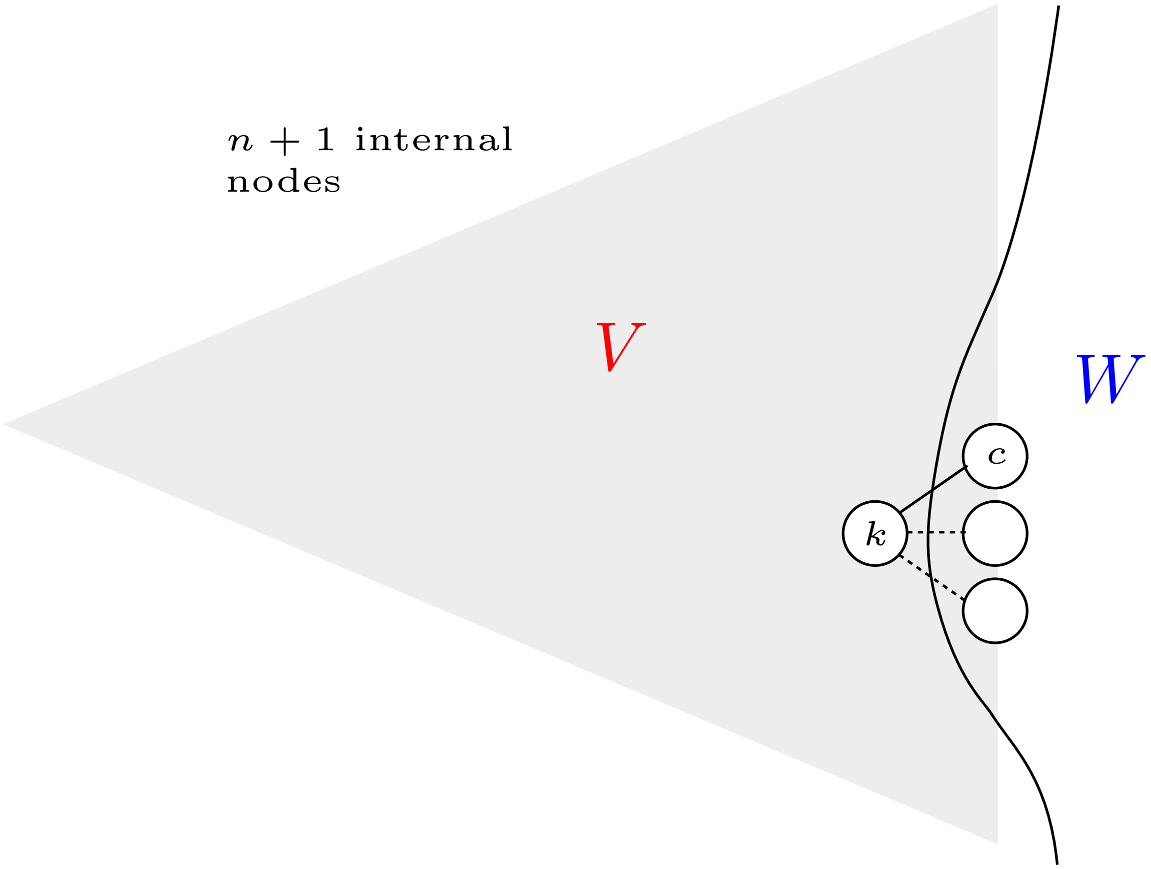
5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n + 1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

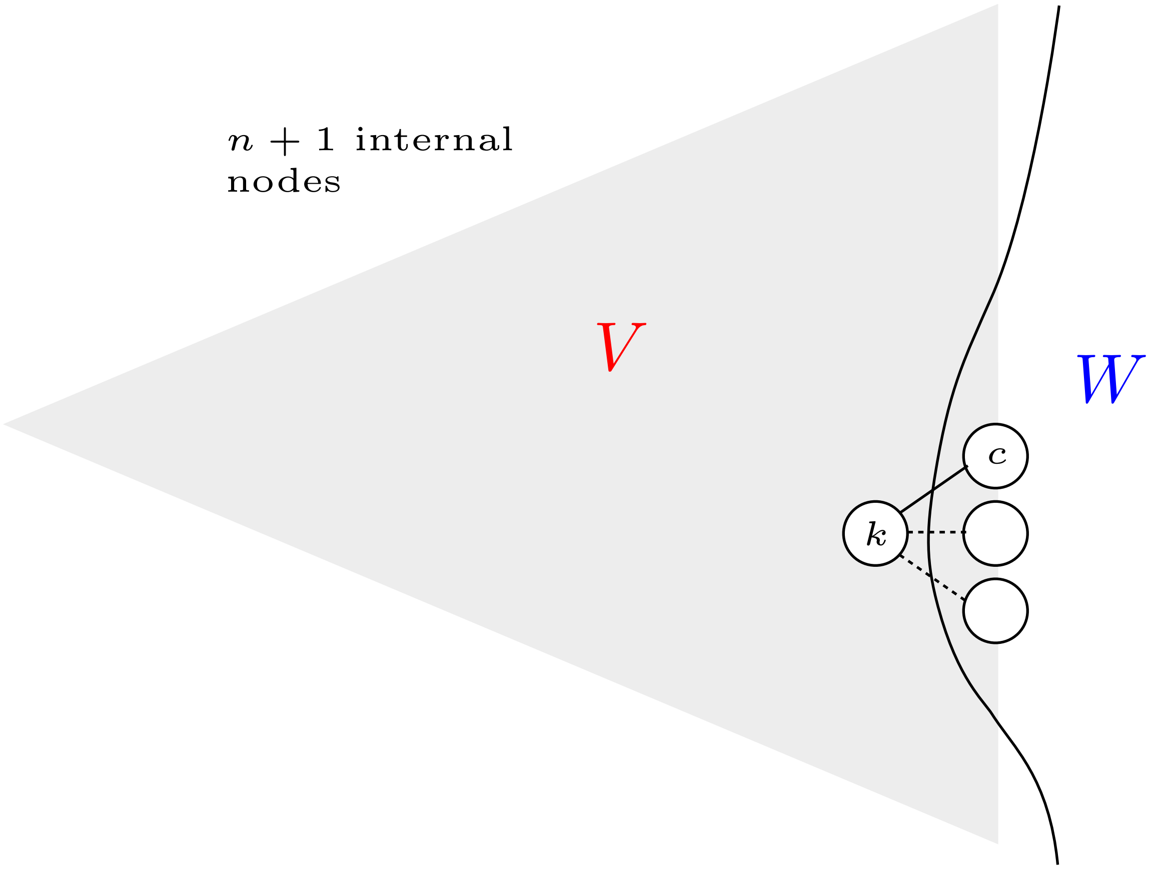
6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

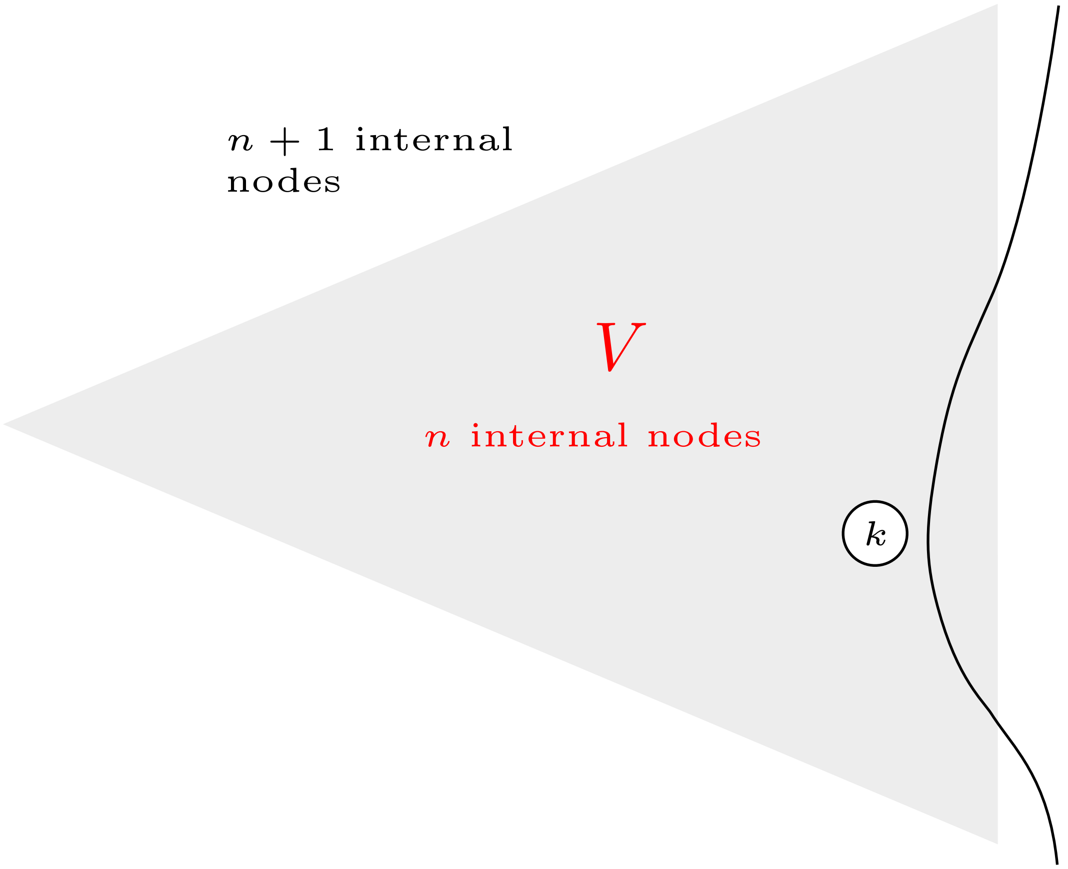
6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n + 1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

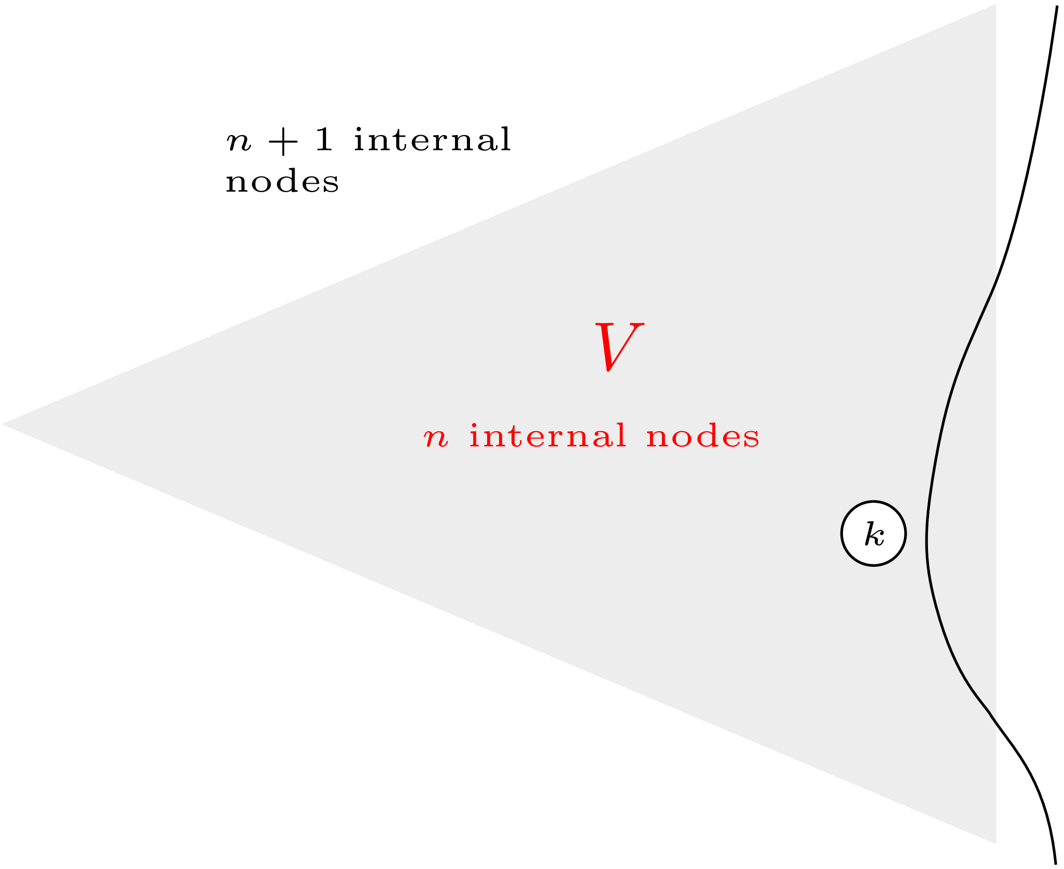
6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

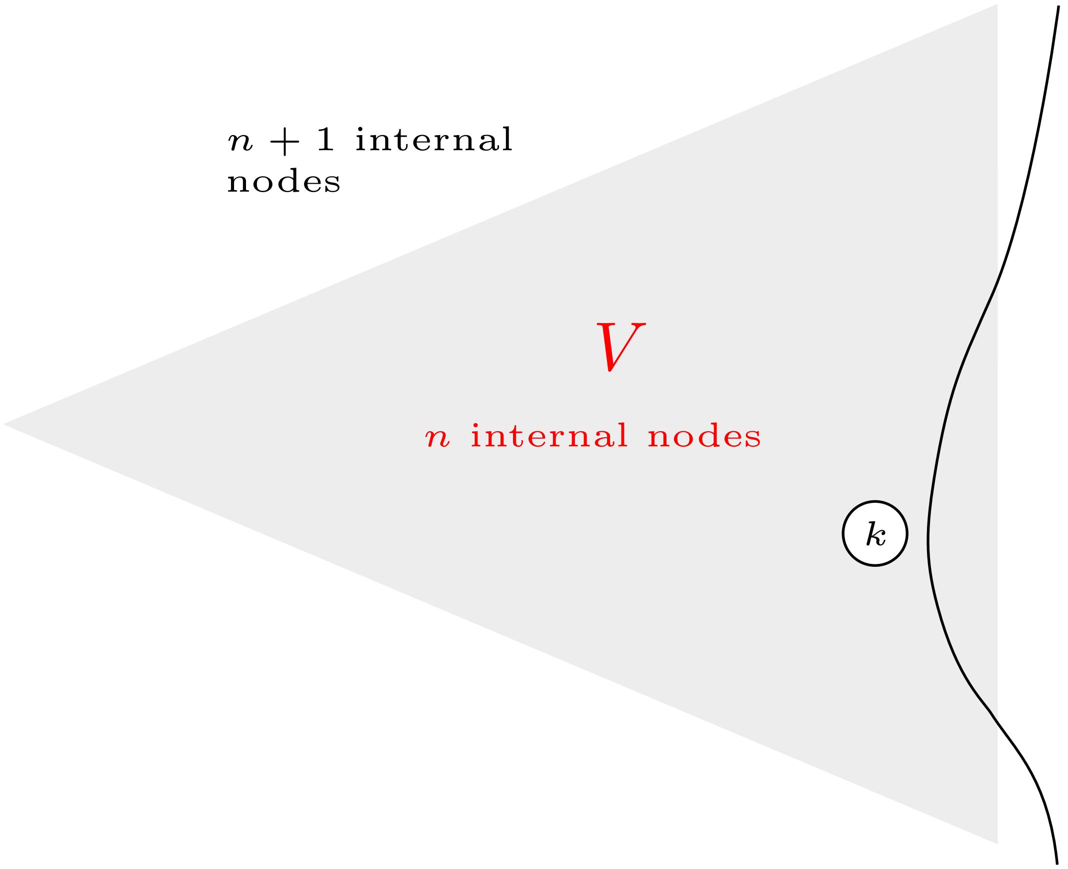
7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

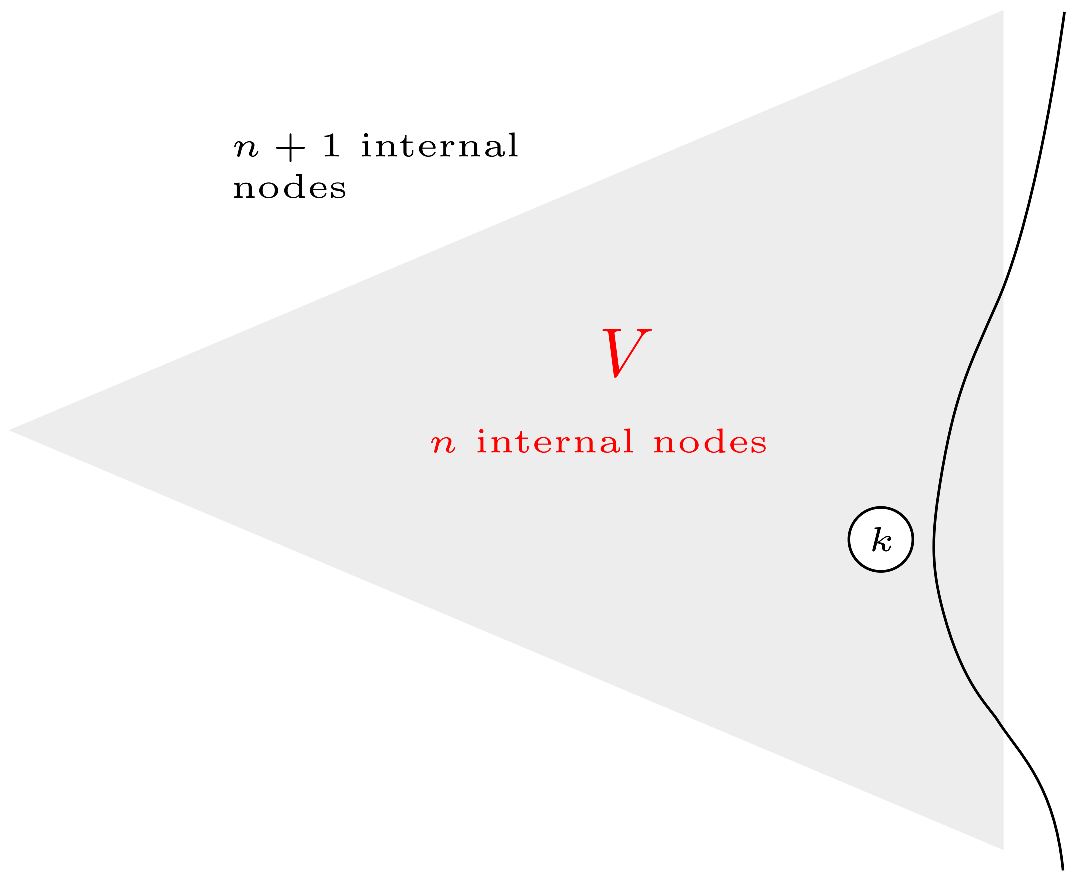
8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \setminus \{k\}} q_{k'} h_{k'}.$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n + 1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

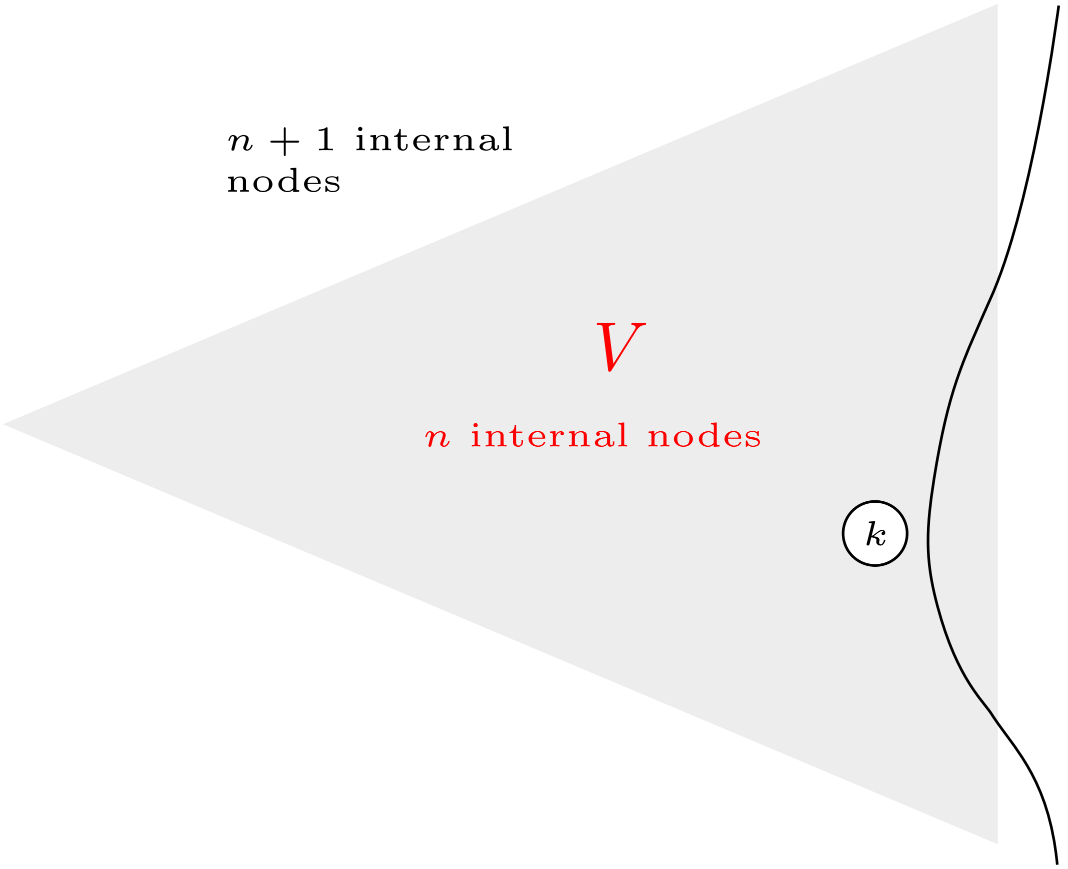
9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \setminus \{k\}} q_{k'} h_{k'}.$$

12. Then we have



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

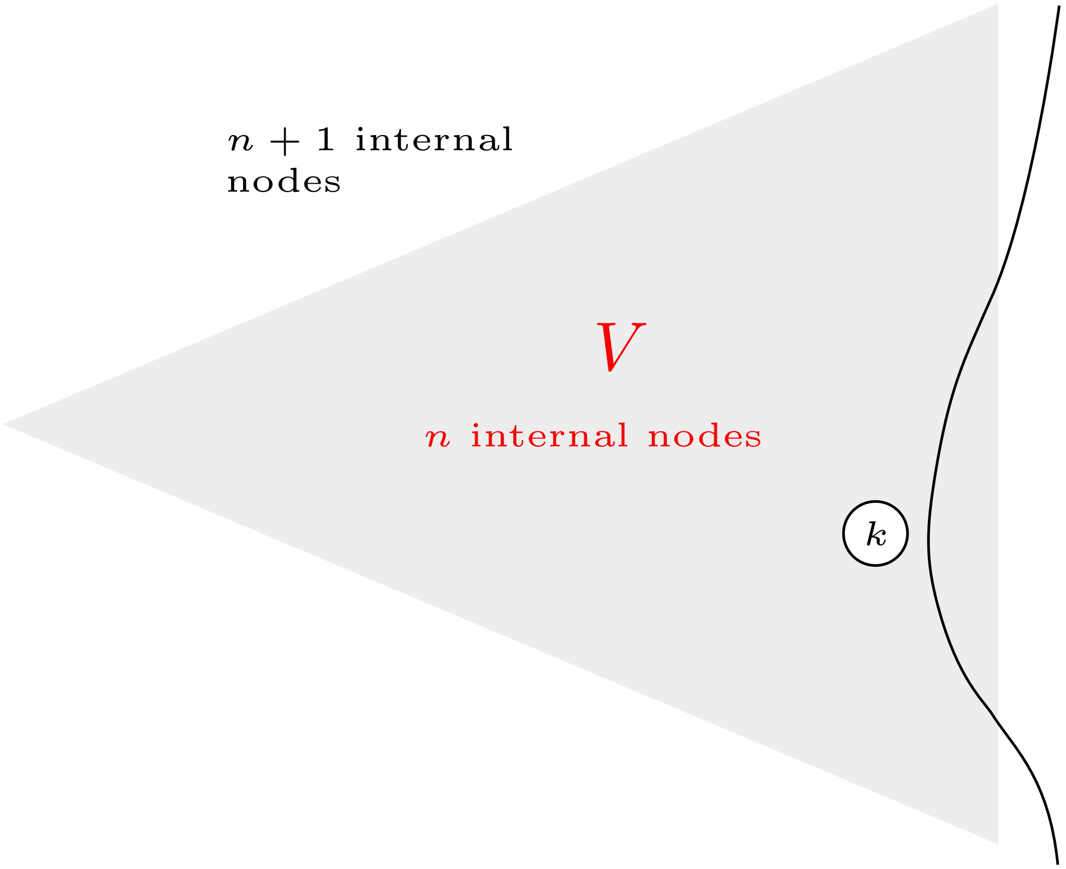
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \setminus \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$H(X) \quad = \quad H(V, W)$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

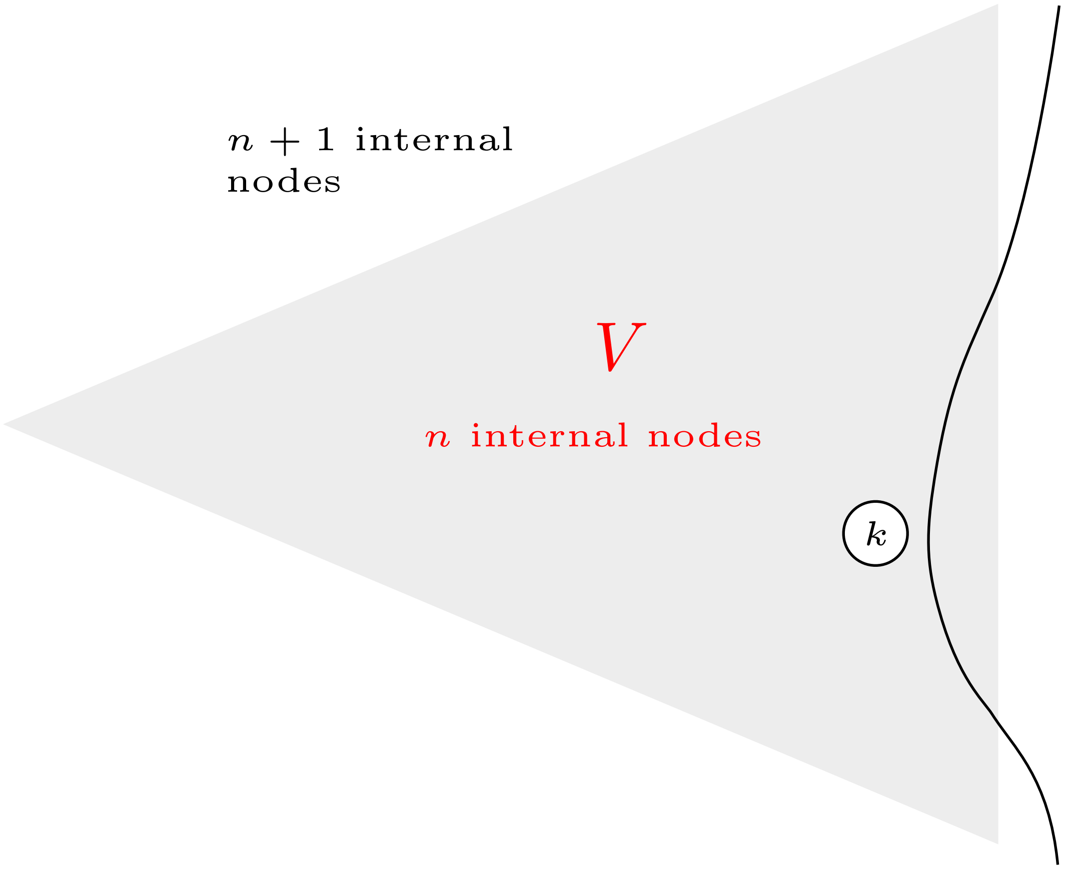
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + H(W|V) \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k\in\mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

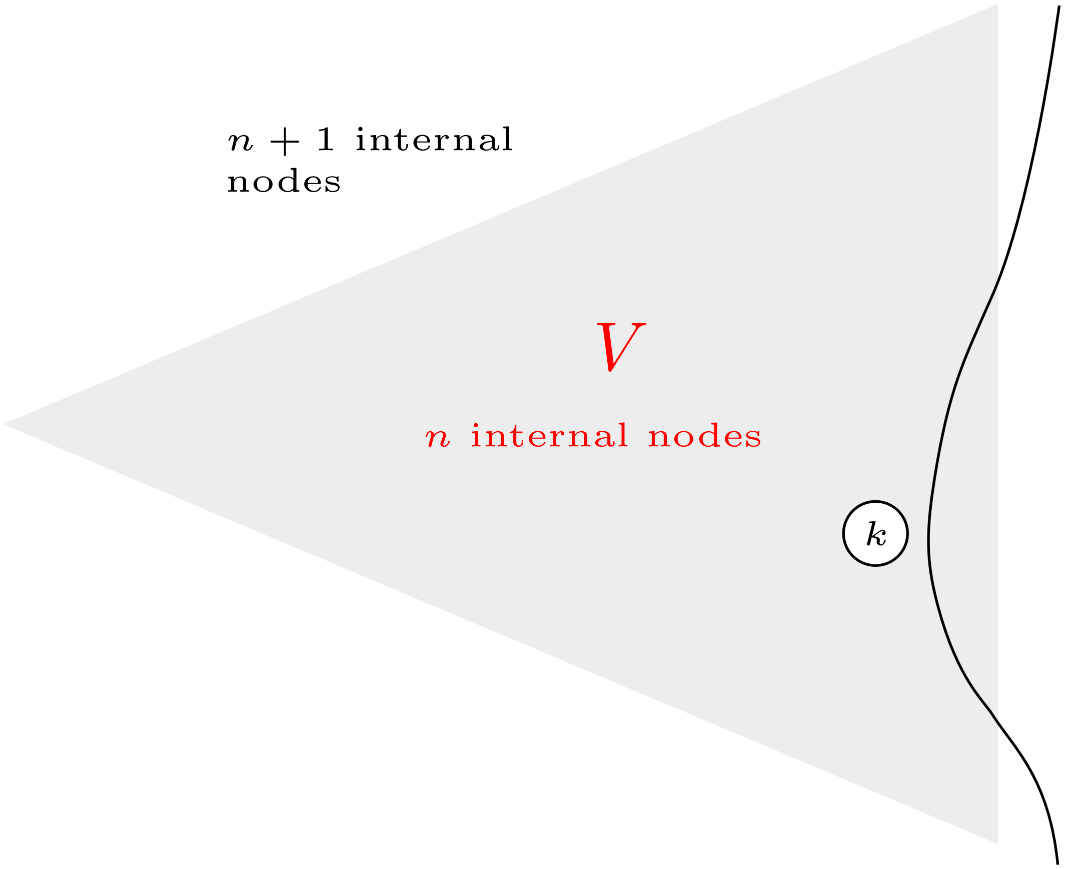
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k'\in\mathcal{I}\backslash\{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= \underline{H(V)} + H(W|V) \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k\in\mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

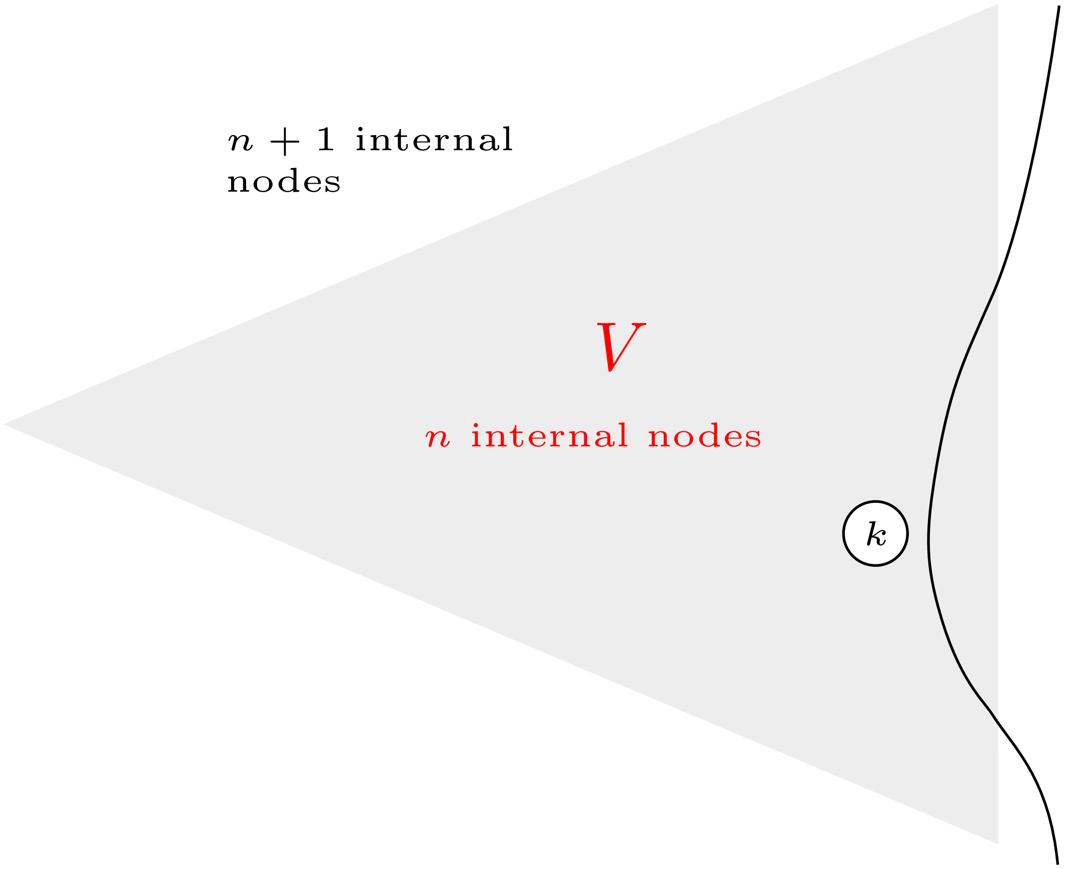
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k'\in\mathcal{I}\backslash\{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= \underline{H(V)} + H(W|V) \\ &= \underline{\sum_{k'\in\mathcal{I}\backslash\{k\}} q_{k'} h_{k'}} + (1-q_k)\cdot 0 + q_k h_k \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

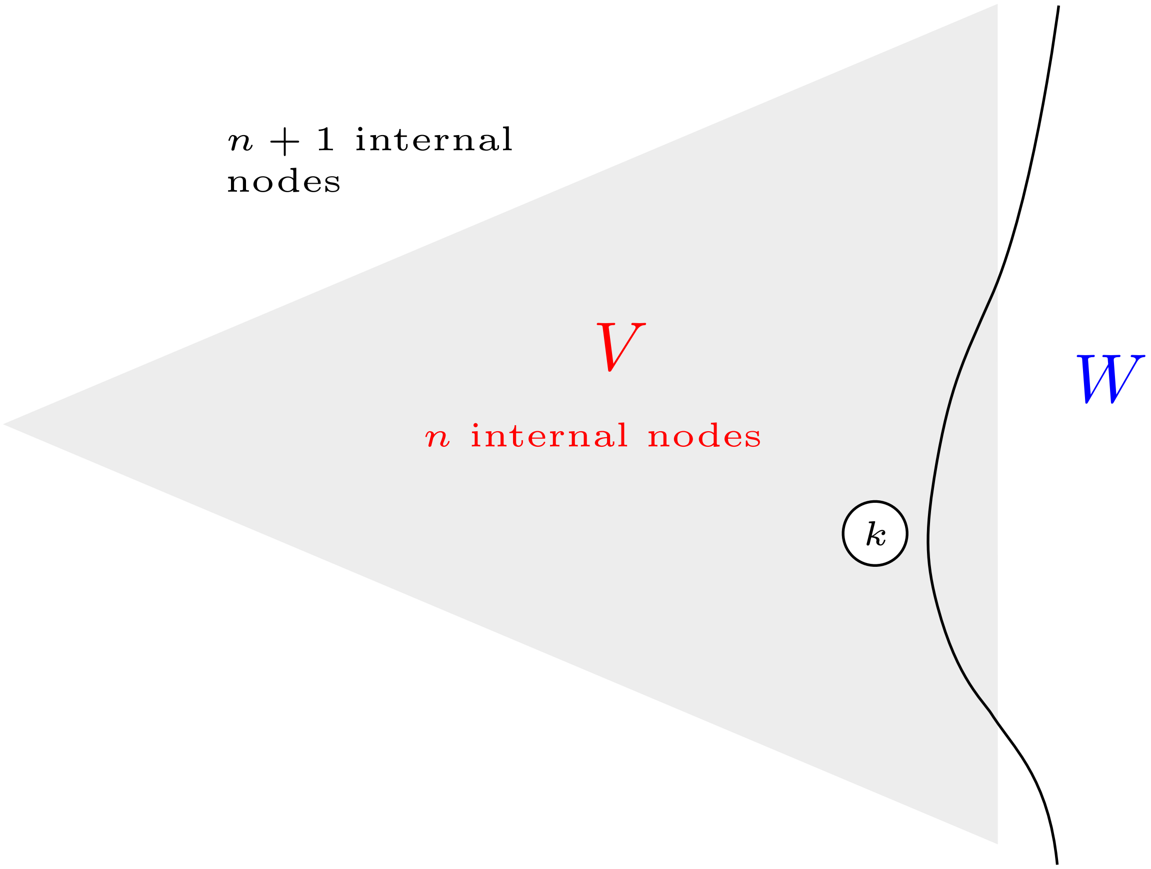
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + \underline{H(W|V)} \\ &= \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'} + (1 - q_k) \cdot 0 + q_k h_k \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

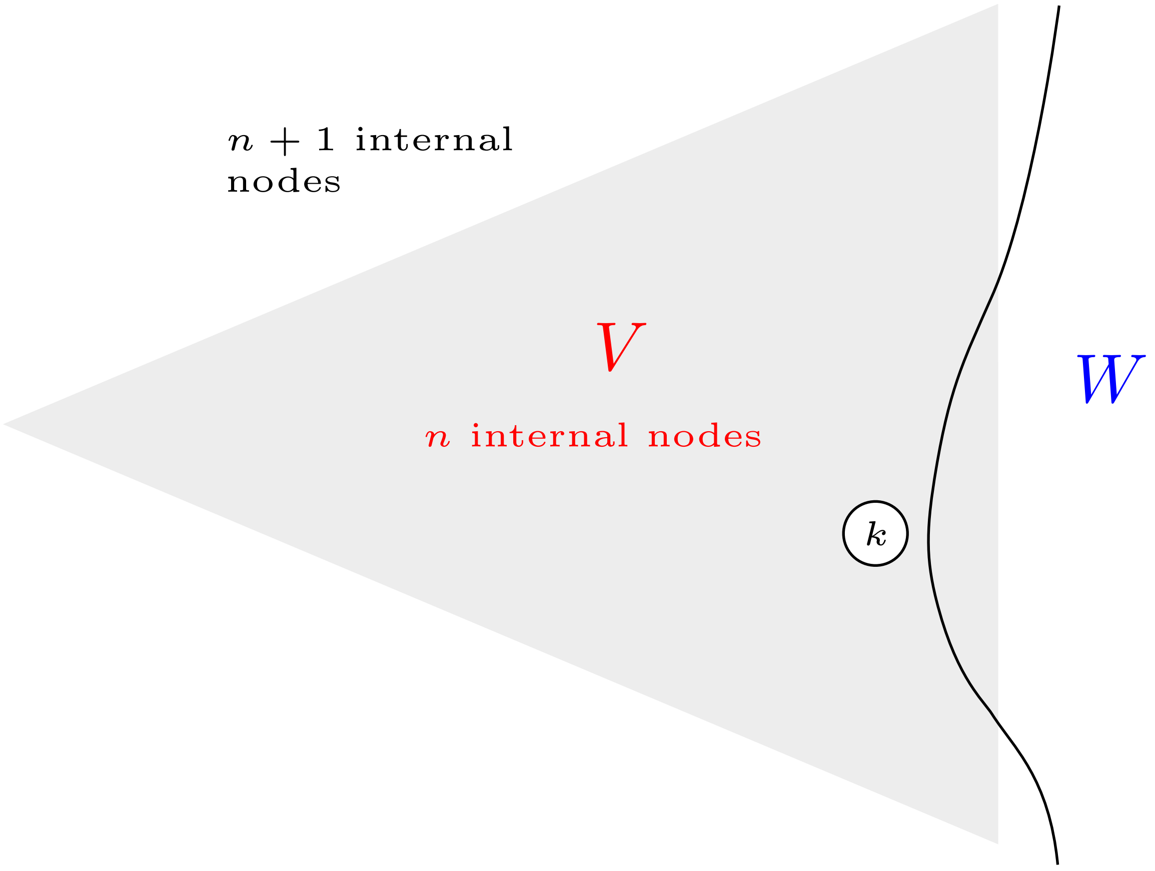
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + \underline{H(W|V)} \\ &= \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'} + \underline{(1 - q_k) \cdot 0} + q_k h_k \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

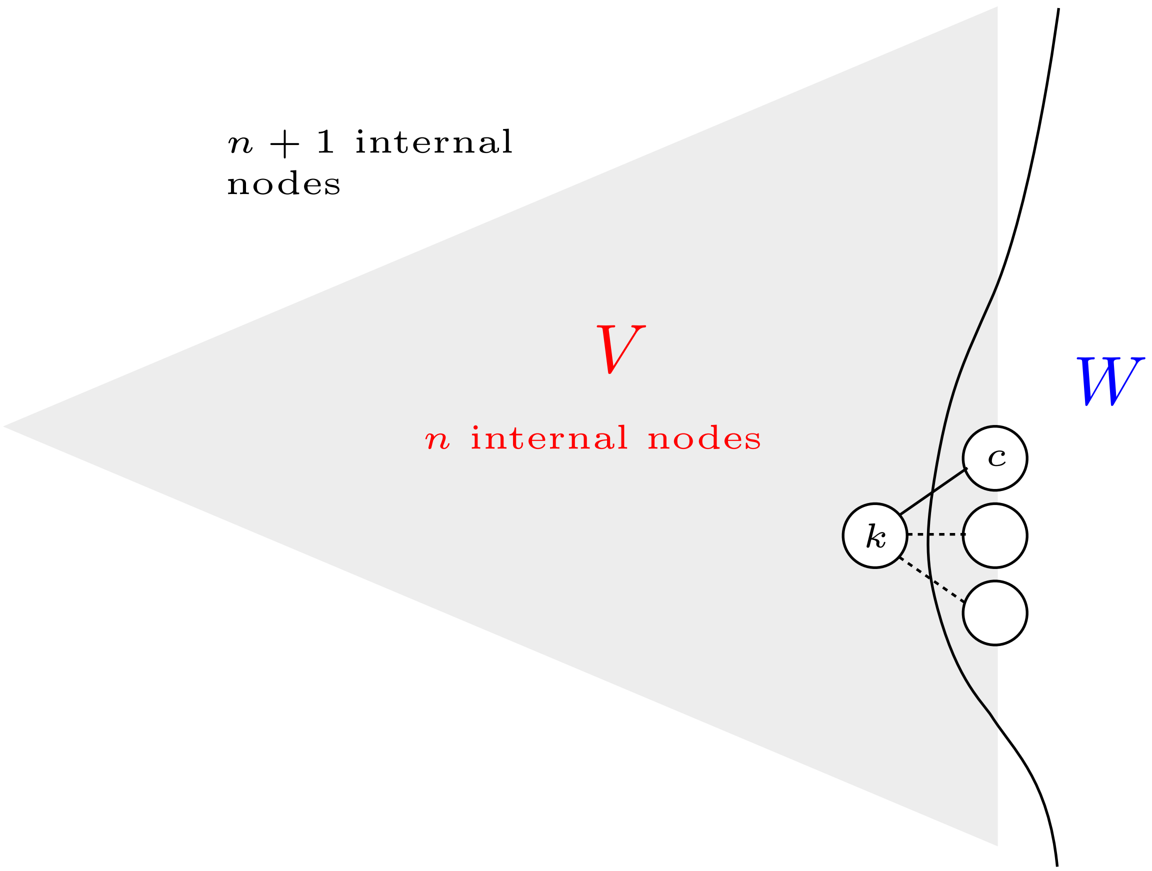
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + \underline{H(W|V)} \\ &= \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'} + (1 - q_k) \cdot 0 + \underline{q_k h_k} \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k\in\mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

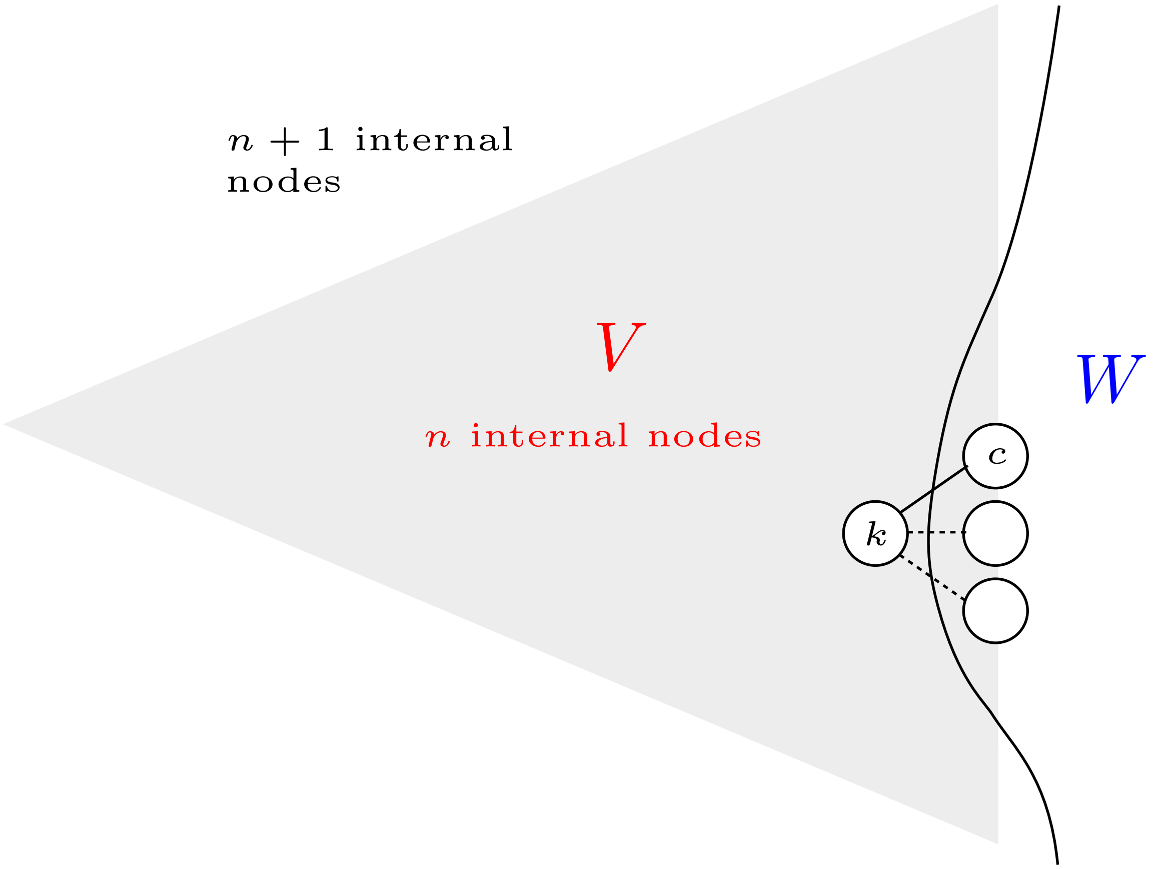
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k'\in\mathcal{I}\backslash\{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + H(W|V) \\ &= \sum_{k'\in\mathcal{I}\backslash\{k\}} q_{k'} h_{k'} + \cancel{(1-q_k)\cdot 0} + q_k h_k \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k\in\mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

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3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k'\in\mathcal{I}\backslash\{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned}
H(X) &= H(V, W) \\
&= H(V) + H(W|V) \\
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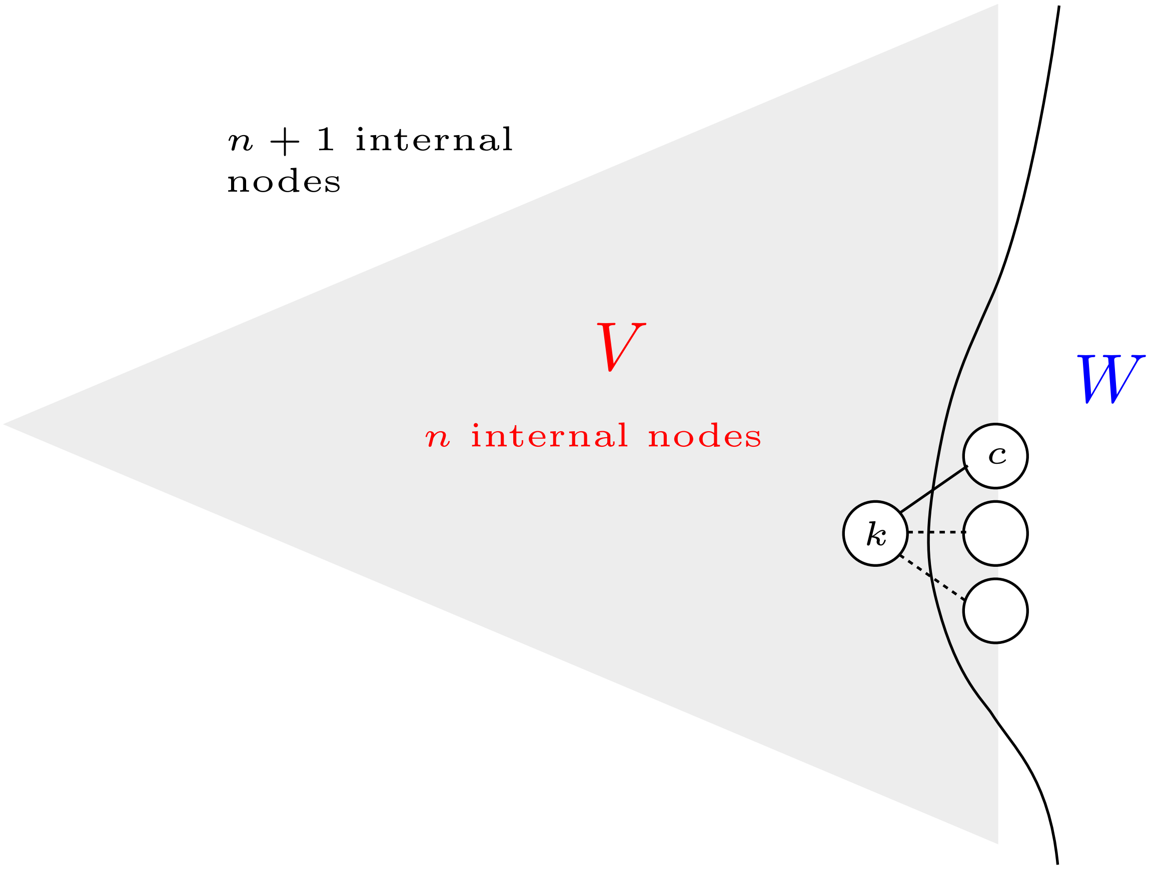
\end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

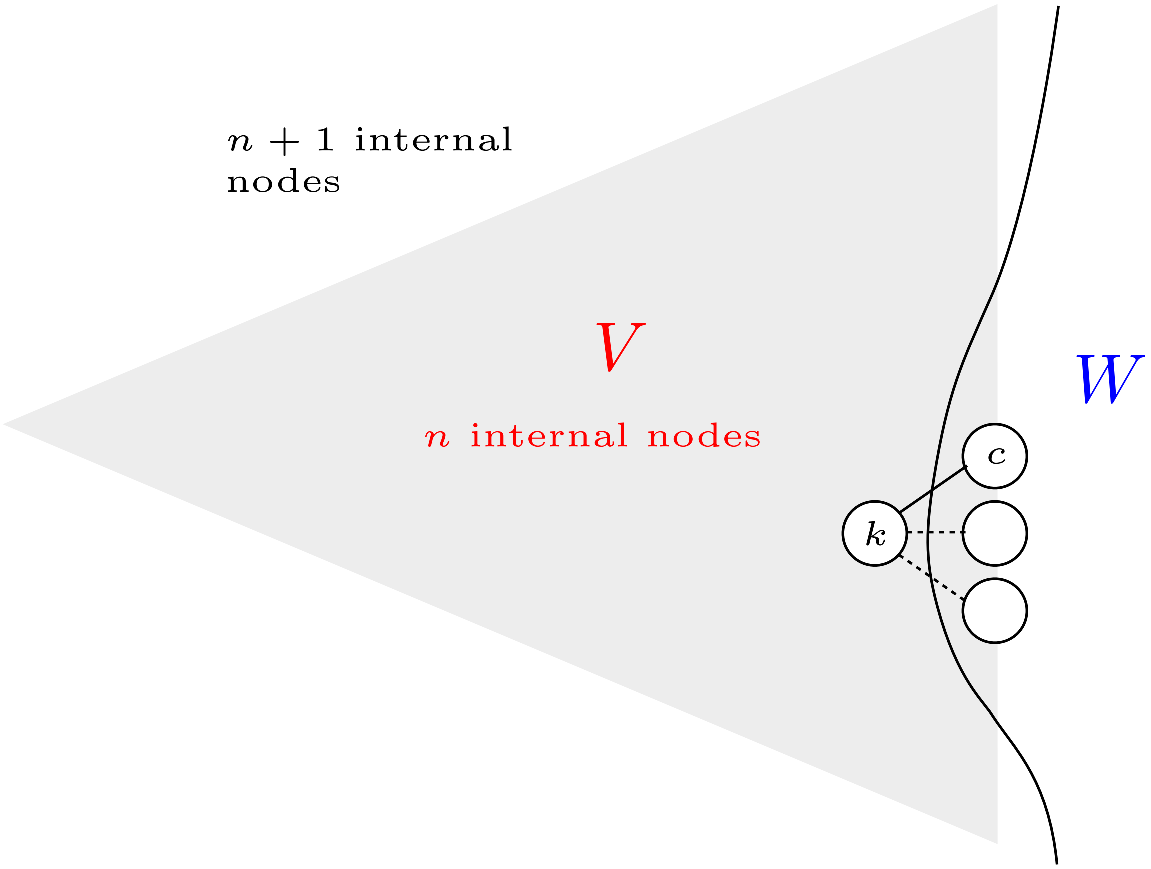
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \setminus \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + H(W|V) \\ &= \sum_{k' \in \mathcal{I} \setminus \{k\}} \underline{q_{k'} h_{k'}} + \cancel{(1 - q_k) \cdot 0} + \underline{q_k h_k} \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

2. If there is only one internal node, it must be the root of the tree. Then the lemma is trivially true upon observing that the reaching probability of the root is equal to 1.

3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

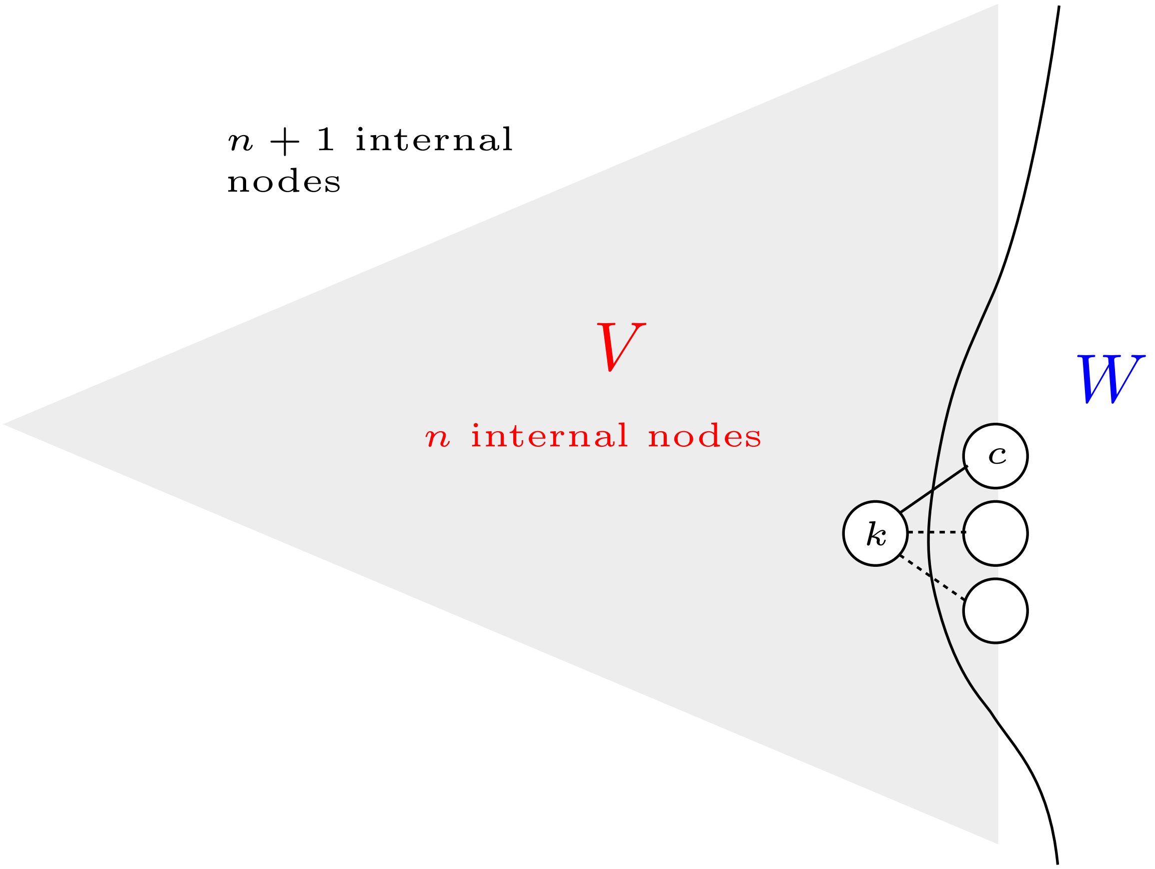
10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + H(W|V) \\ &= \sum_{k' \in \mathcal{I} \backslash \{k\}} \underline{q_{k'} h_{k'}} + \cancel{(1 - q_k) \cdot 0} + \underline{q_k h_k} \\ &= \sum_{\underline{k' \in \mathcal{I}}} q_{k'} h_{k'}. \end{aligned}$$



Lemma 4.19 $H_D(X) = \sum_{k \in \mathcal{I}} q_k h_k$.

Proof

1. We prove the lemma by induction on the number of internal nodes of the code tree.

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3. Assume the lemma is true for all code trees with n internal nodes, and consider a code tree with $n+1$ internal nodes.

4. Let k be an internal node such that k is the parent of a leaf c with maximum order.

5. Each sibling of c may or may not be a leaf. If it is not a leaf, then it cannot be the ascendent of another leaf because we assume that c is a leaf with maximum order.

6. Now consider revealing the outcome of X in two steps. In the first step, if the outcome of X is not a leaf descending from node k, we identify the outcome exactly, otherwise we identify the outcome to be a child of node k. We call this random variable V.

7. If we do not identify the outcome exactly in the first step, which happens with probability q_k, we further identify in the second step which of the children (child) of node k the outcome is (there is only one child of node k which can be the outcome if all the siblings of c are not leaves). We call this random variable W.

8. If the second step is not necessary, we assume that W takes a constant value with probability 1.

9. Then $X = (V, W)$.

10. The outcome of V can be represented by a code tree with n internal nodes which is obtained by pruning the original code tree at node k.

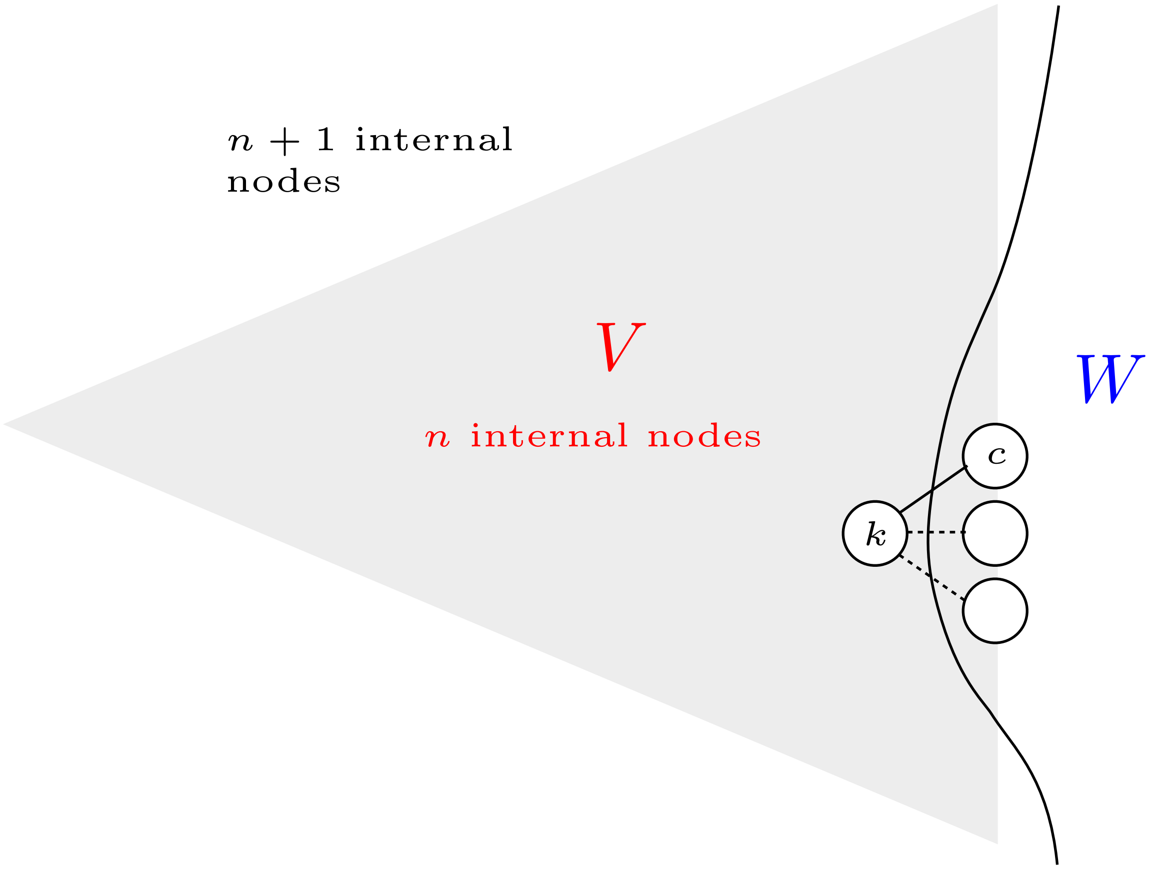
11. By the induction hypothesis,

$$H(V) = \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'}.$$

12. Then we have

$$\begin{aligned} H(X) &= H(V, W) \\ &= H(V) + H(W|V) \\ &= \sum_{k' \in \mathcal{I} \backslash \{k\}} q_{k'} h_{k'} + (1 - q_k) \cdot 0 + q_k h_k \\ &= \sum_{k' \in \mathcal{I}} q_{k'} h_{k'}. \end{aligned}$$

The lemma is proved.



Lemma 4.20 $L = \sum_{k \in \mathcal{I}} q_k$.

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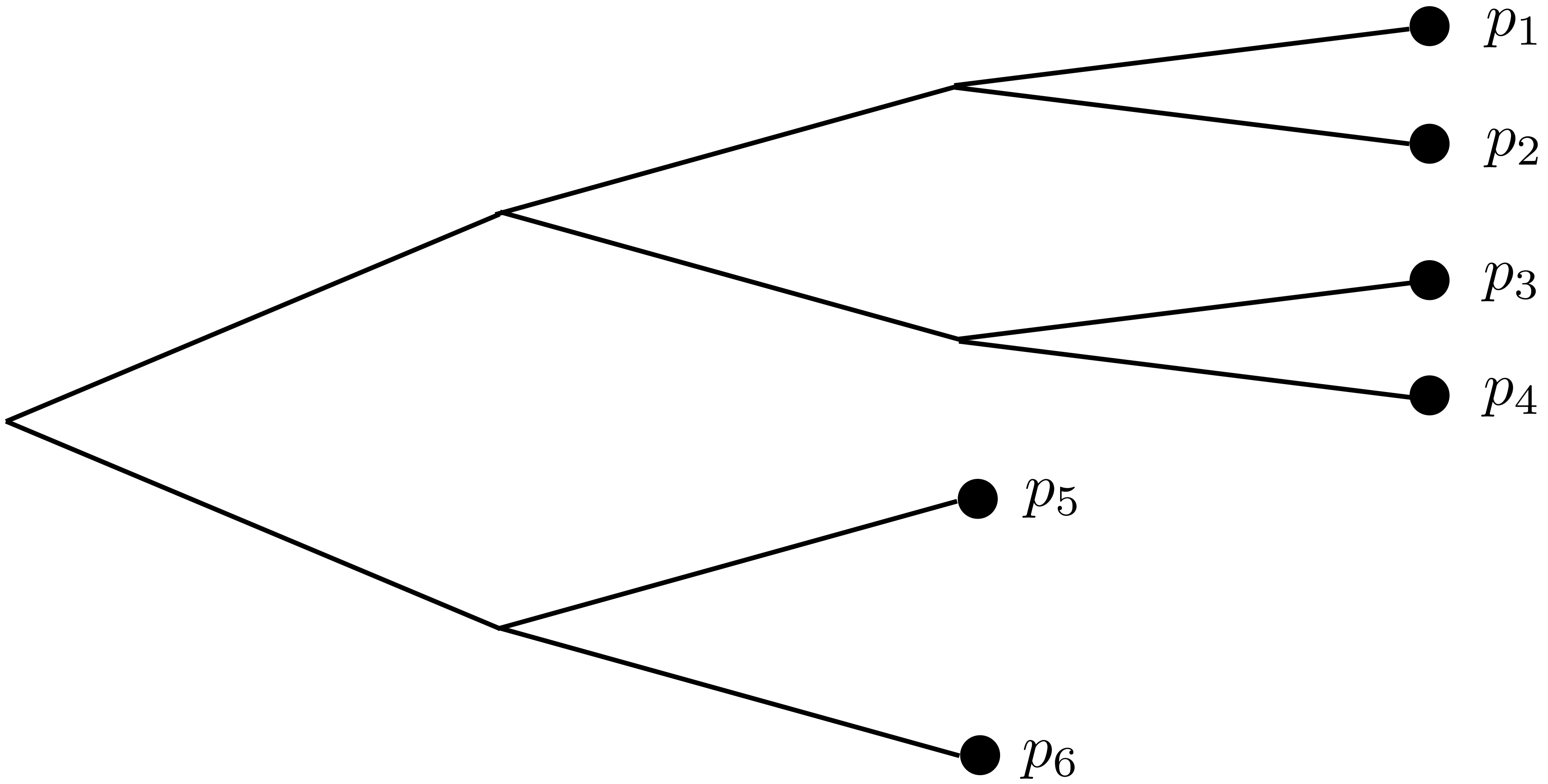
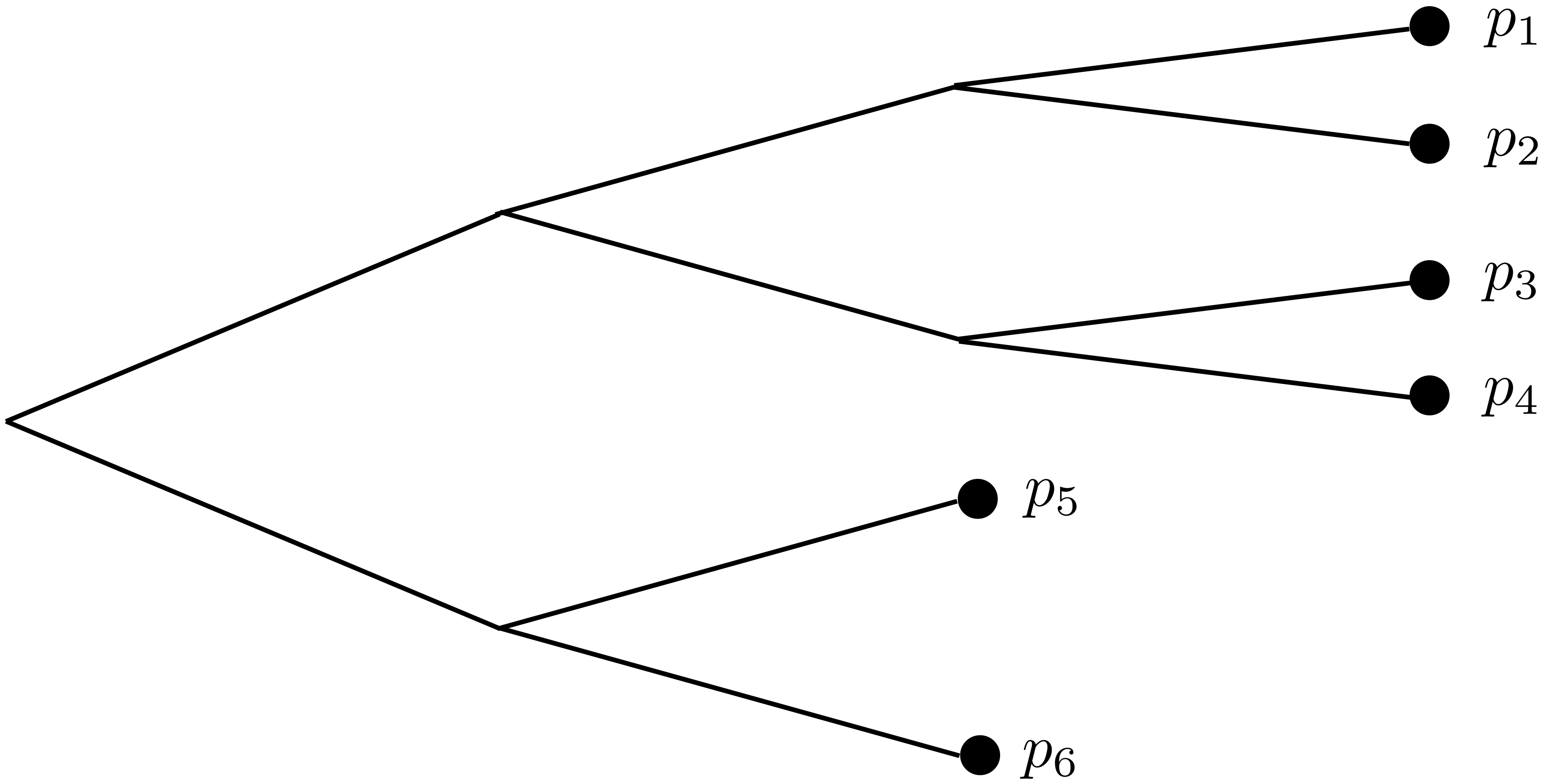


Lemma 4.20 $L = \sum_{k \in \mathcal{I}} q_k$.

$$\boxed{L = 3p_1 + 3p_2 + 3p_3 + 3p_4 + 2p_5 + 2p_6}$$



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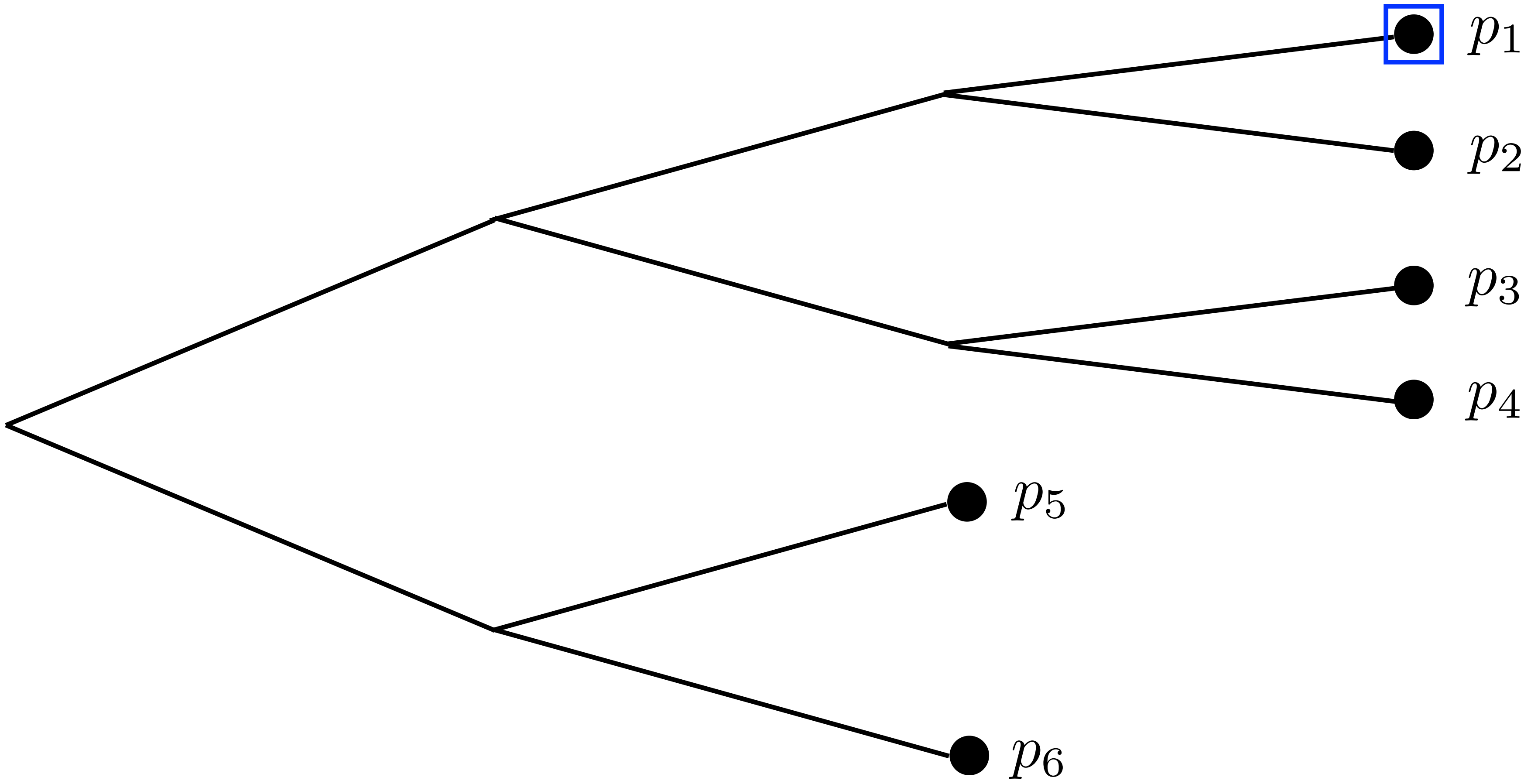

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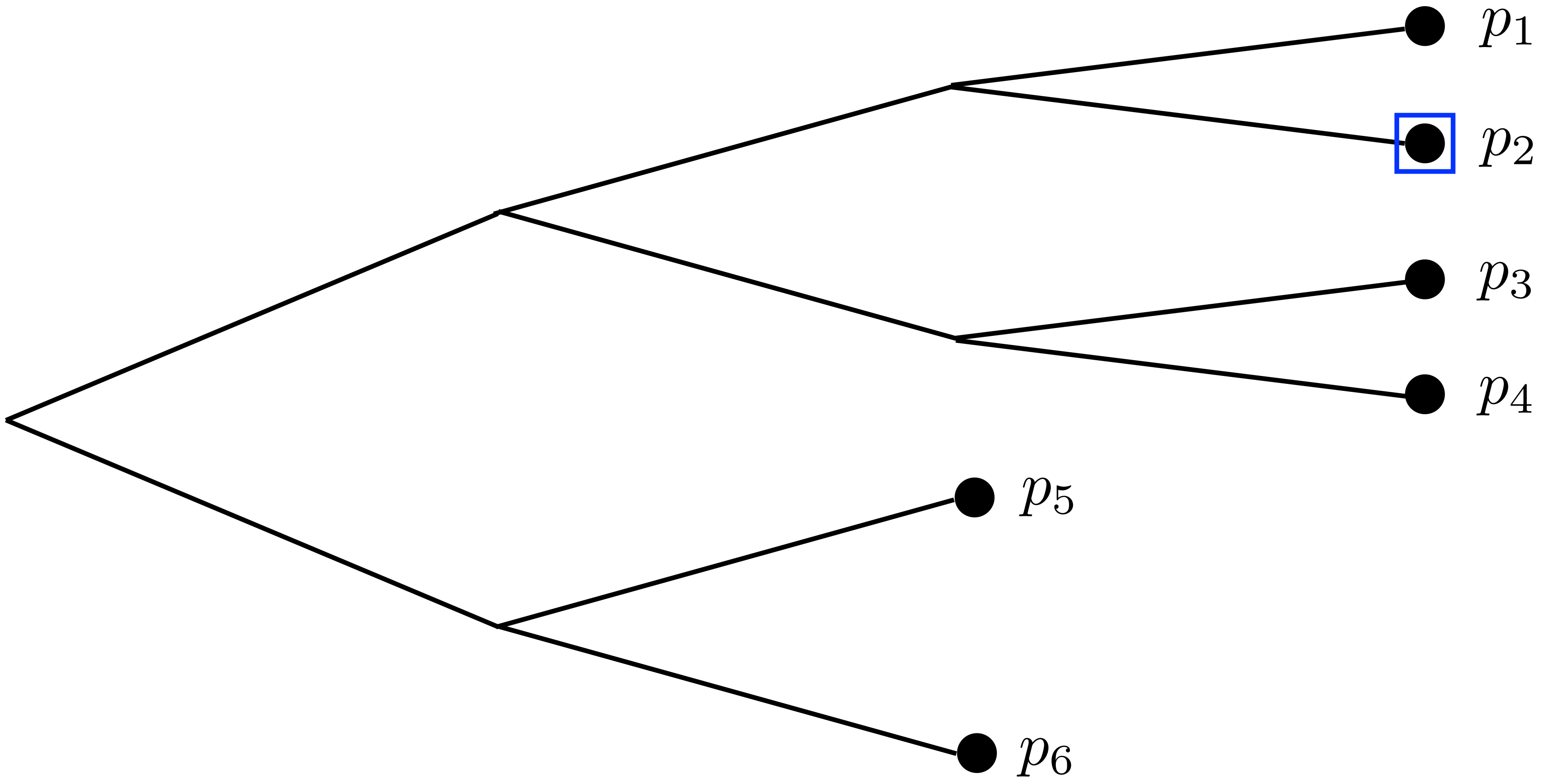

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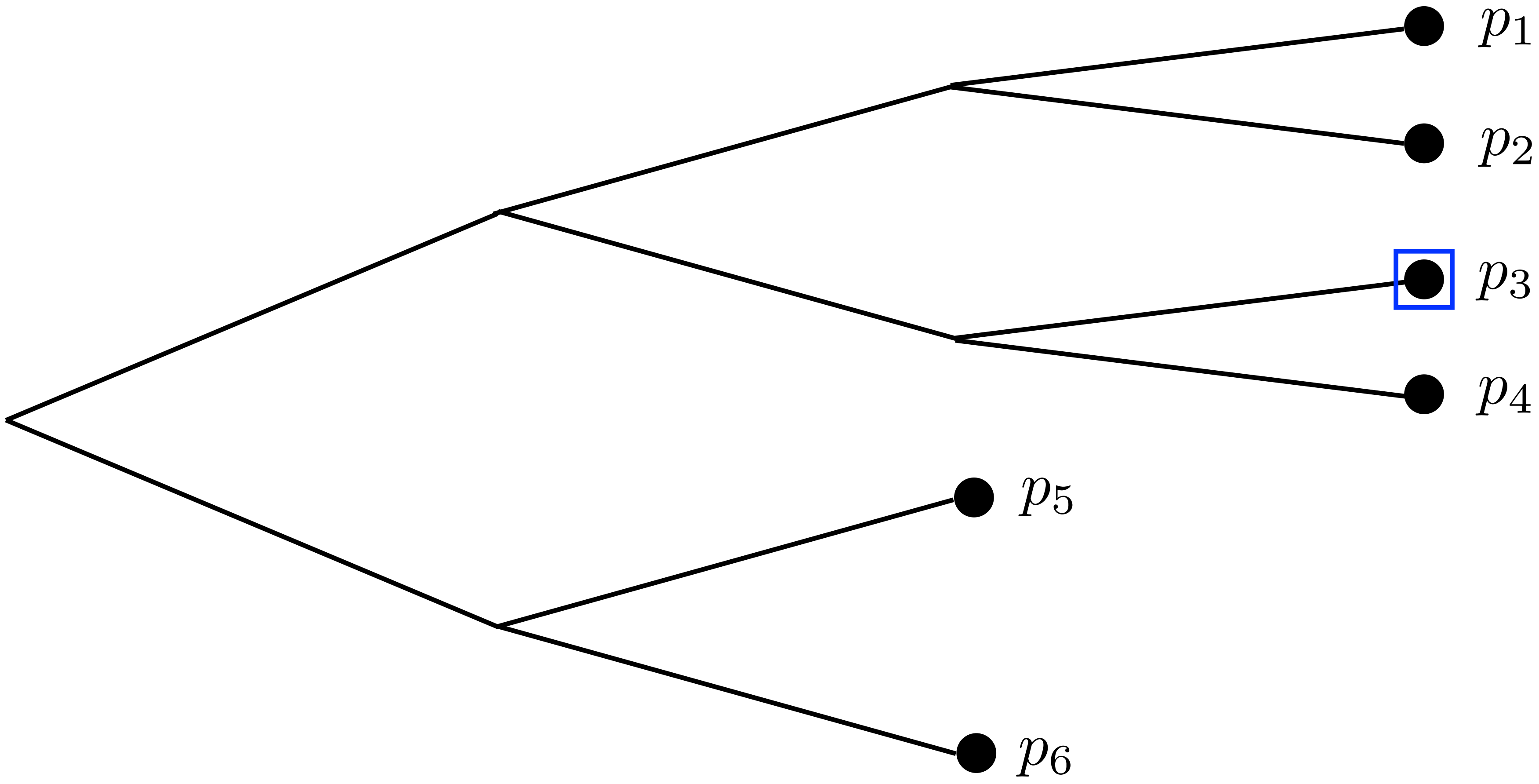

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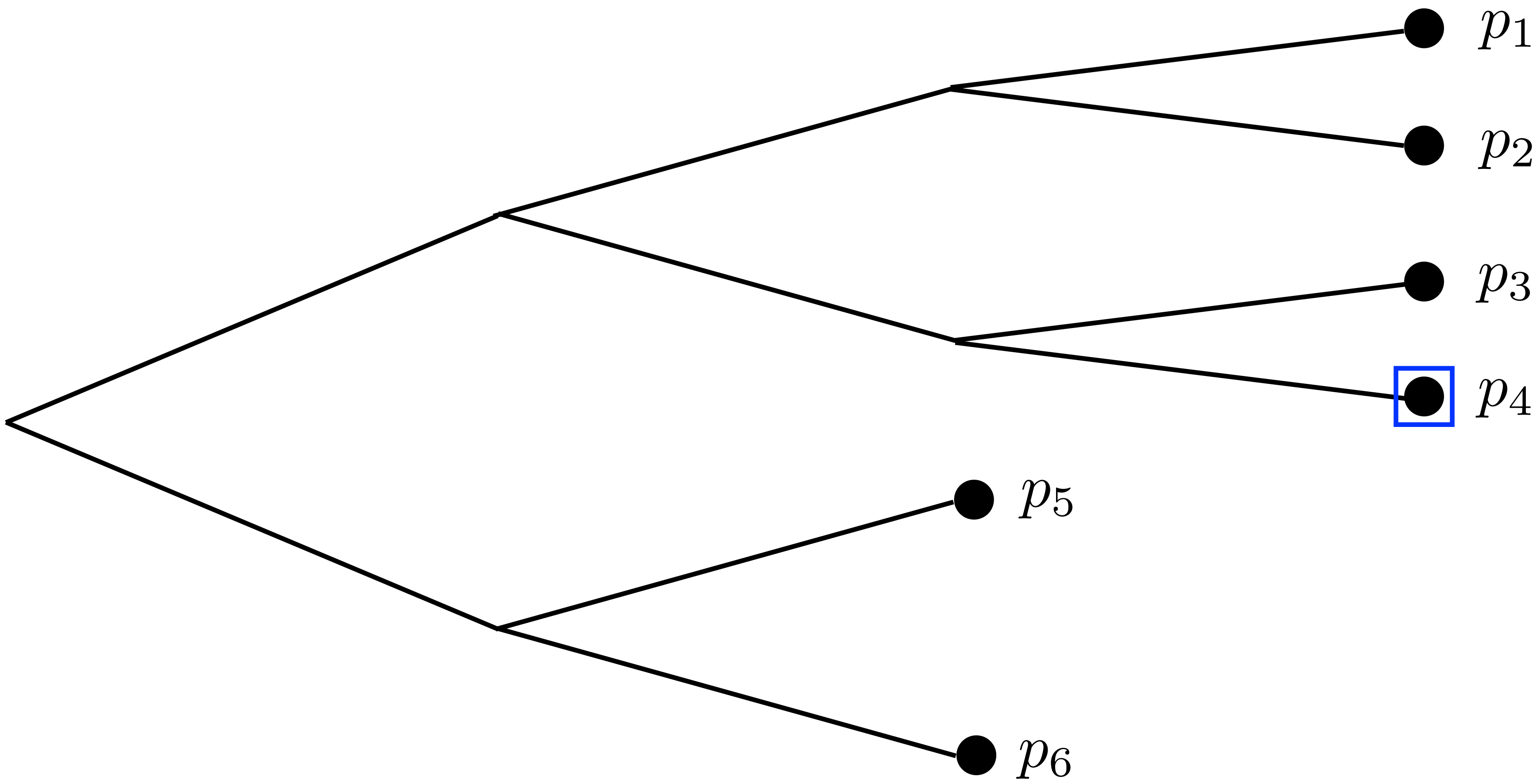

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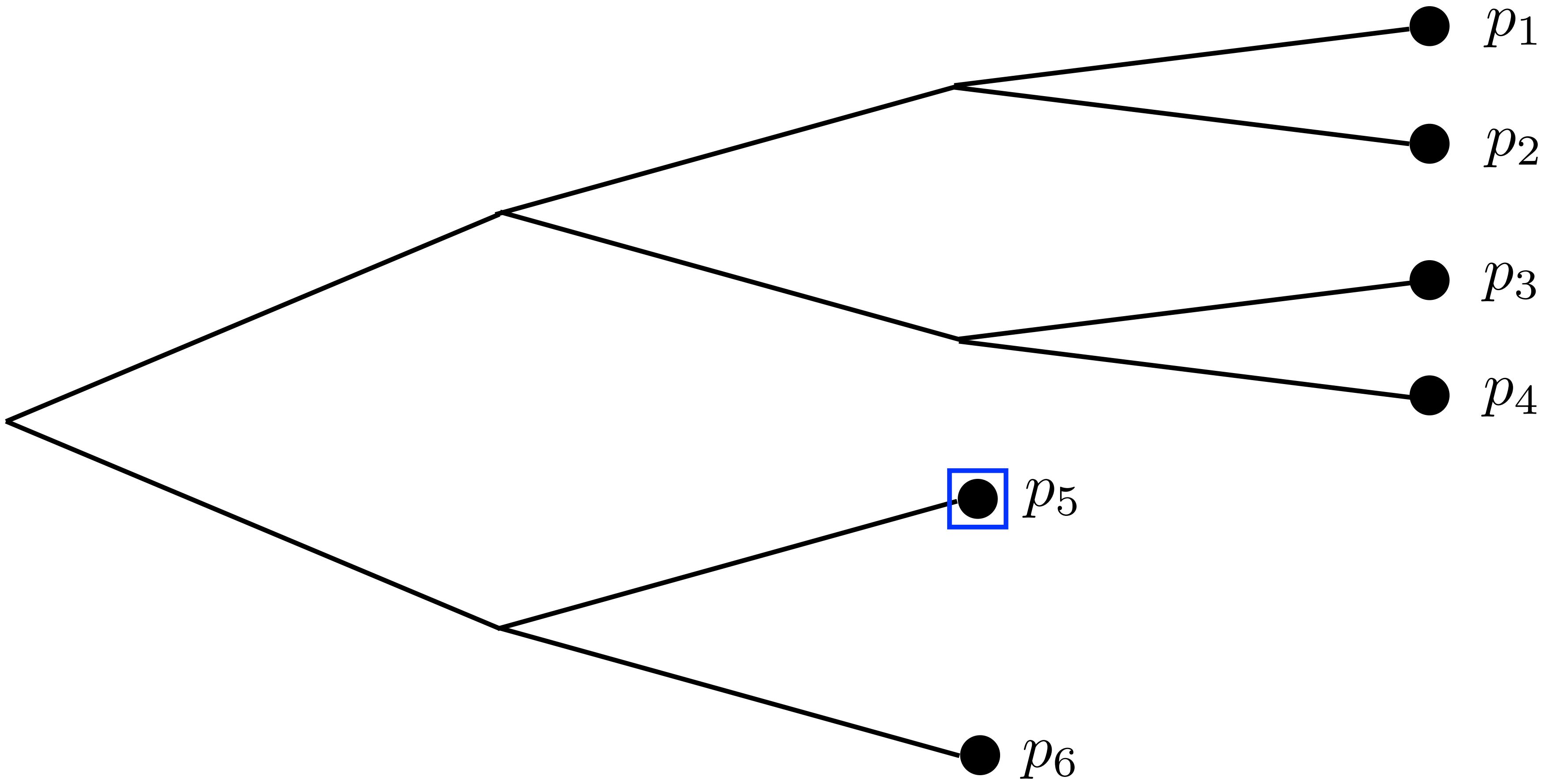

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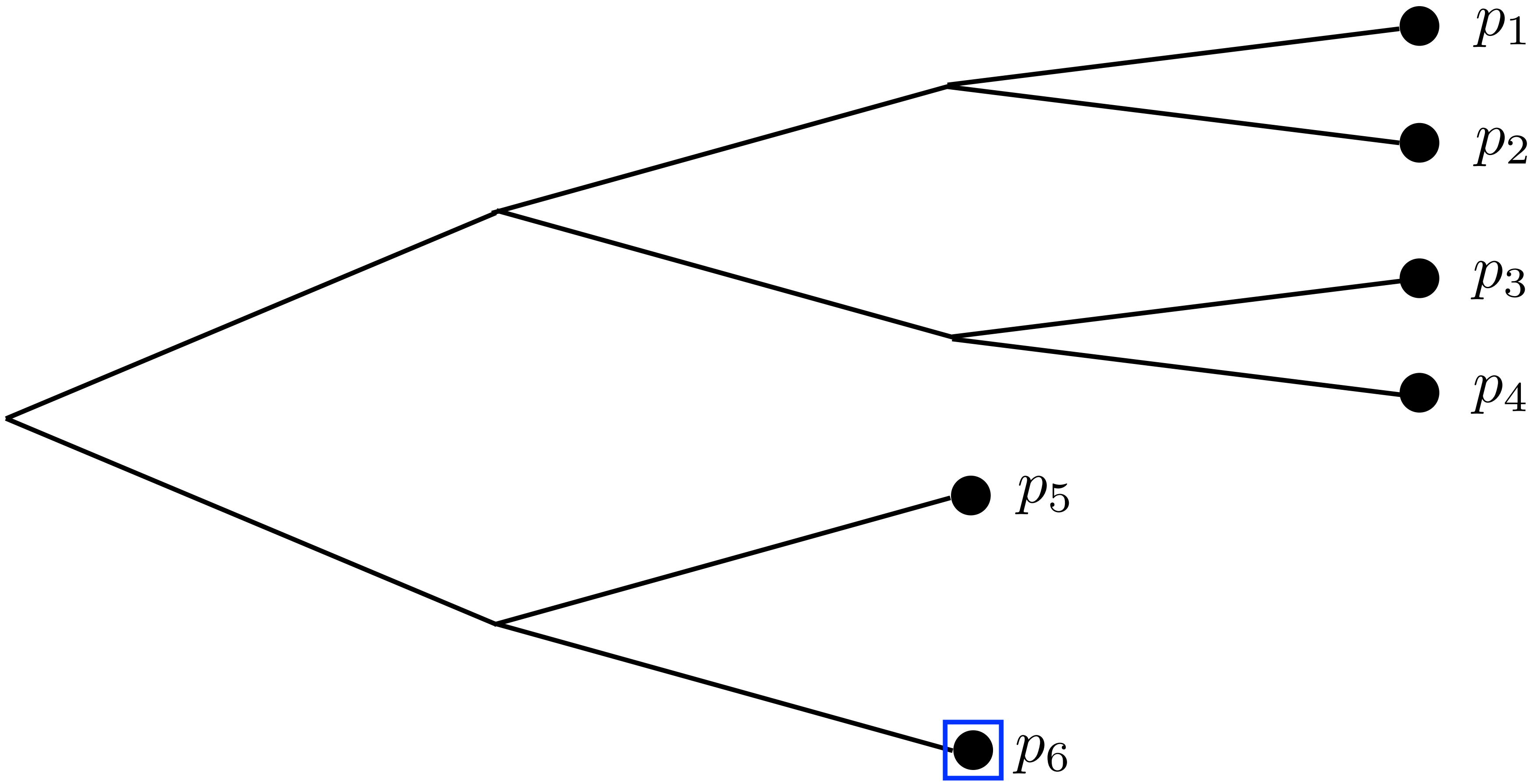

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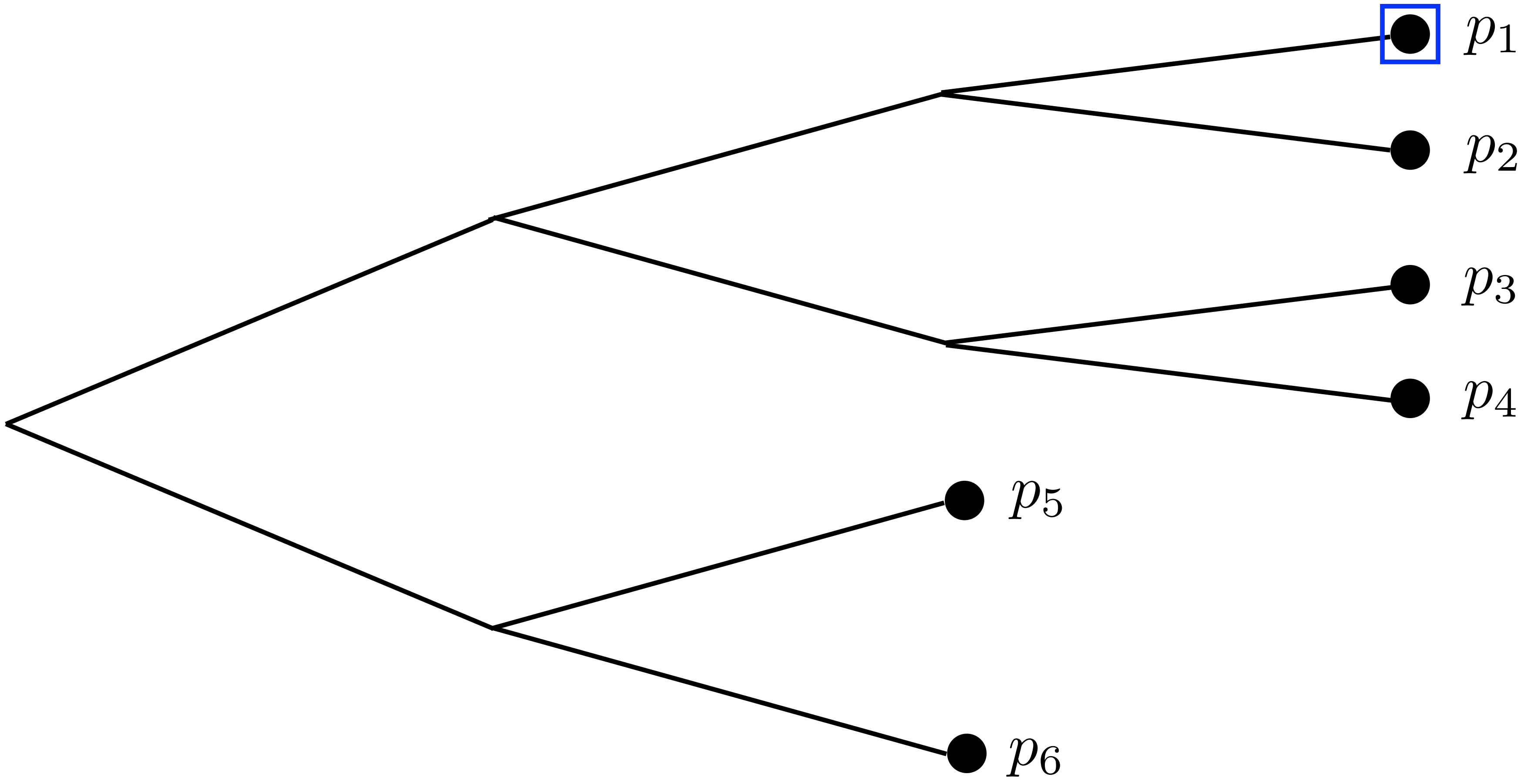

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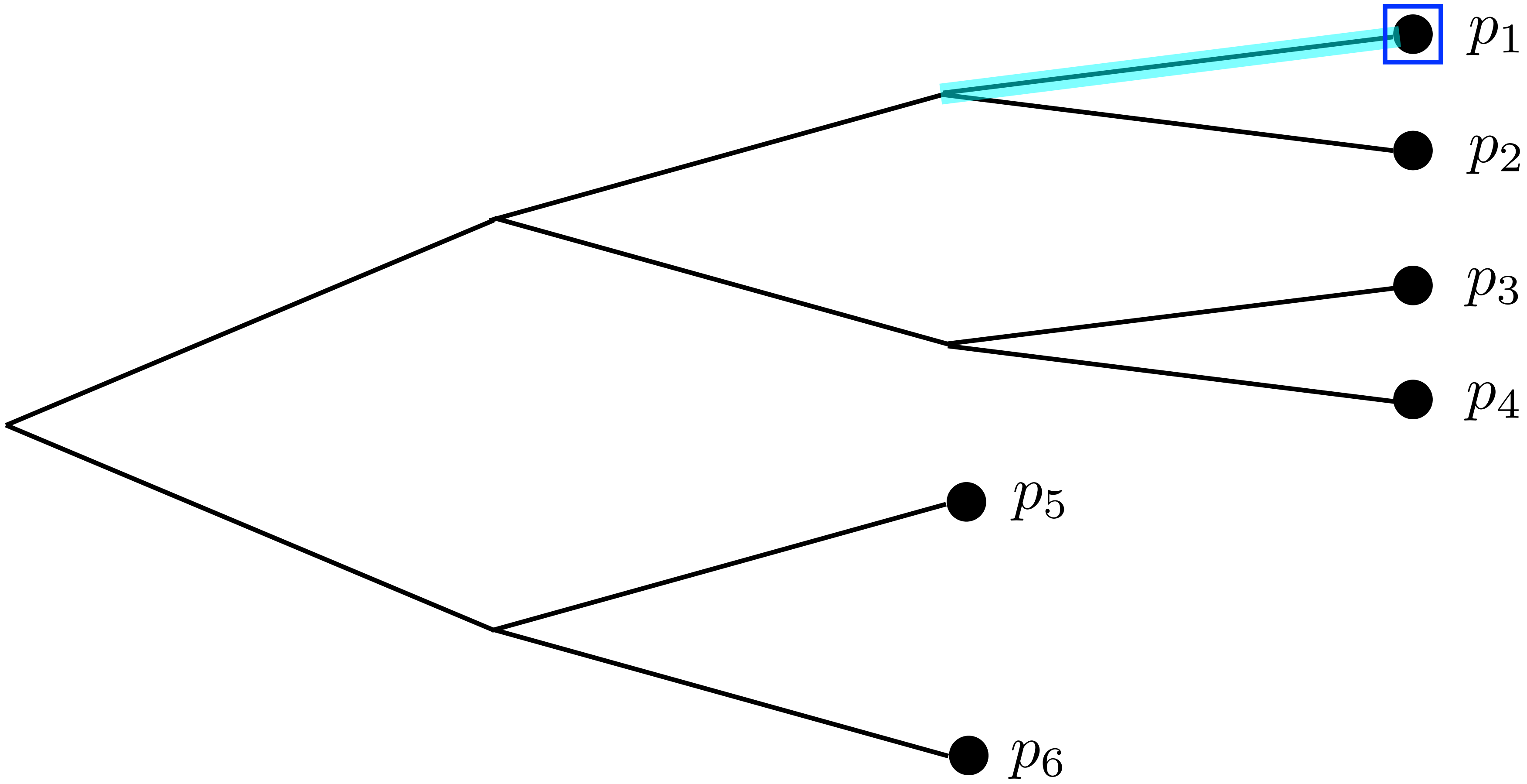

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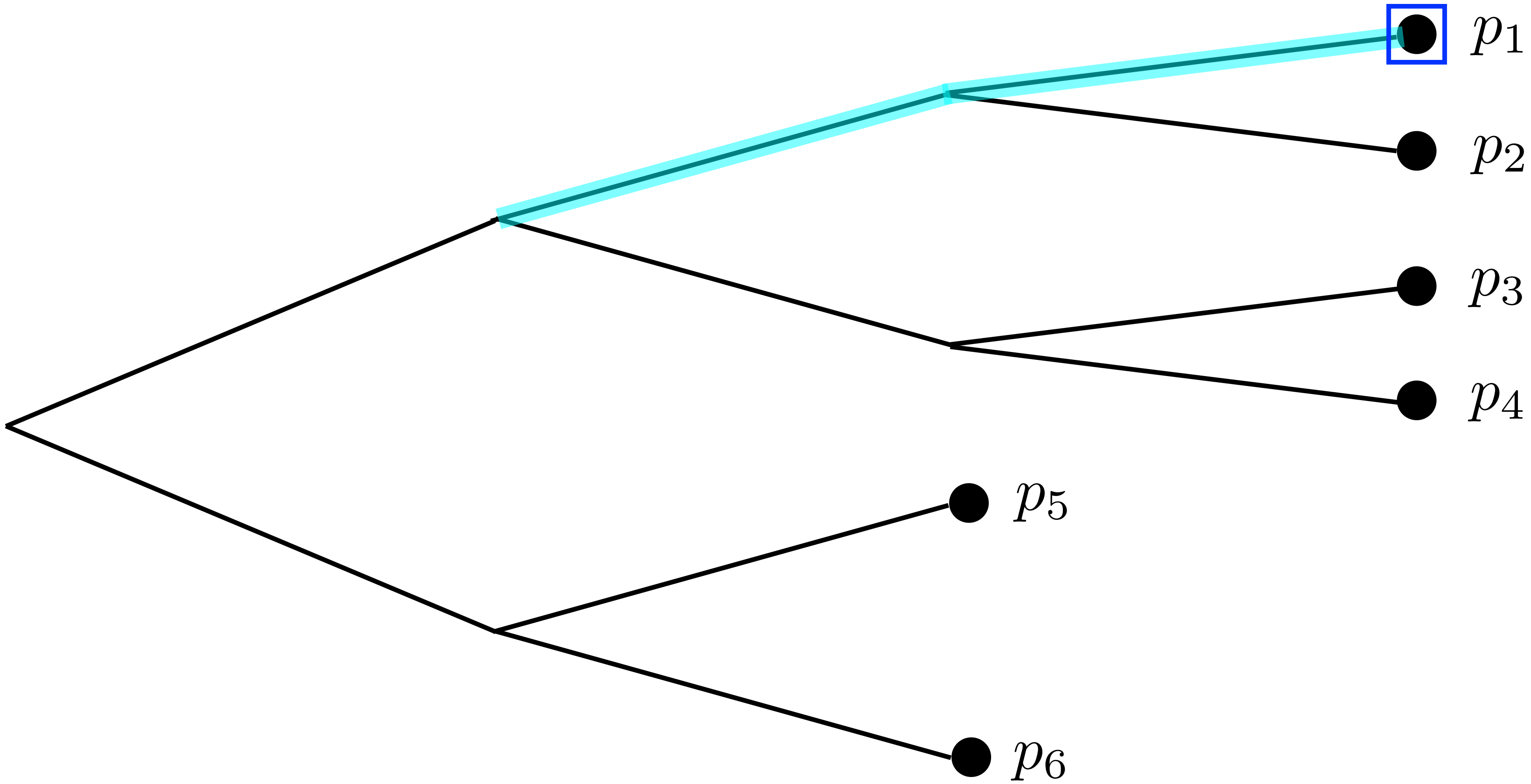

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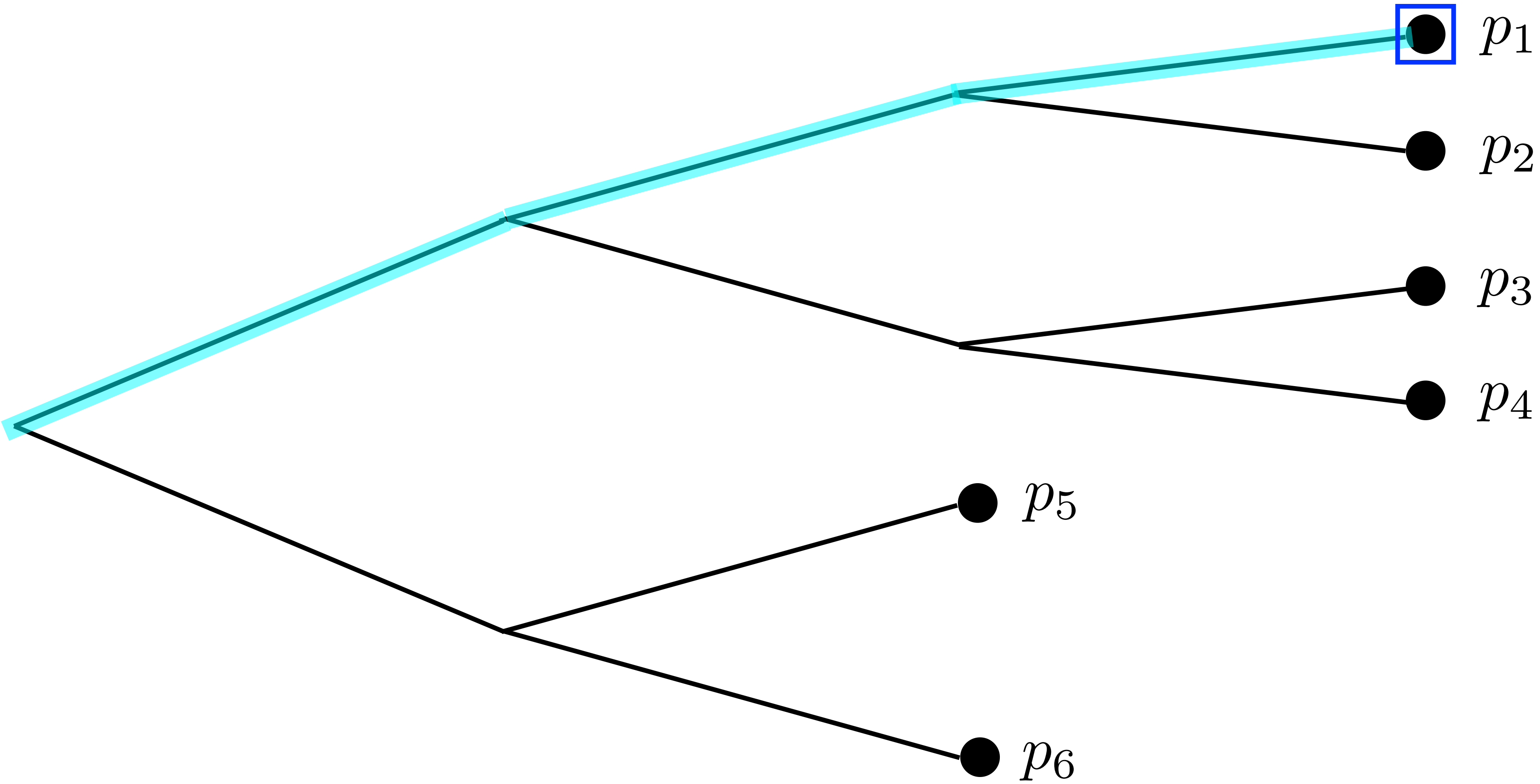

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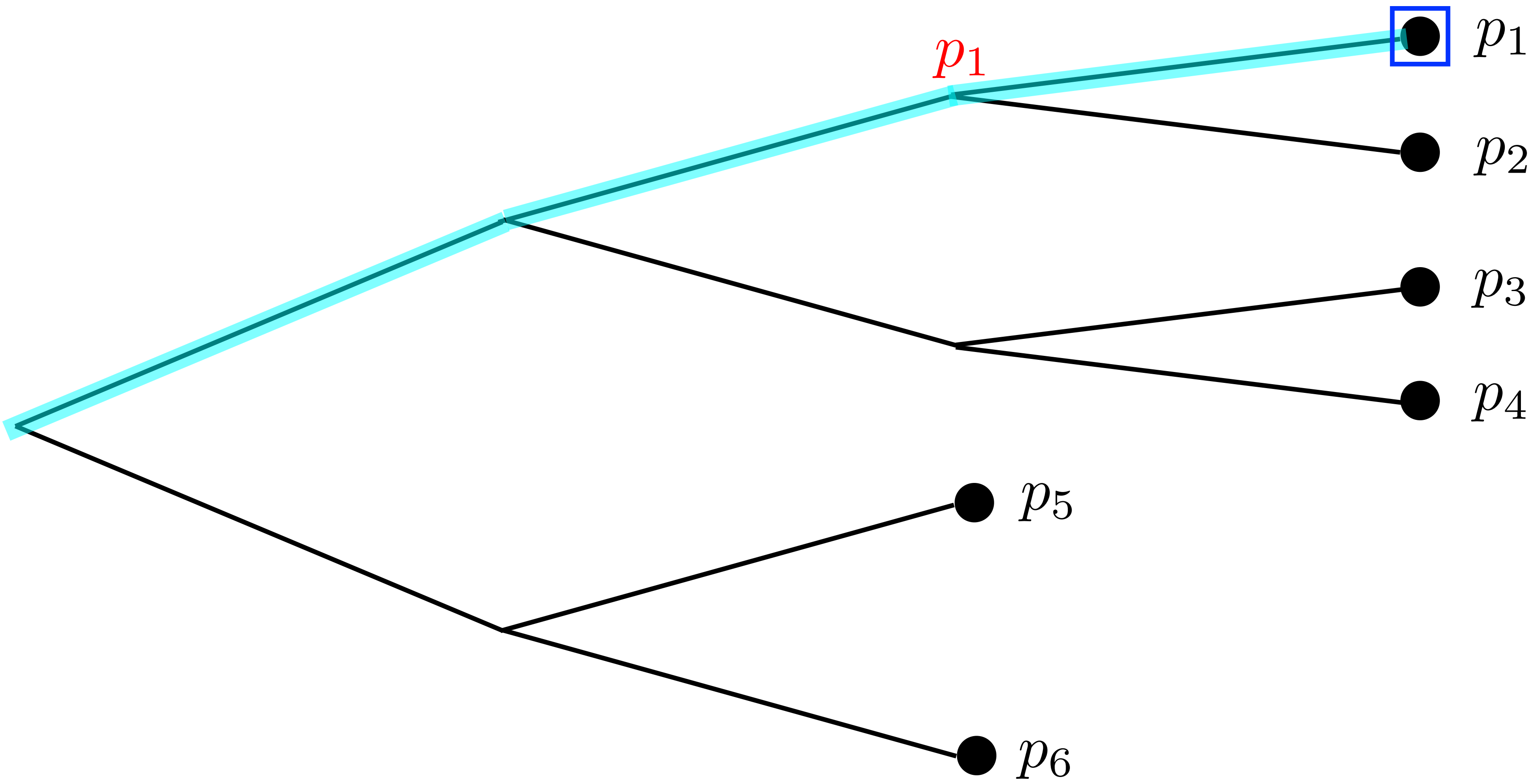

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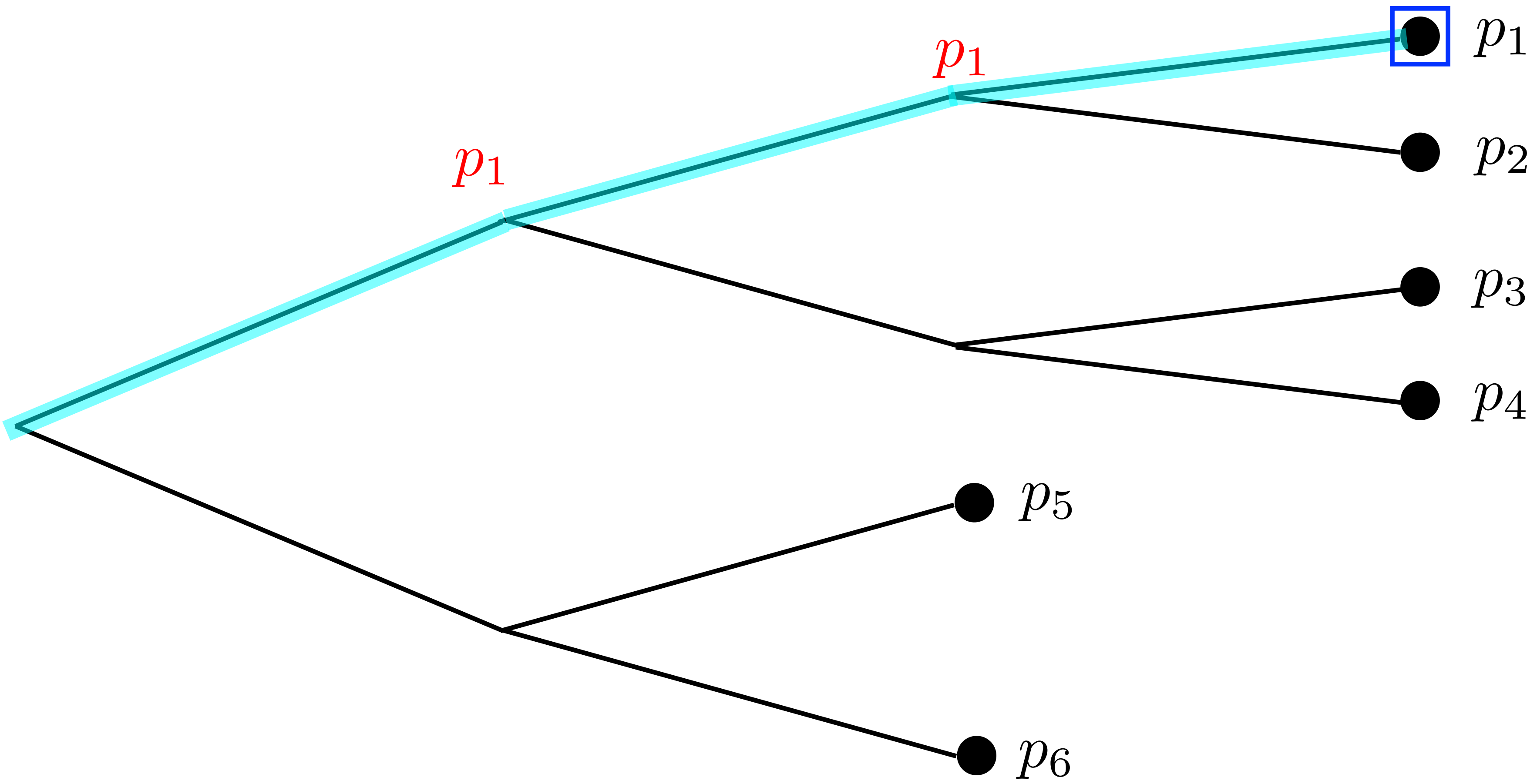

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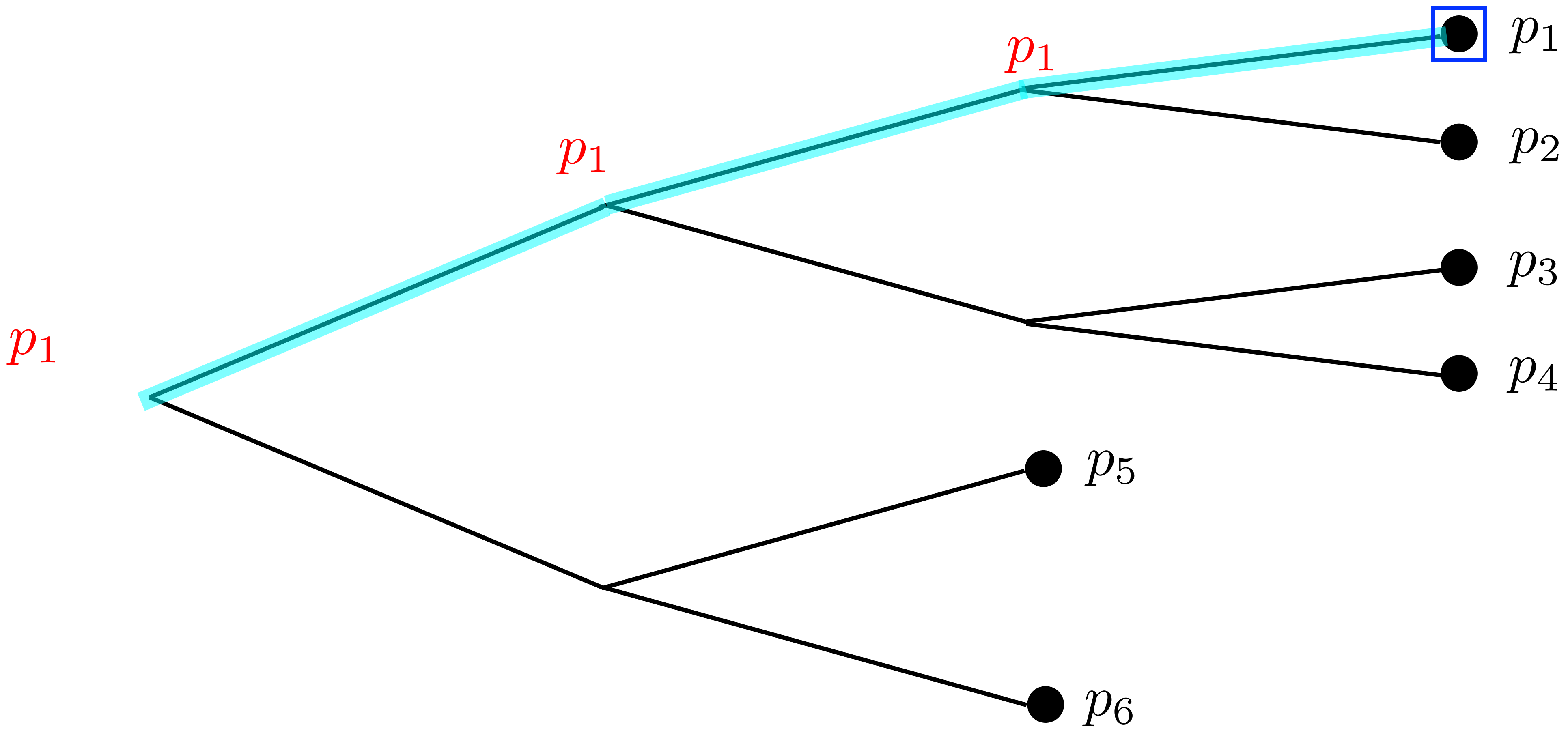

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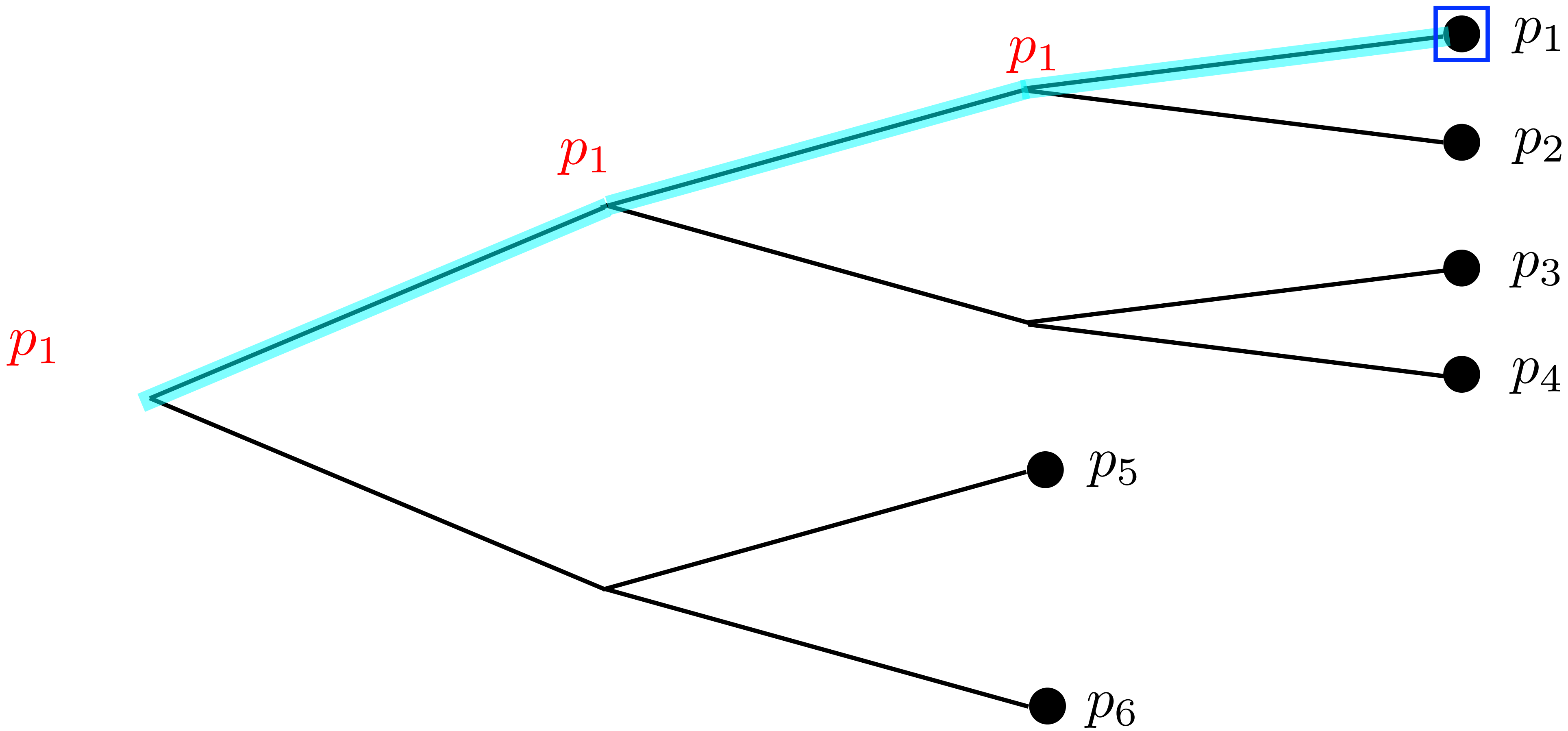

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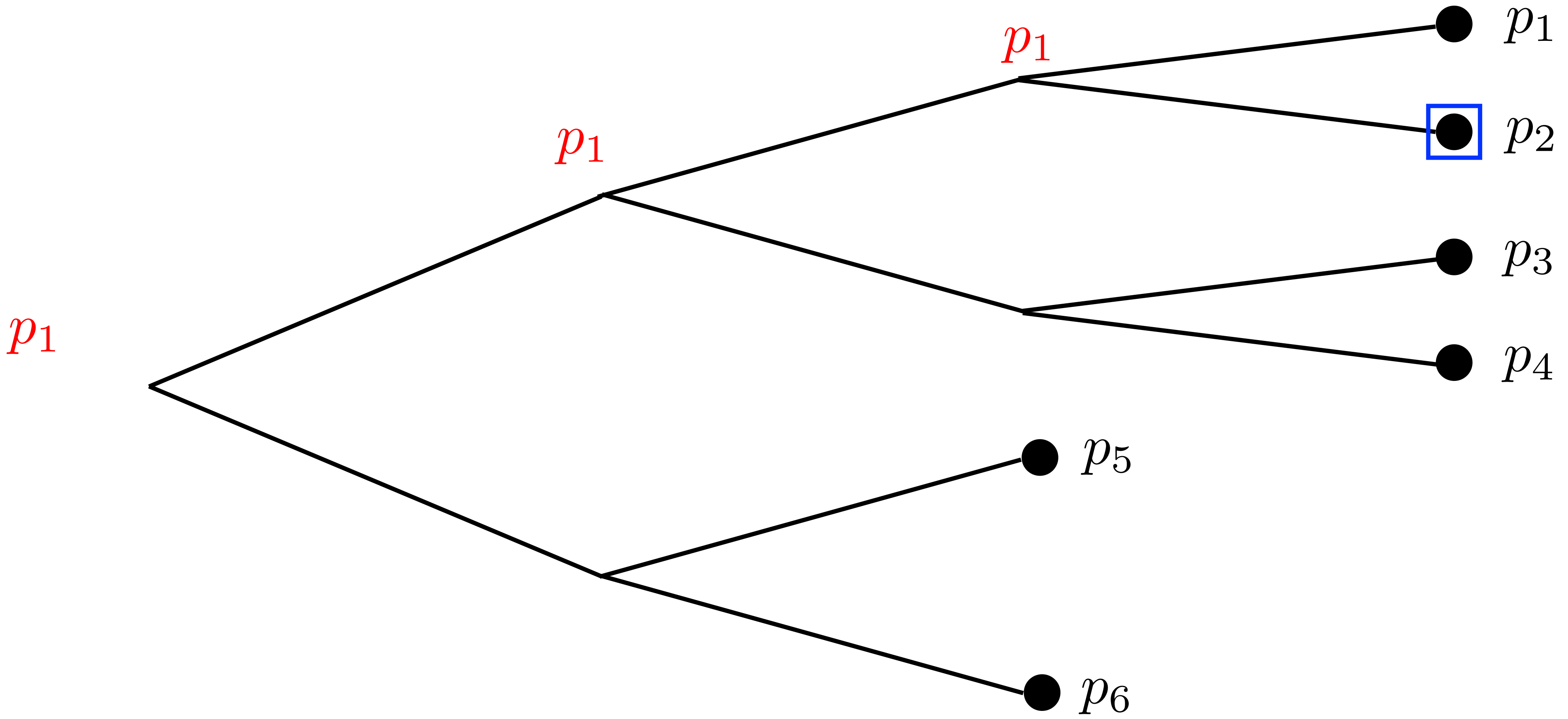

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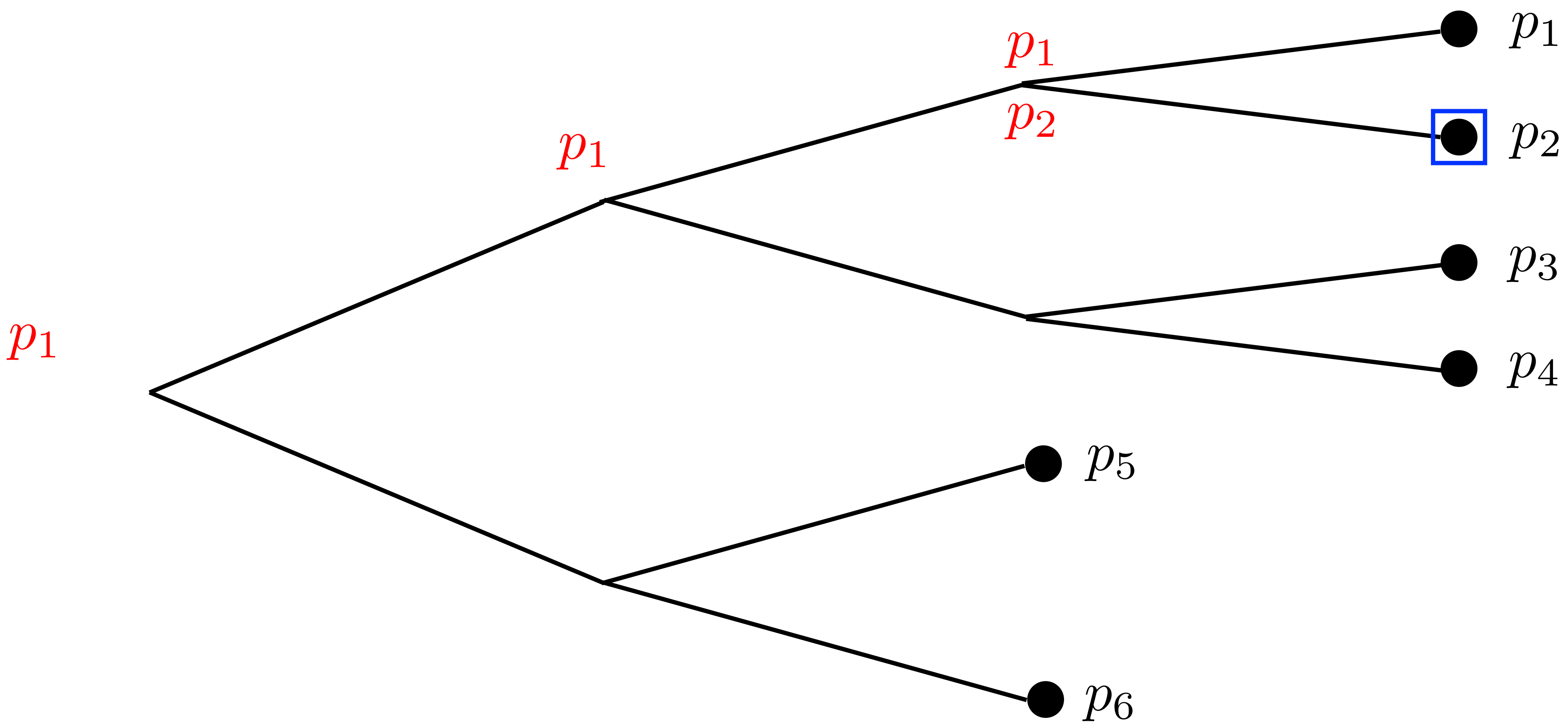

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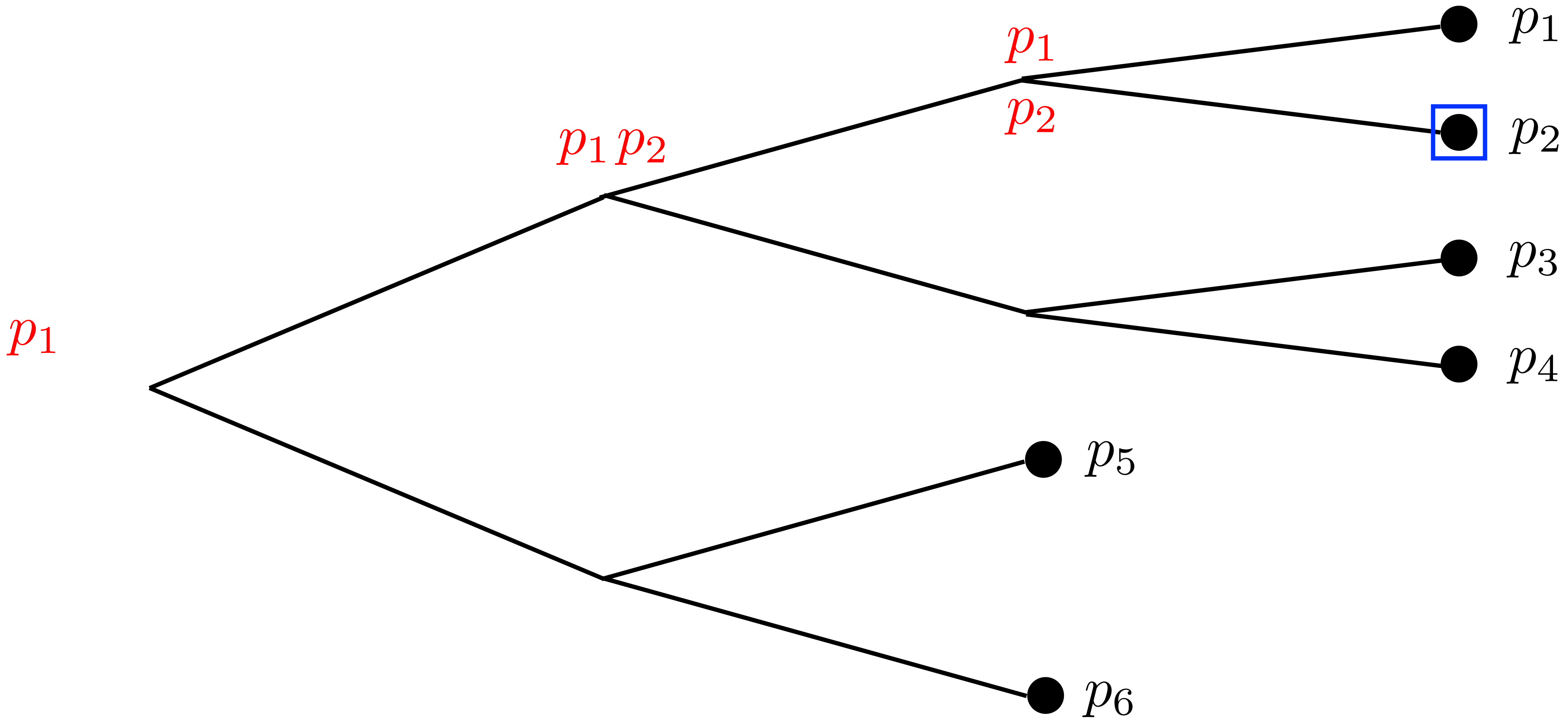

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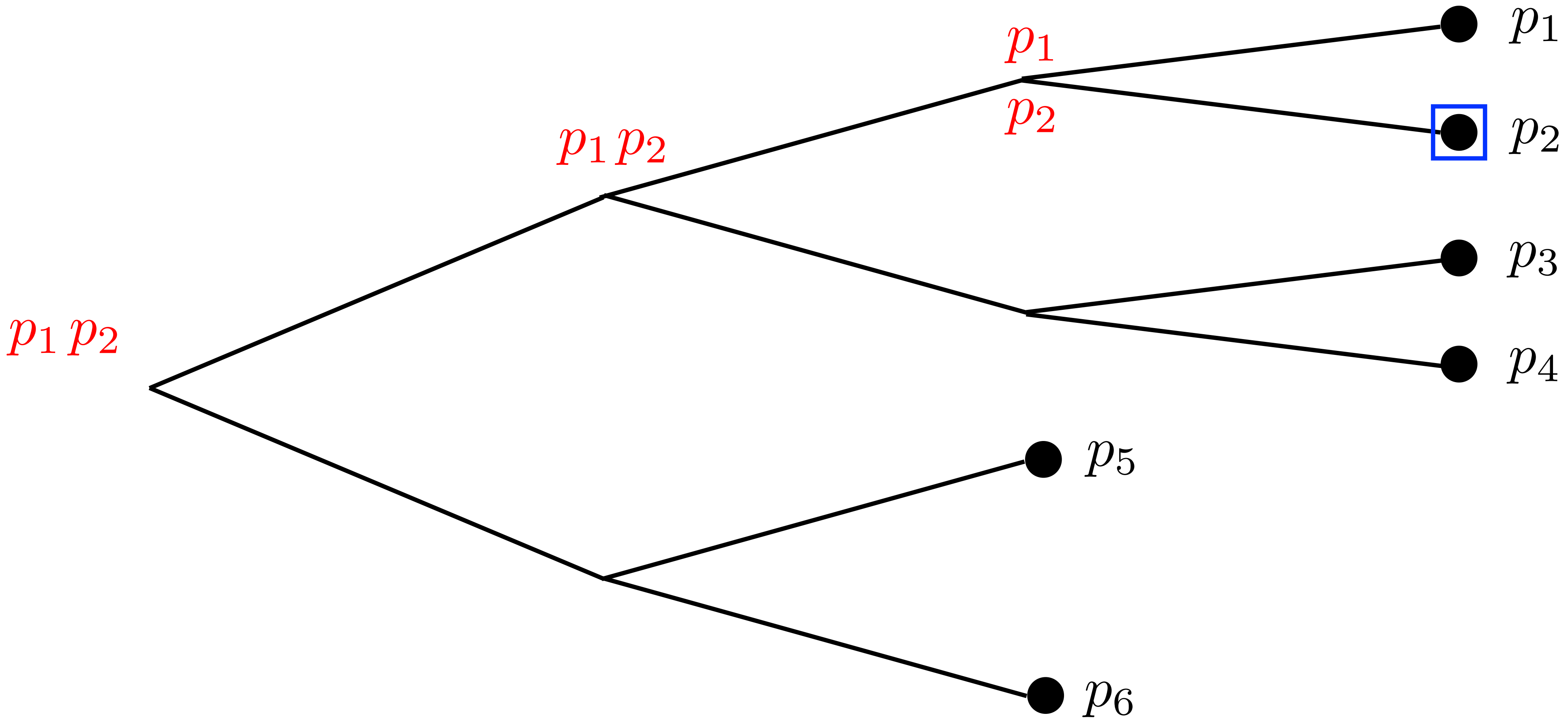

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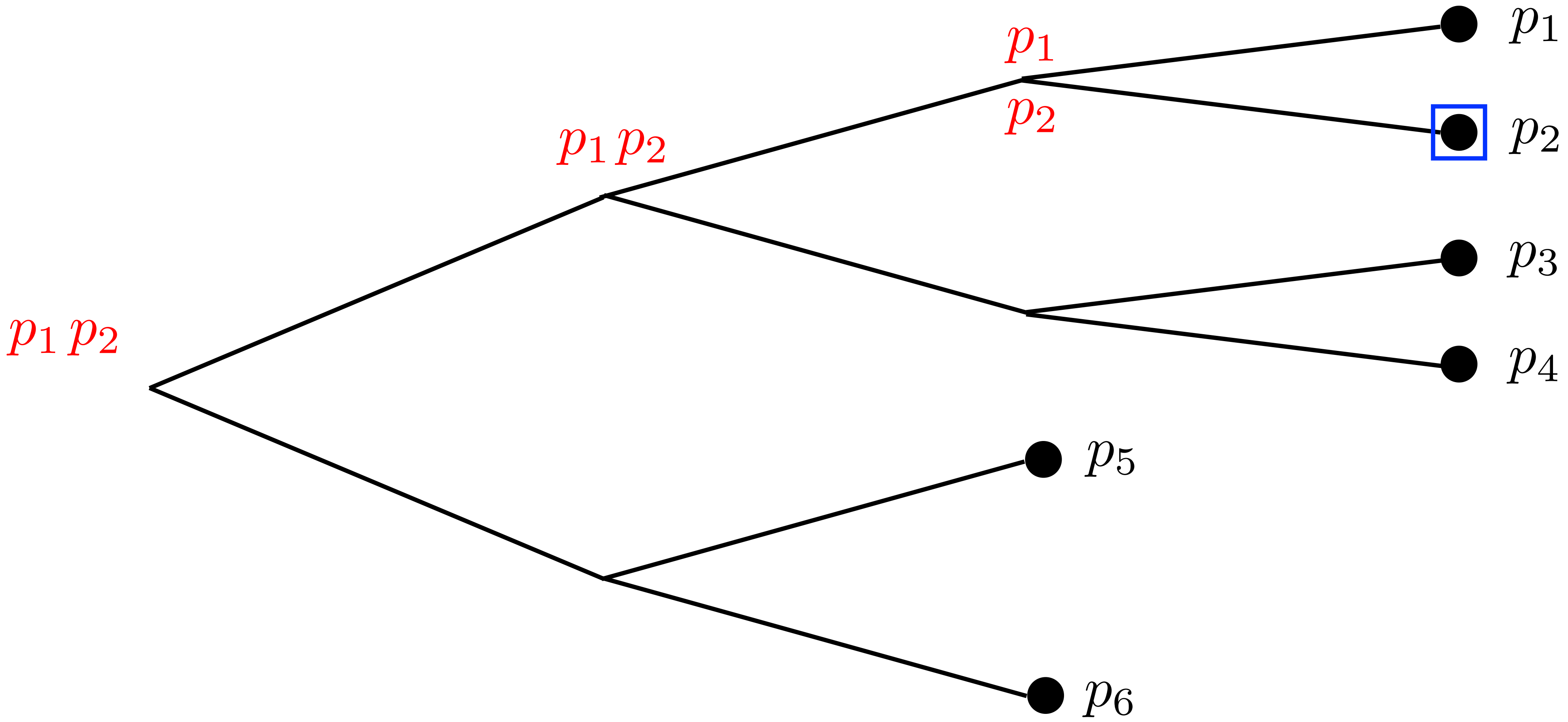

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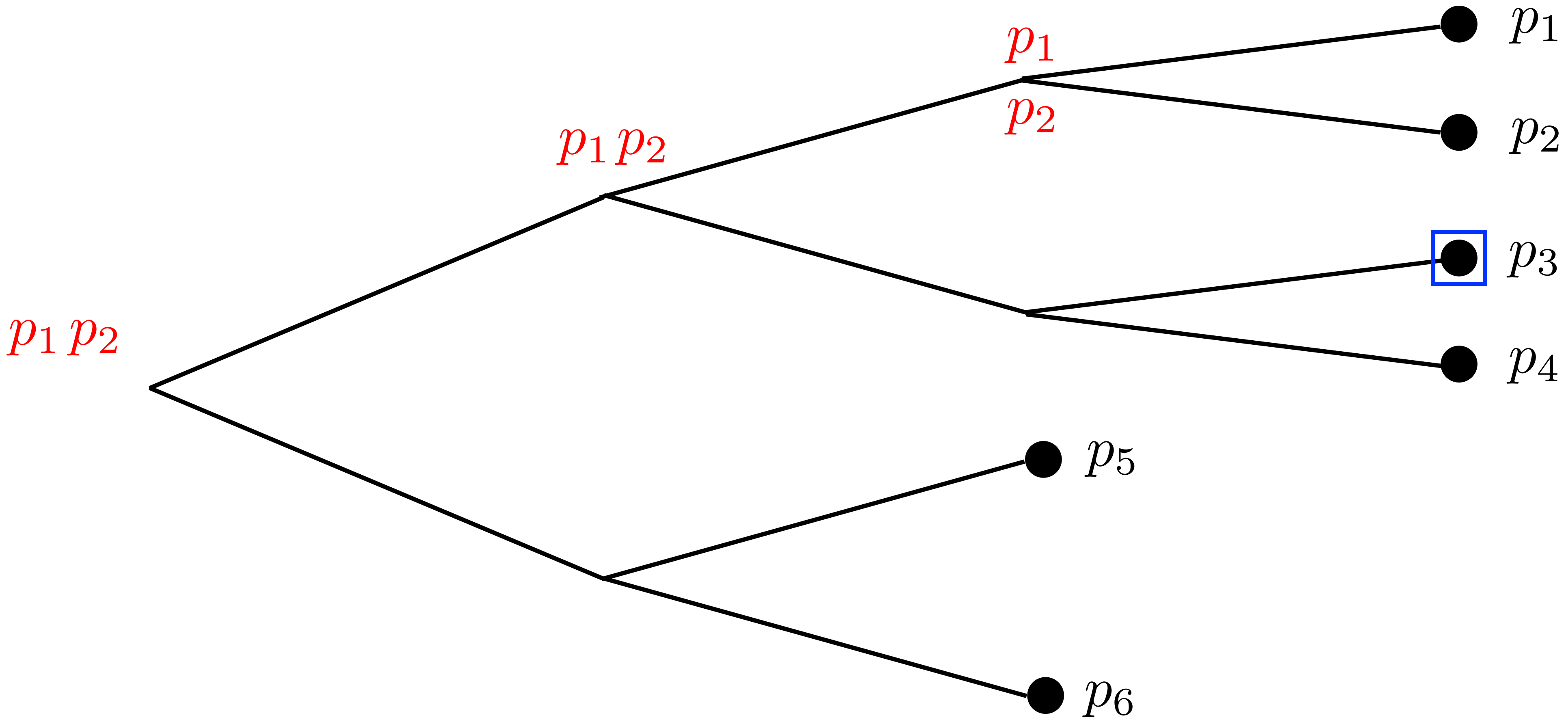

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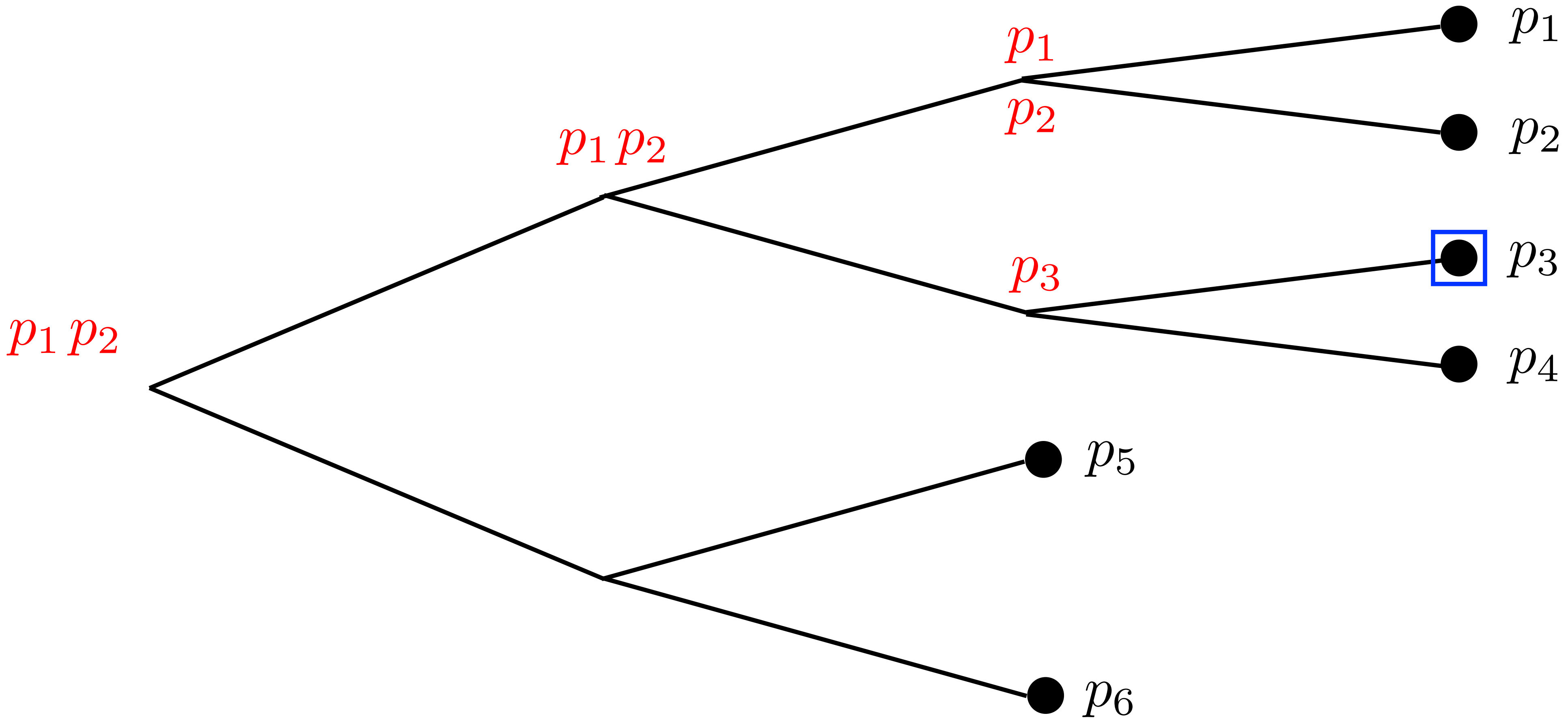

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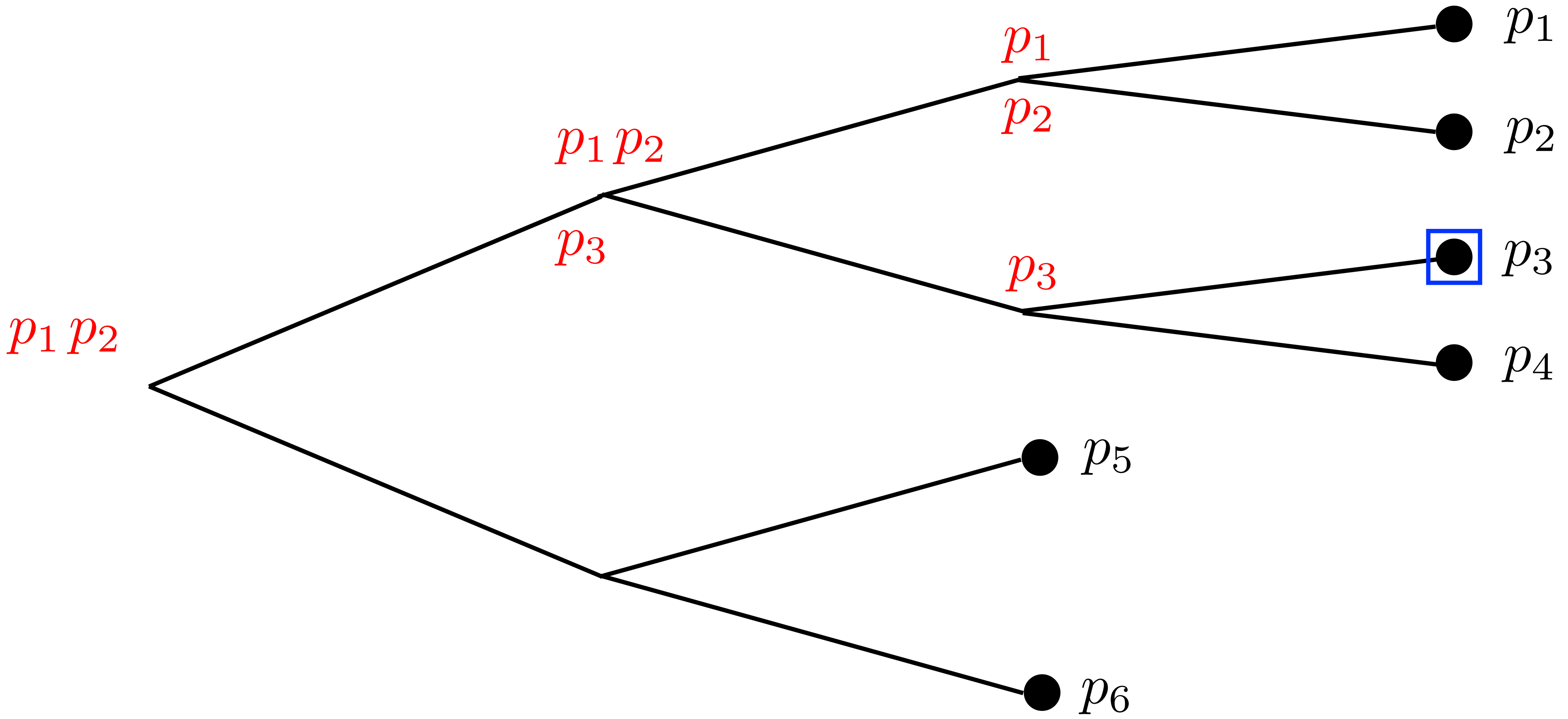

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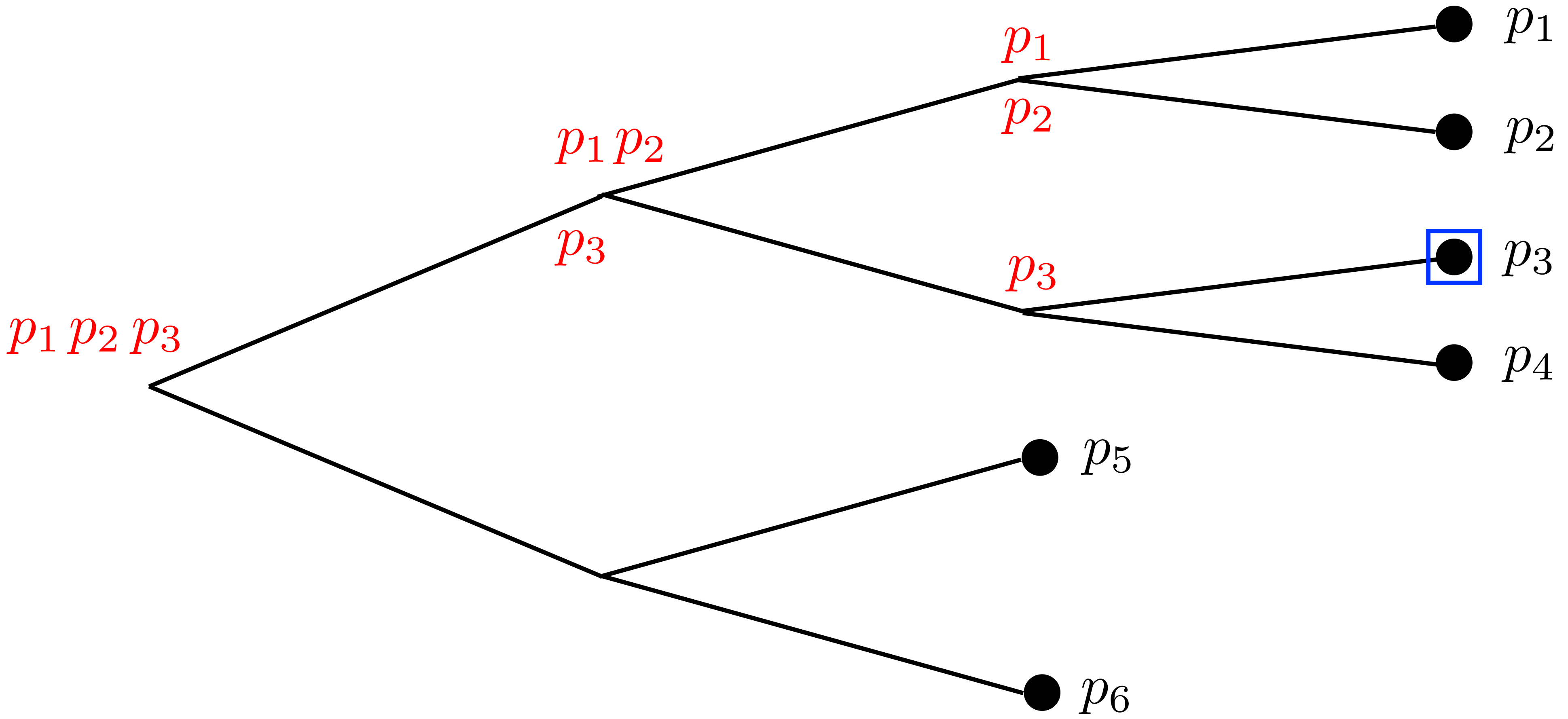

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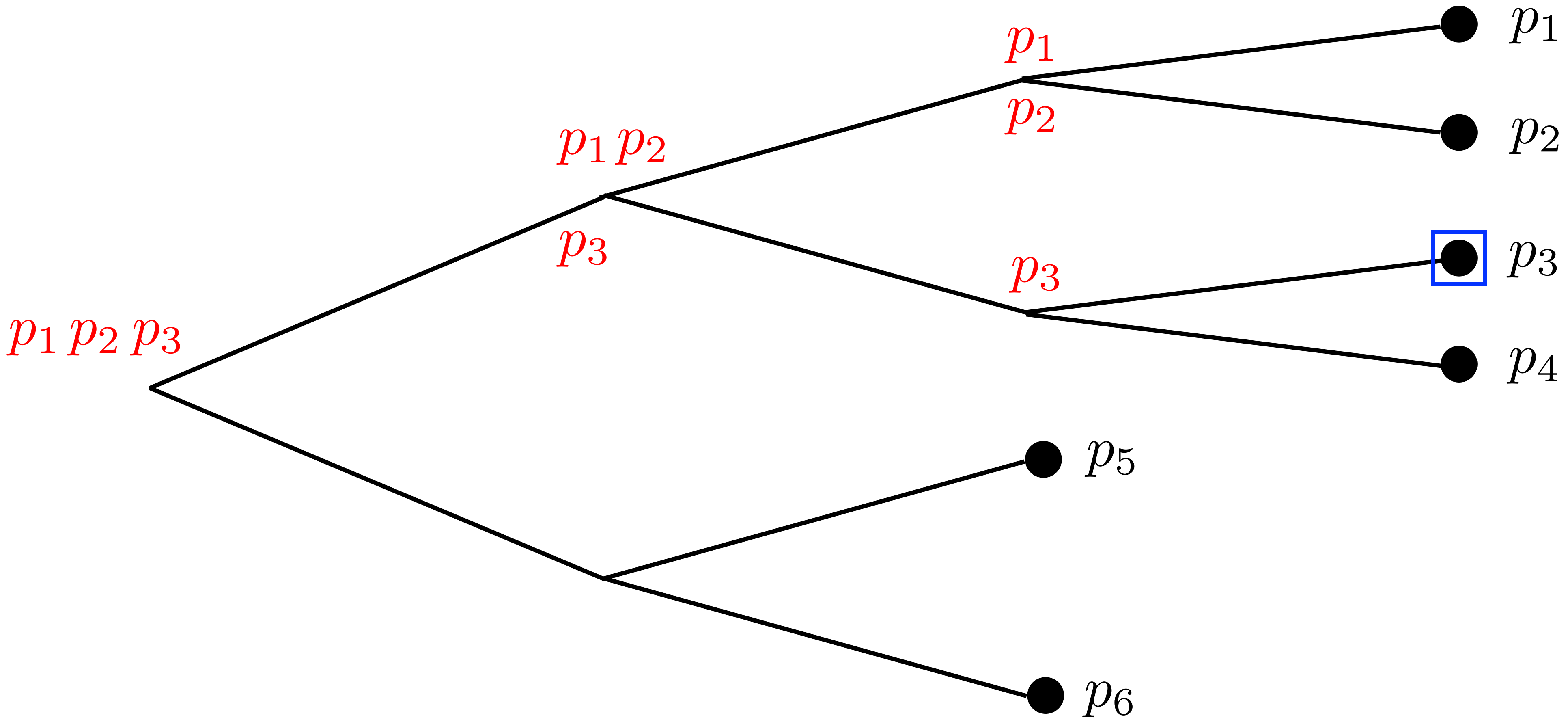

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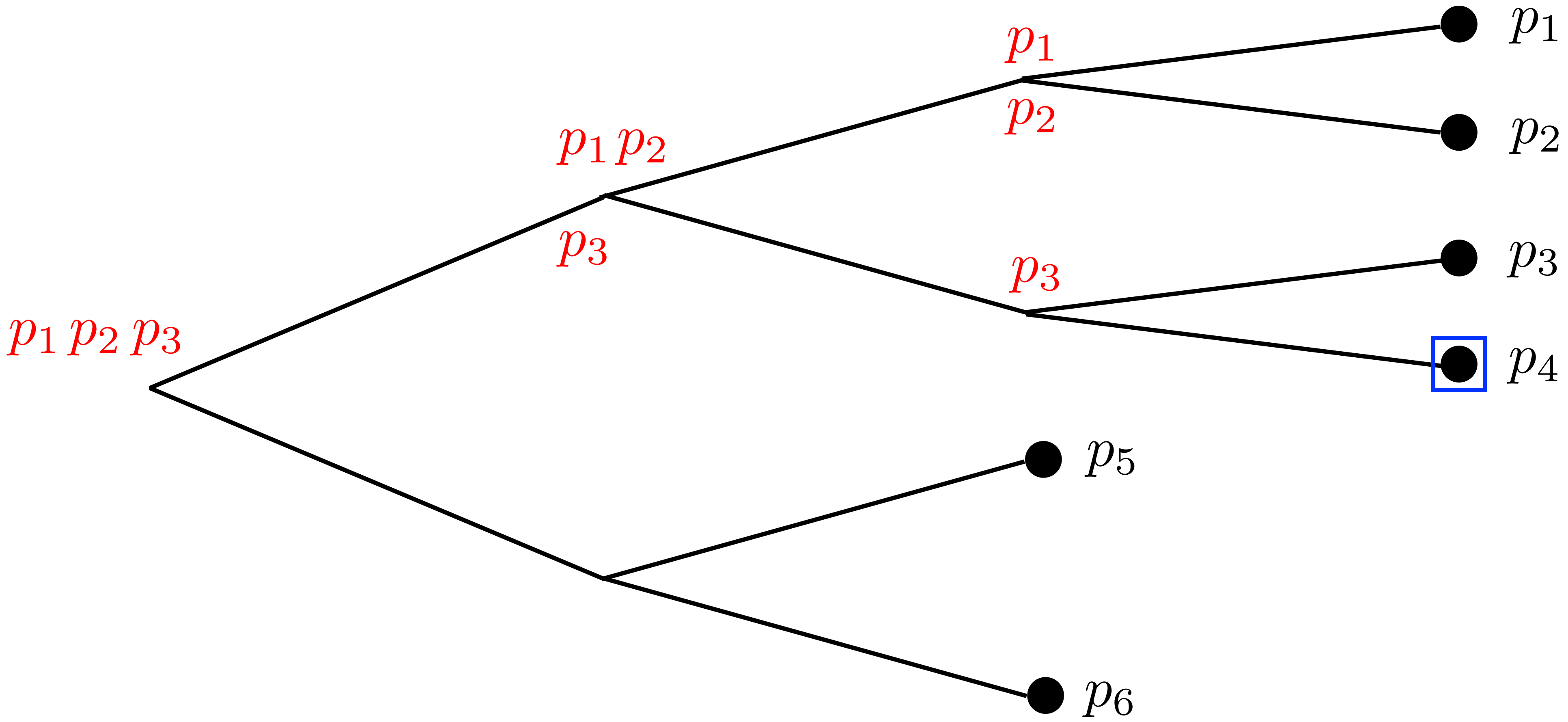

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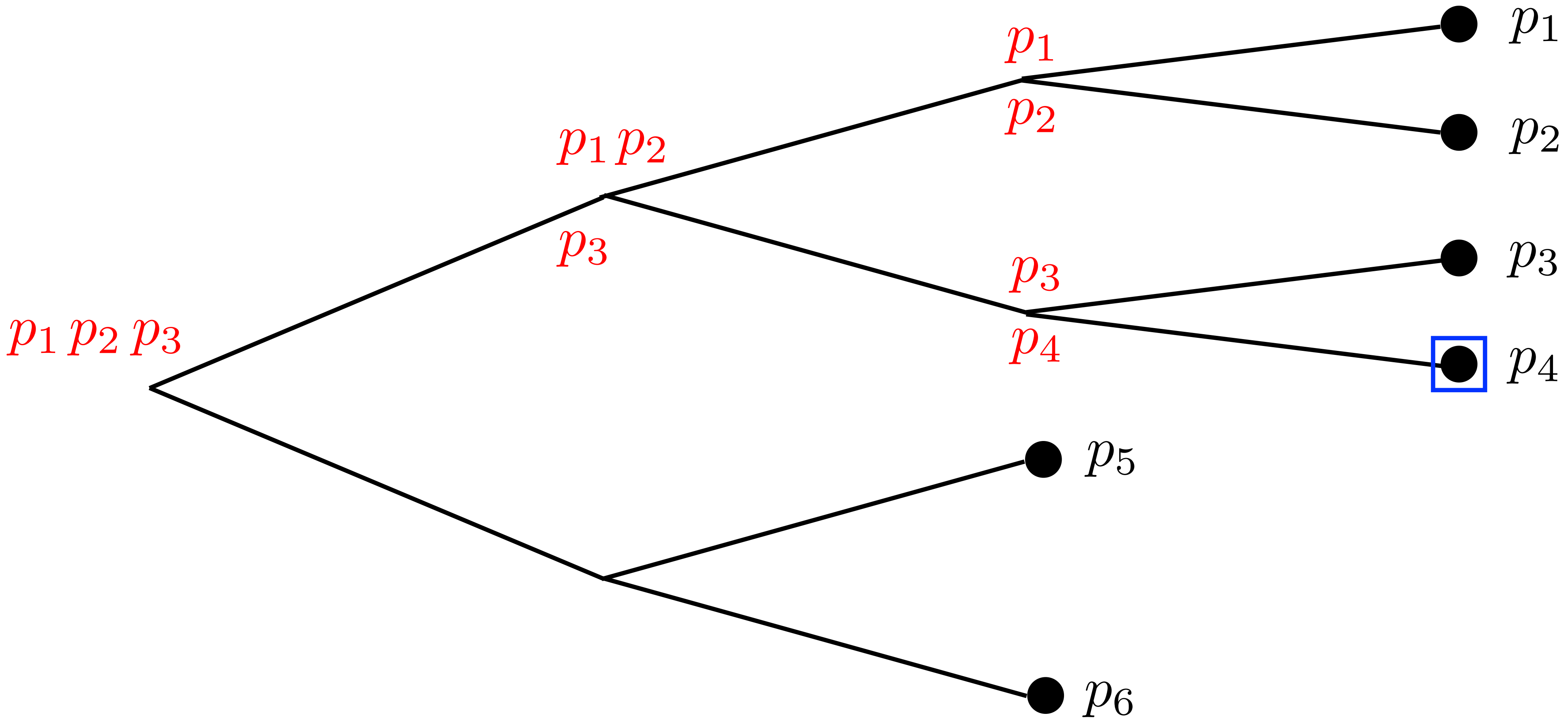

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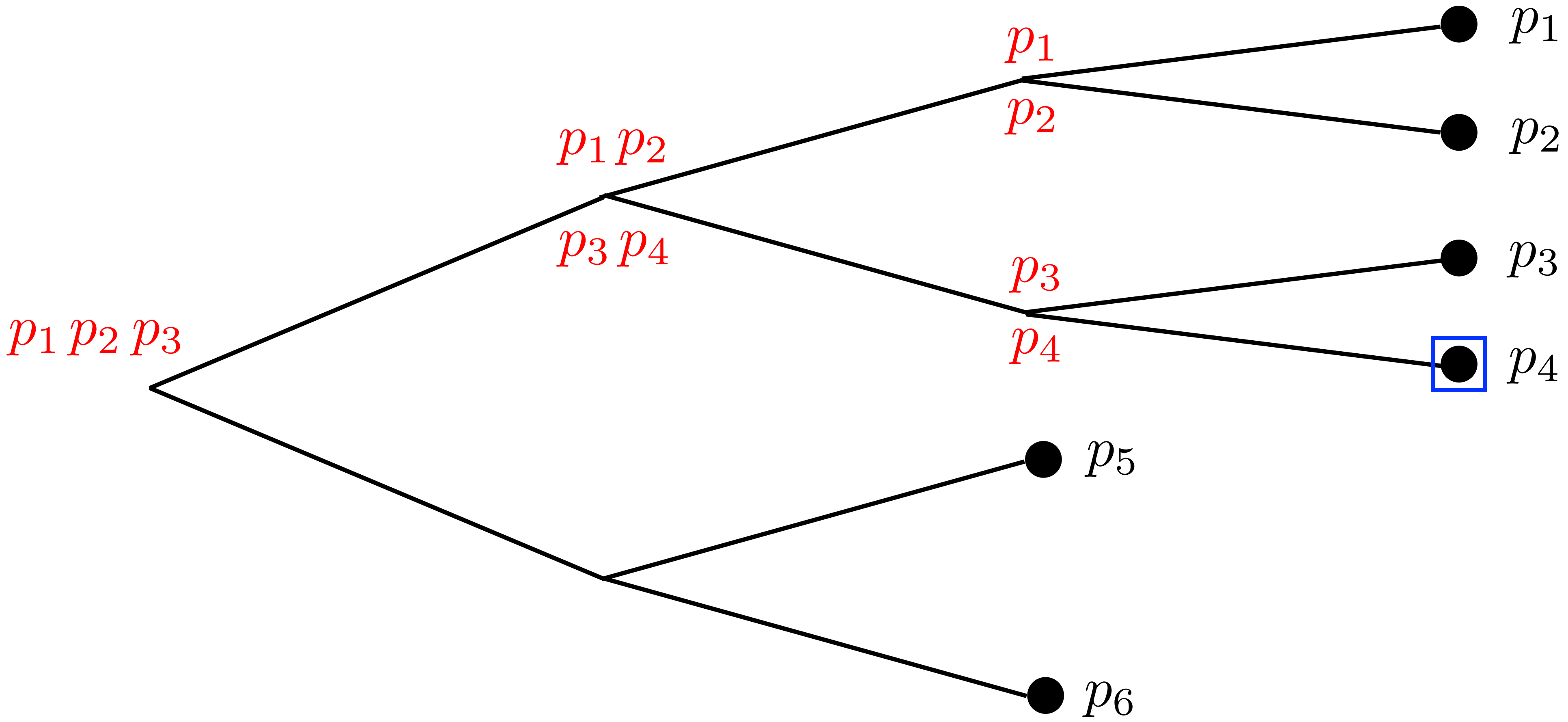

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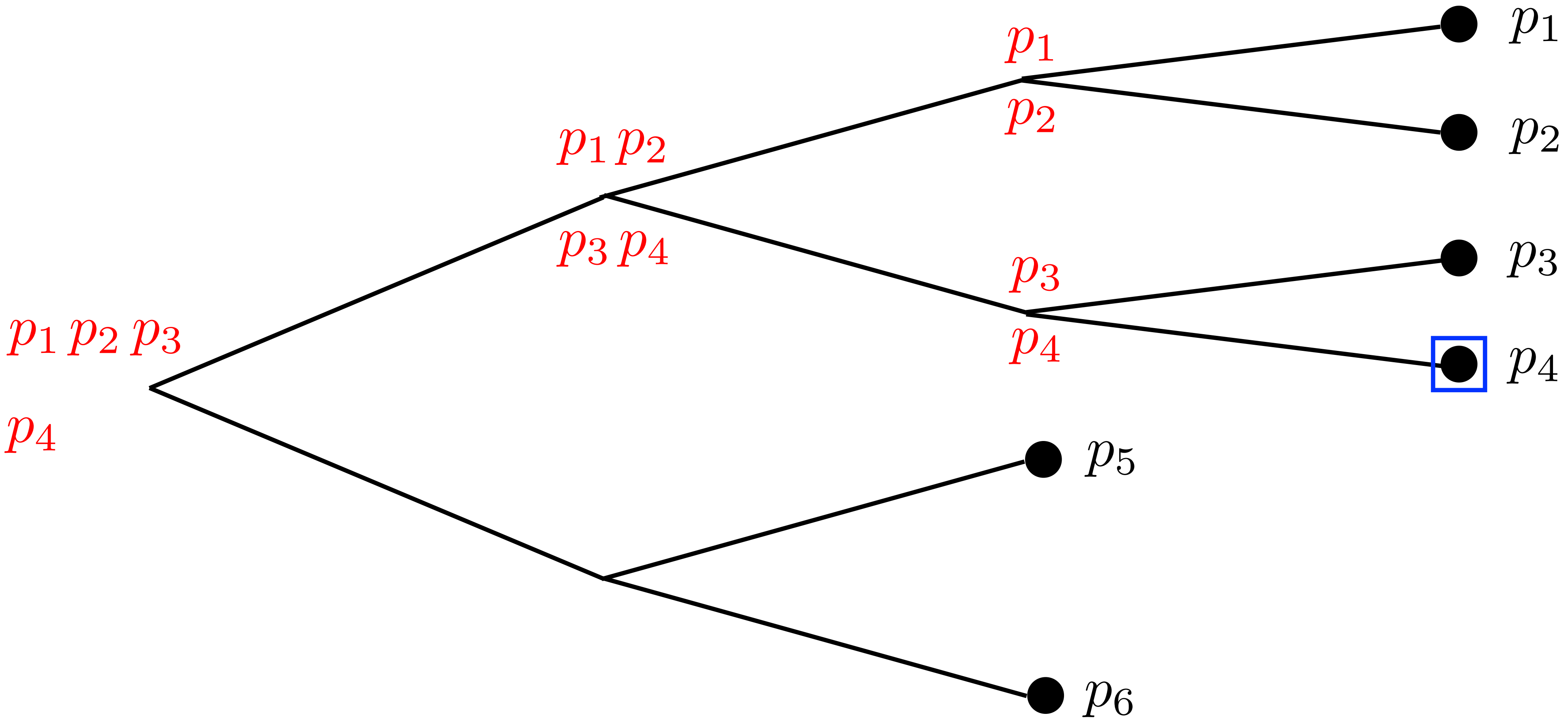

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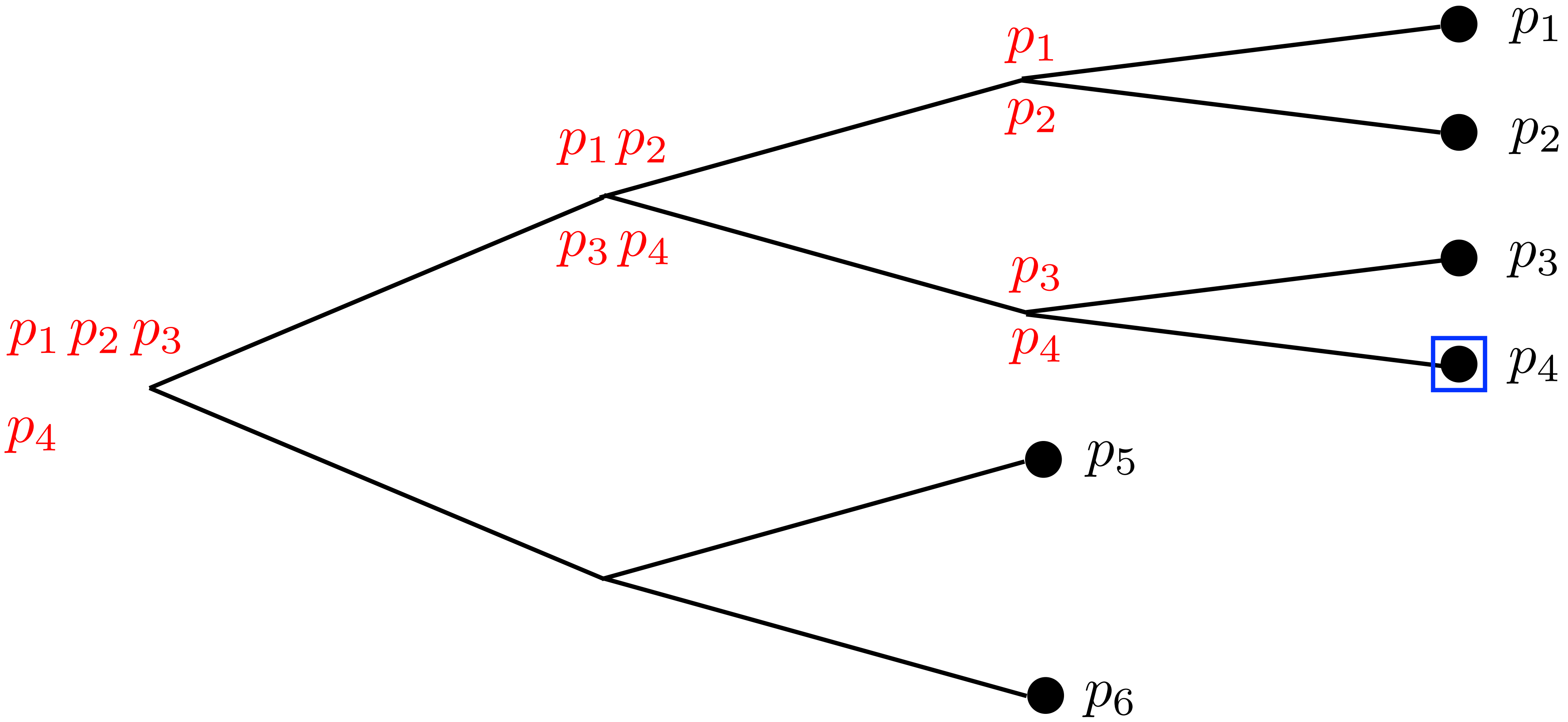

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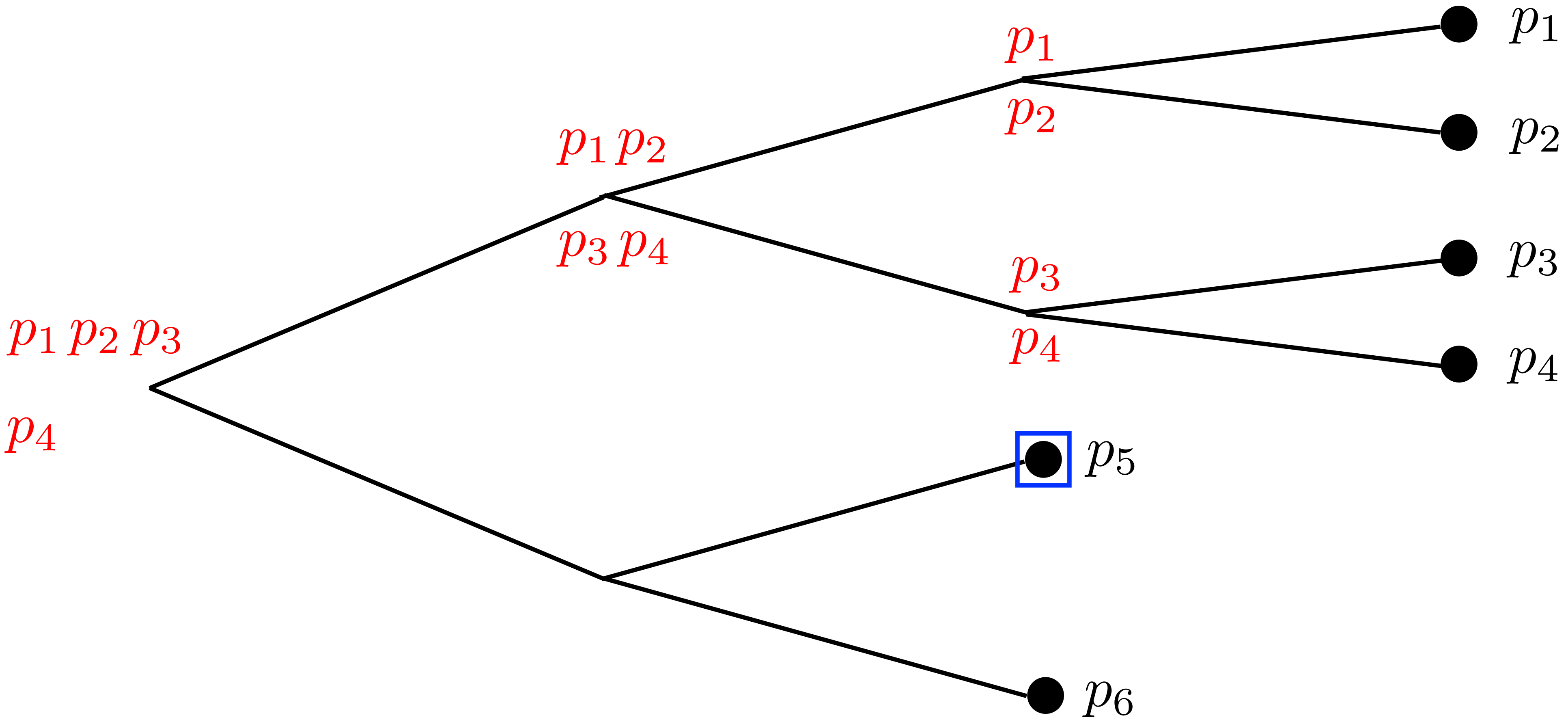

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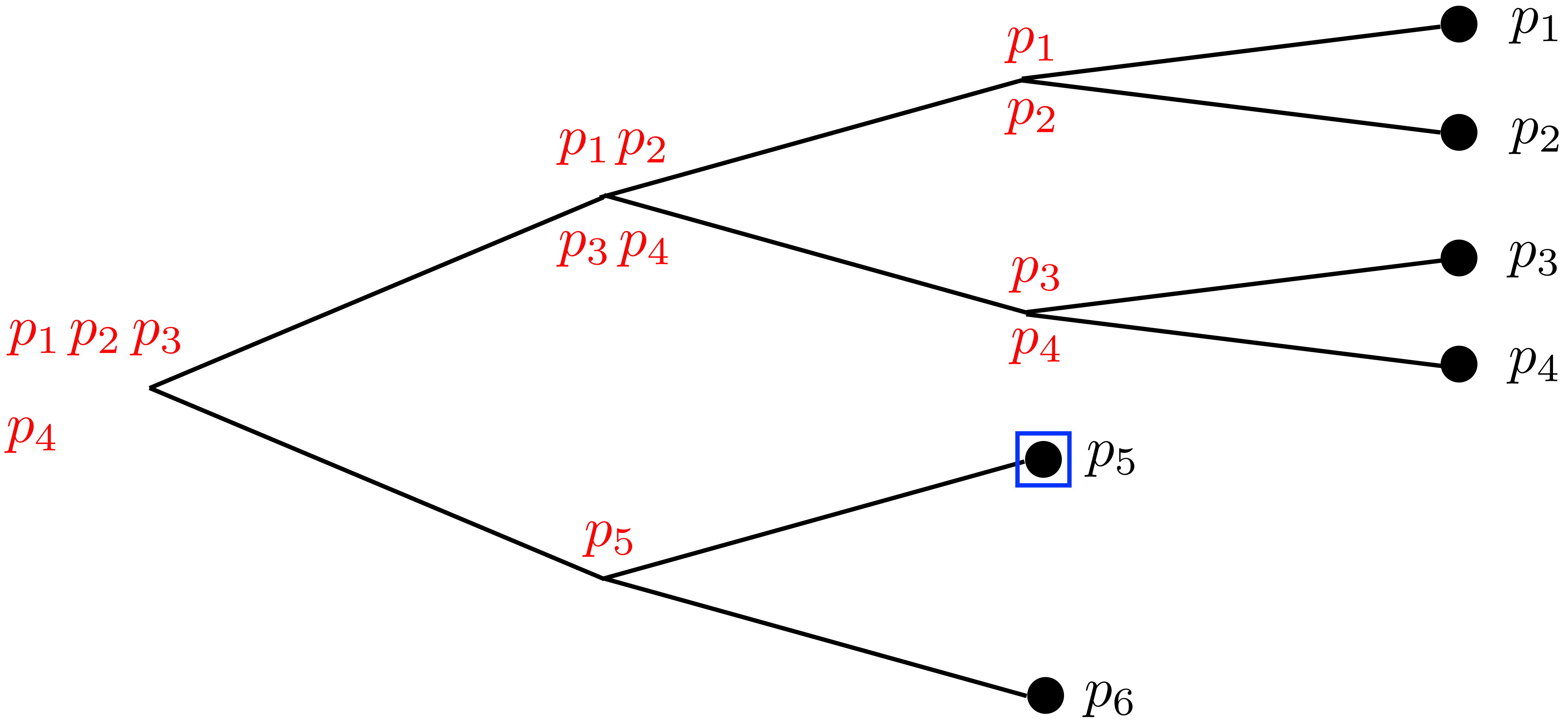

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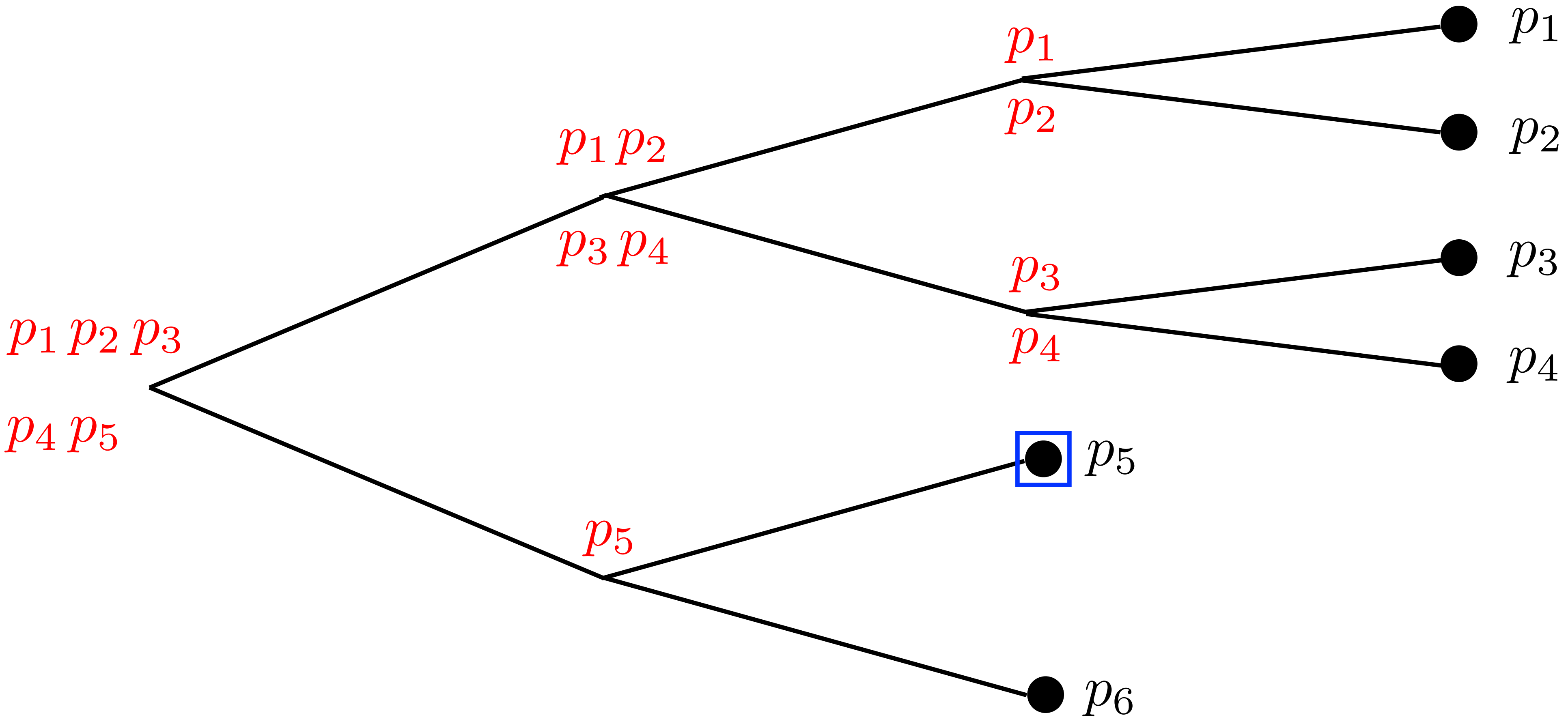

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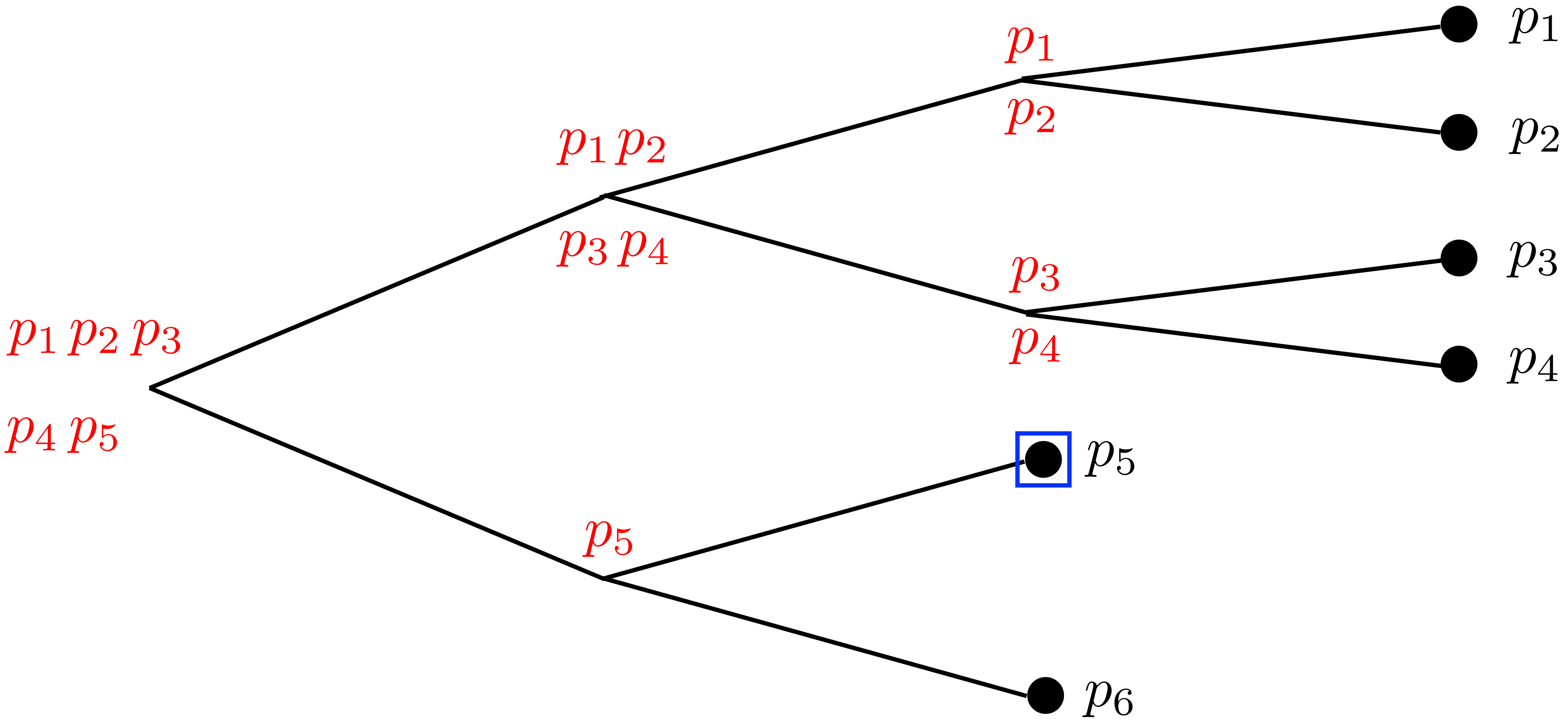

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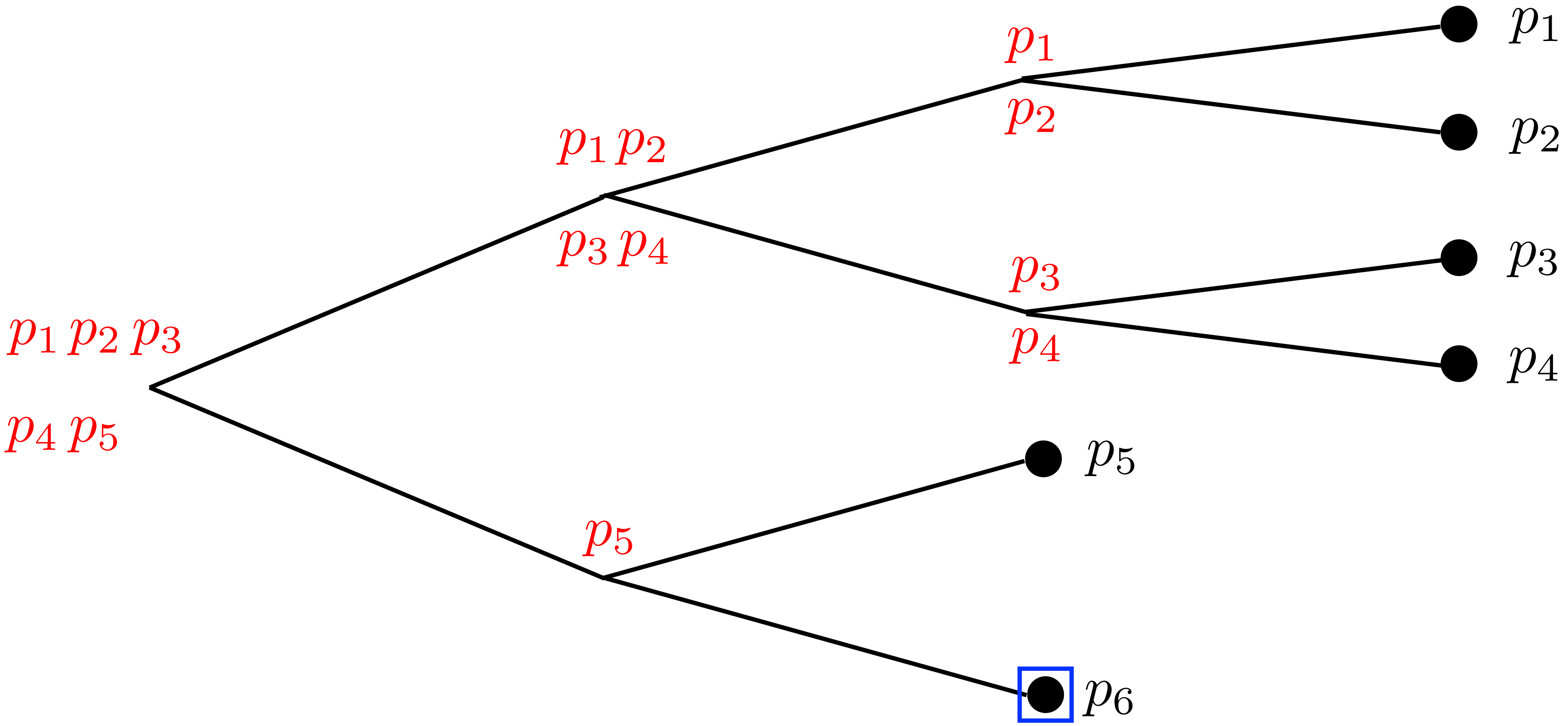

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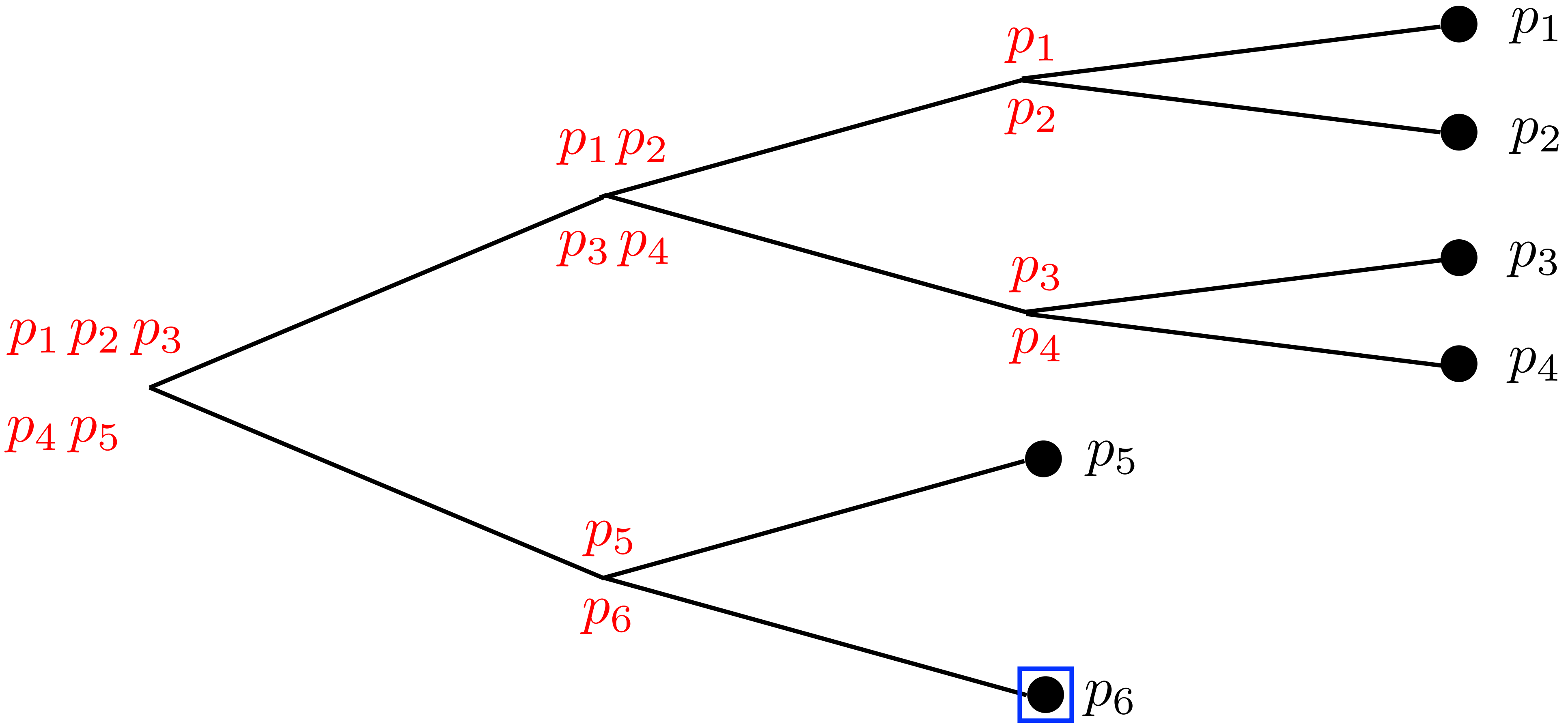

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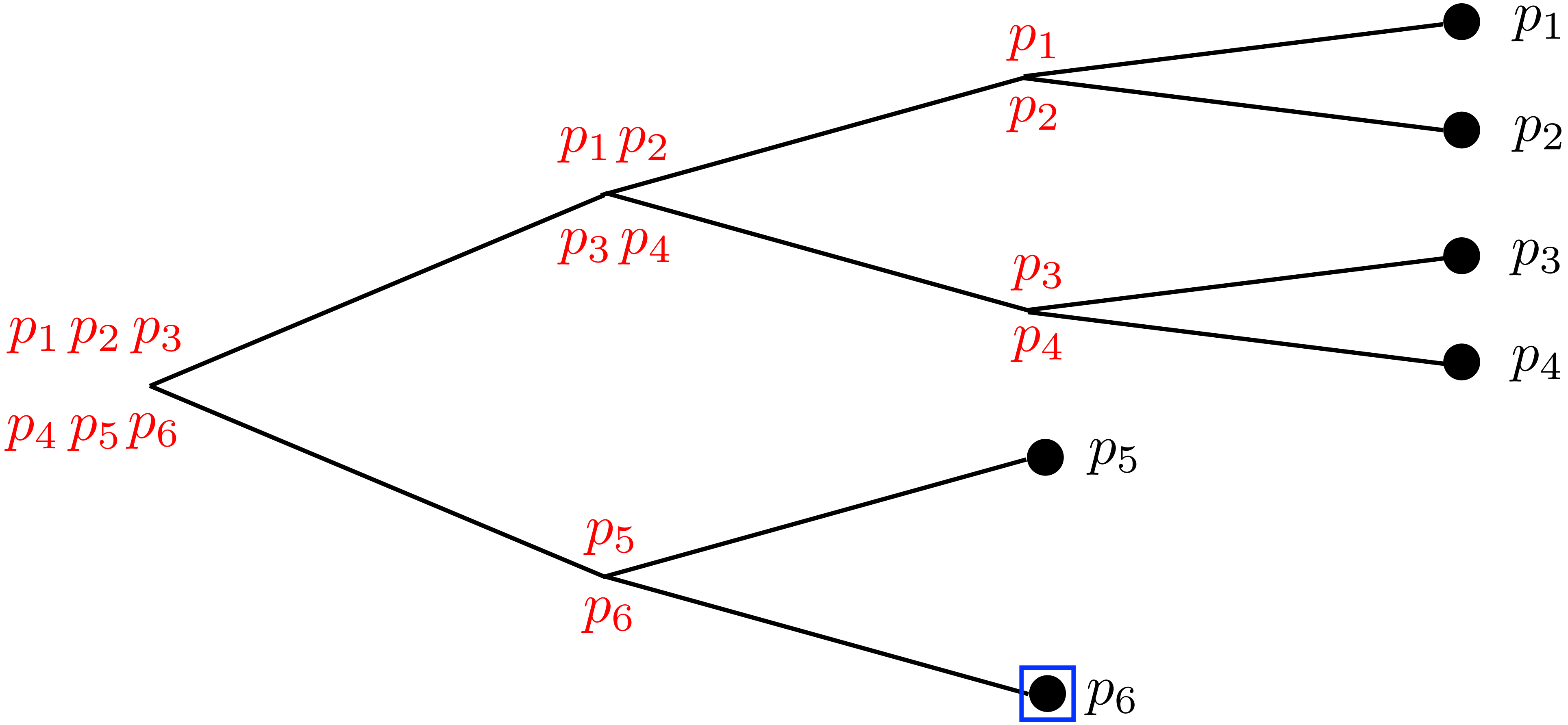

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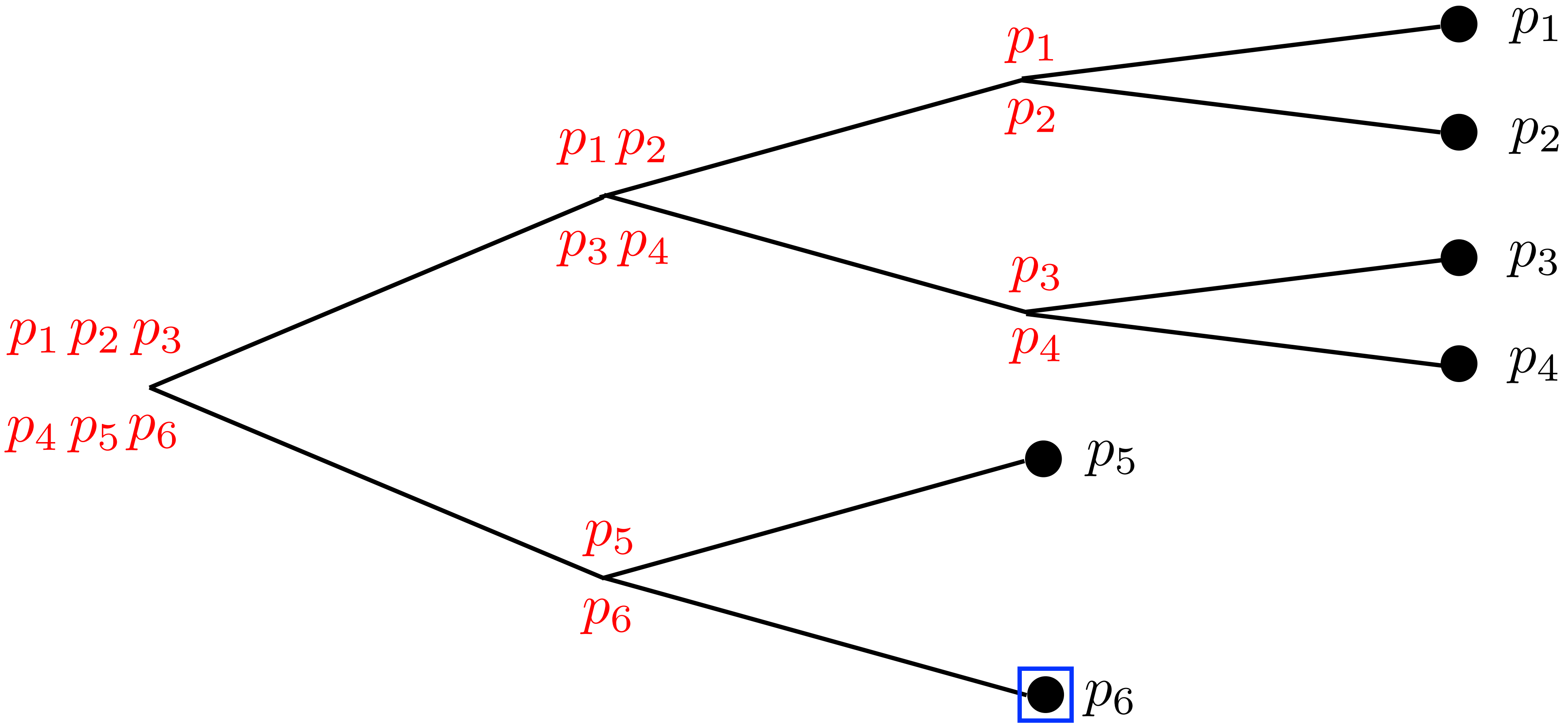

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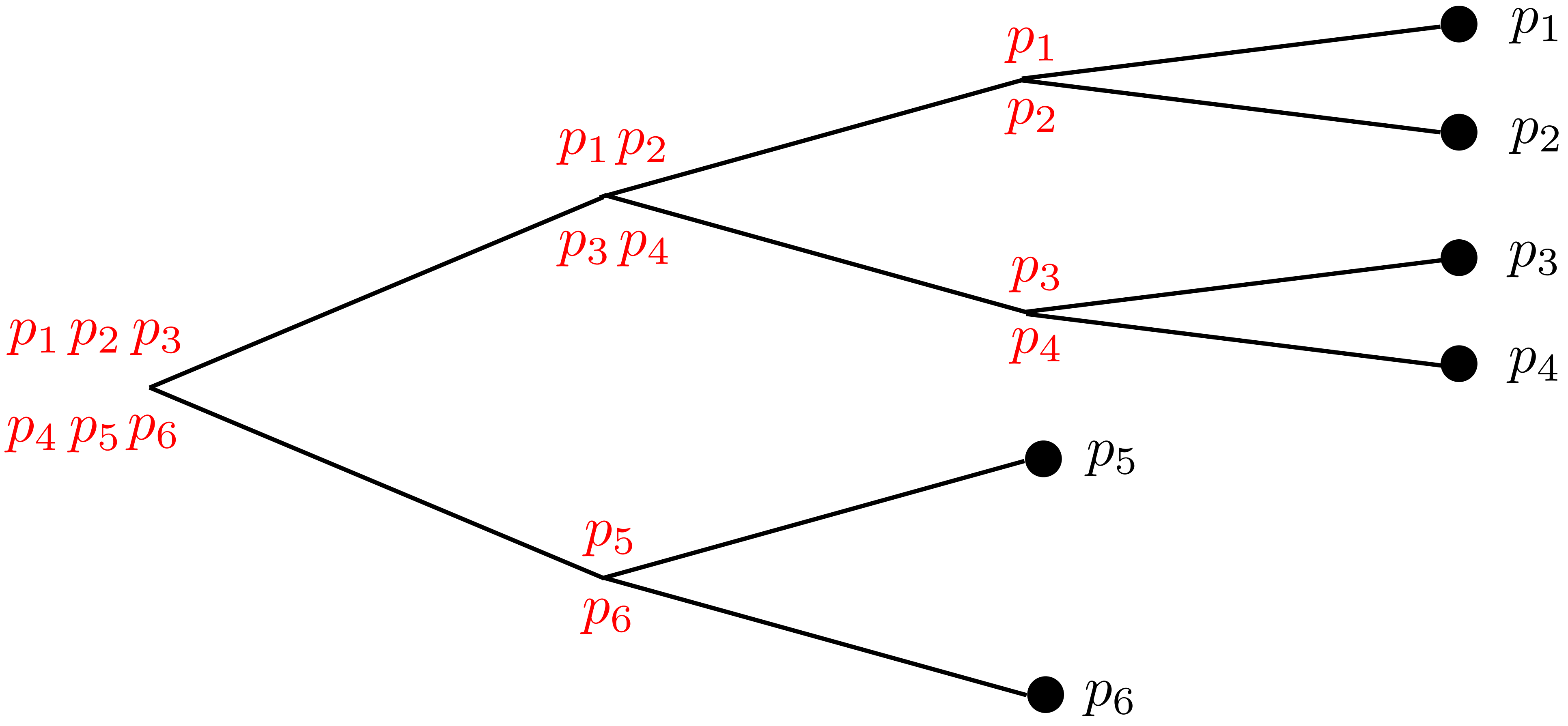

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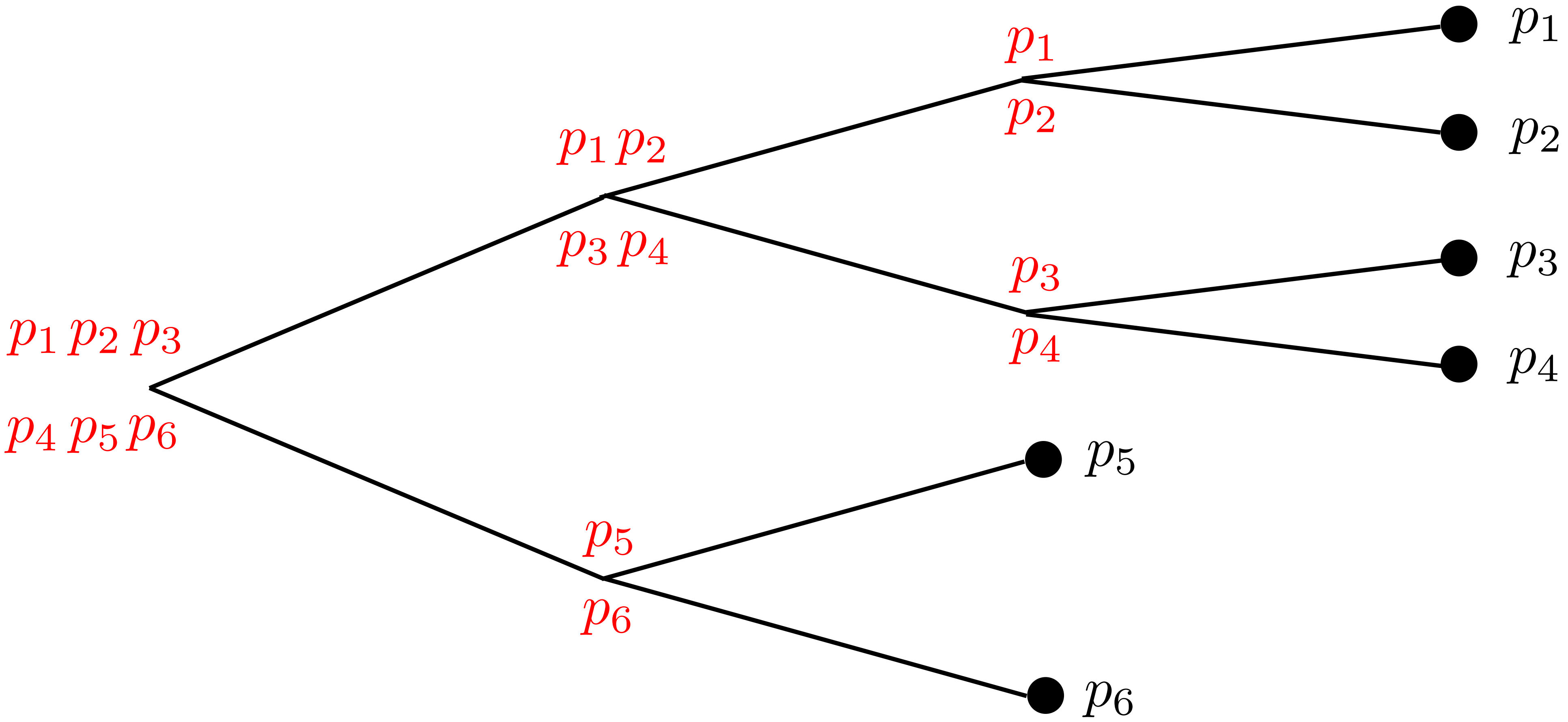

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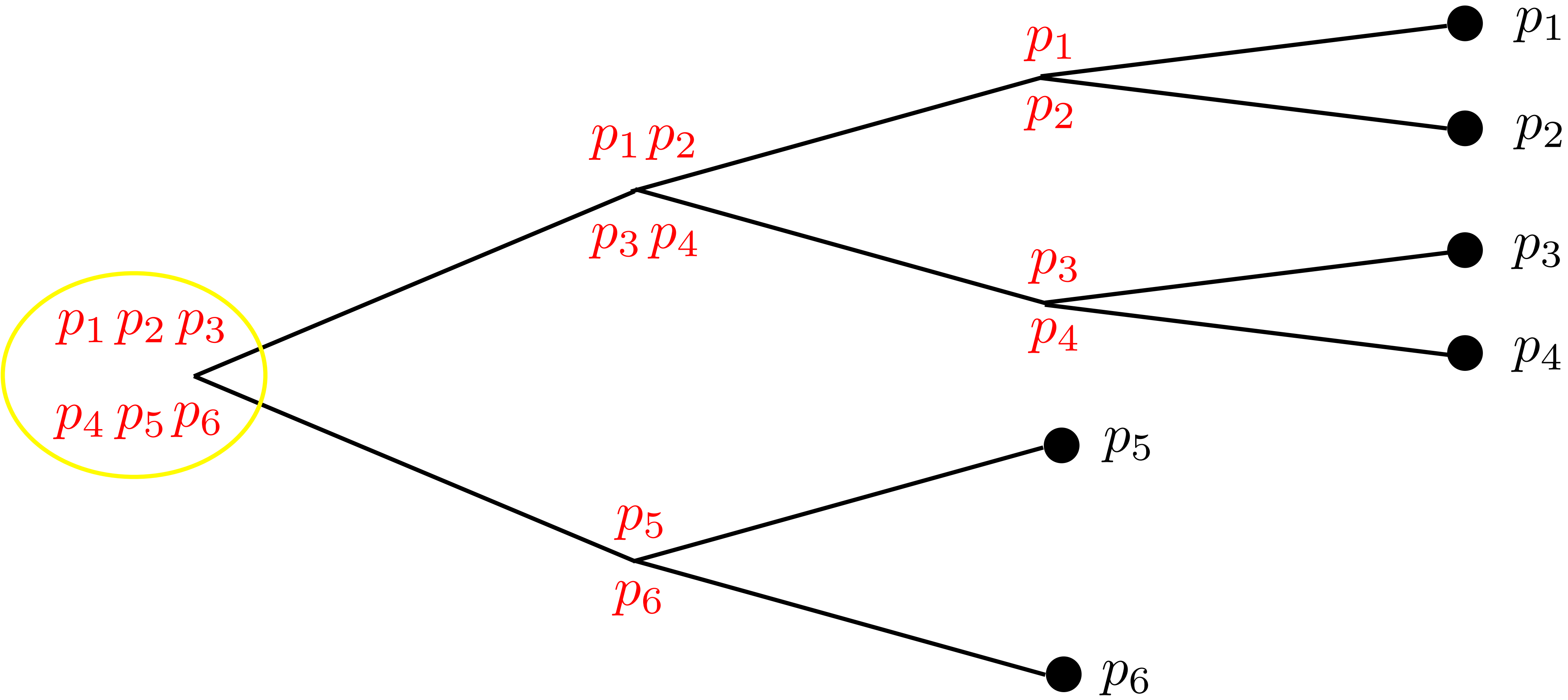

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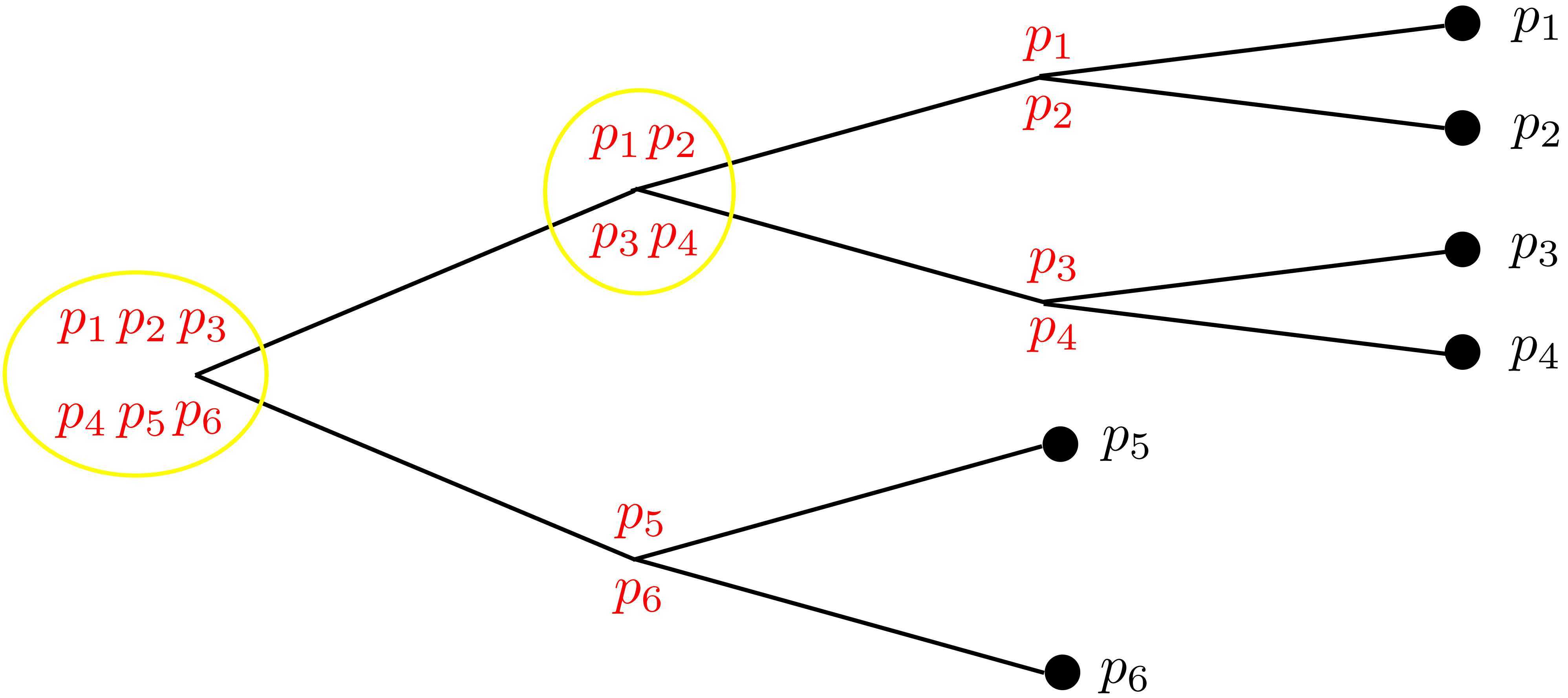

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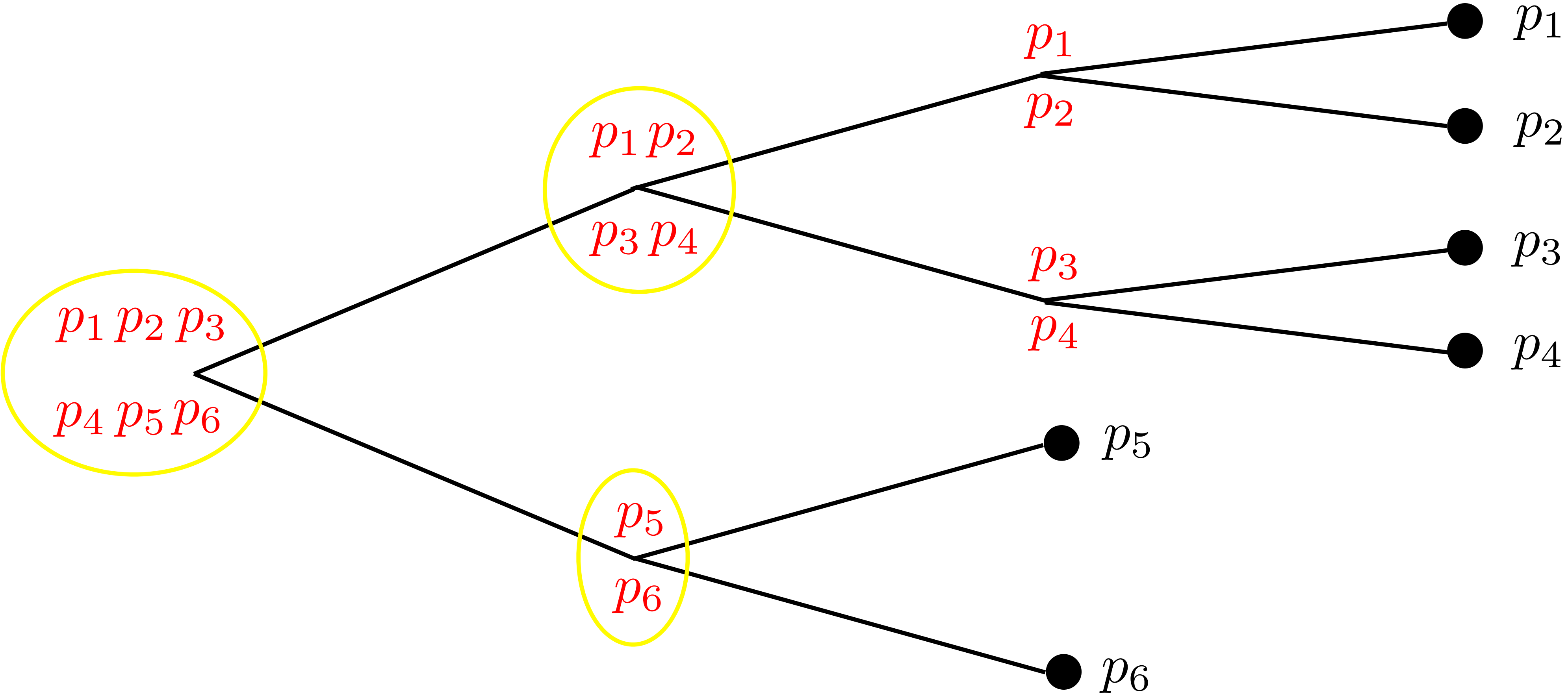

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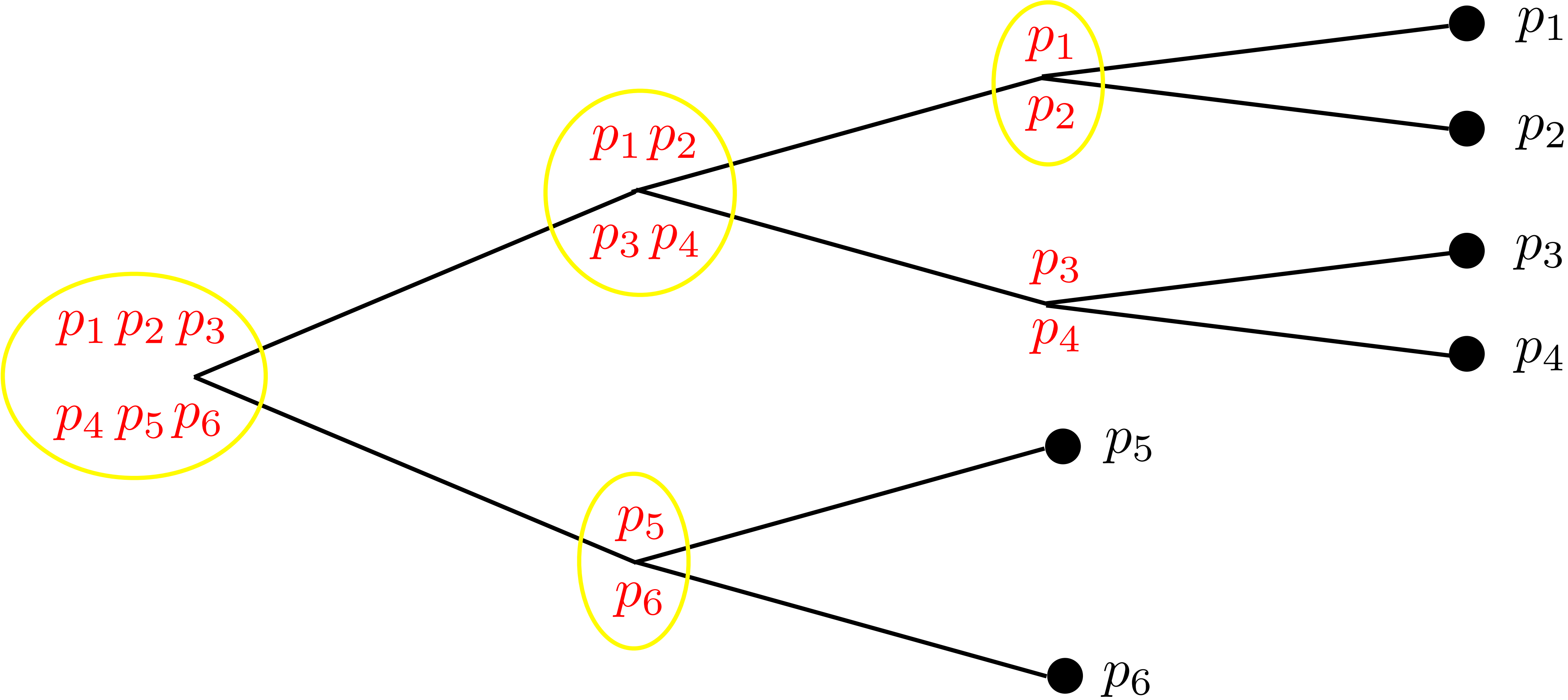

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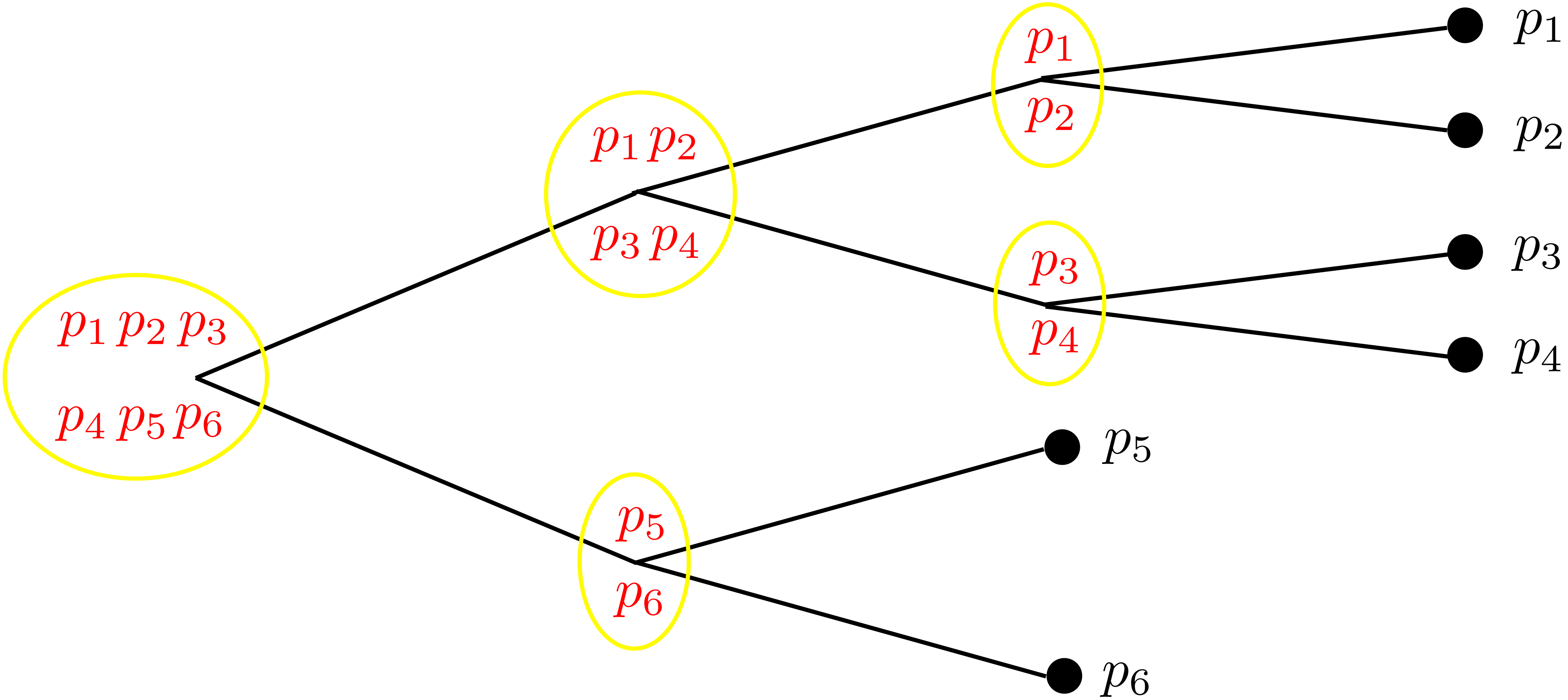

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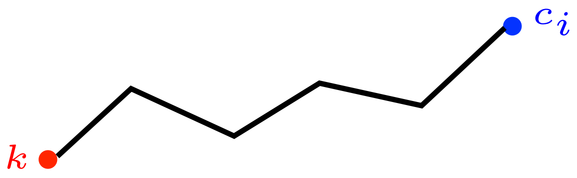

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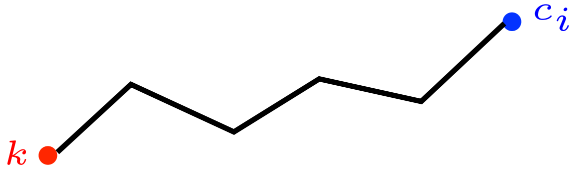

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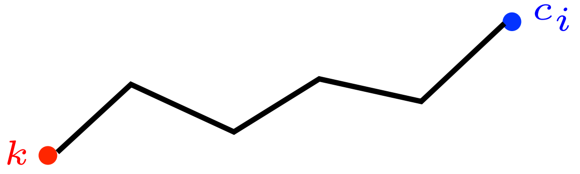

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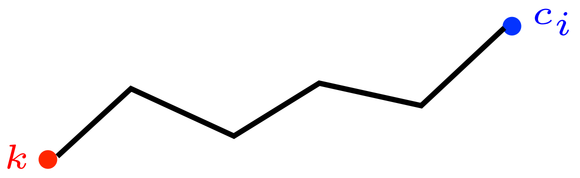

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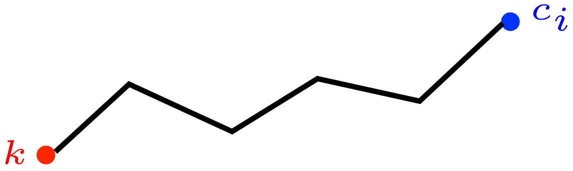

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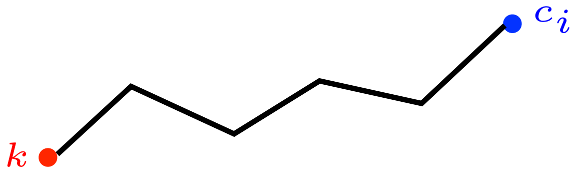

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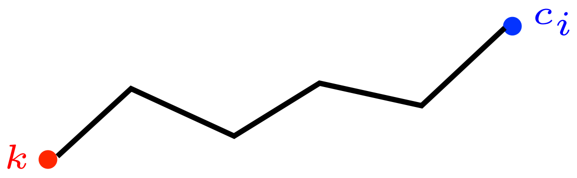

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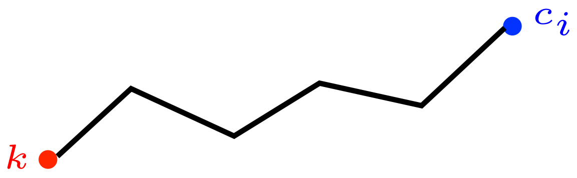

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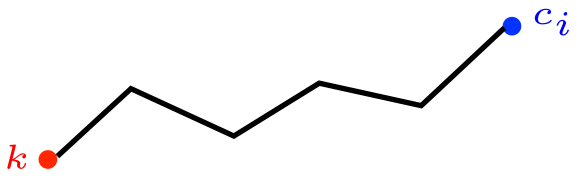

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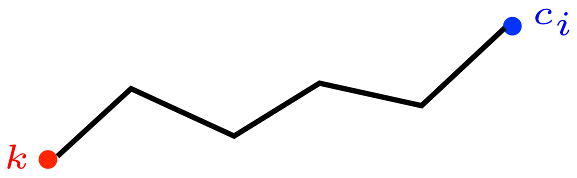

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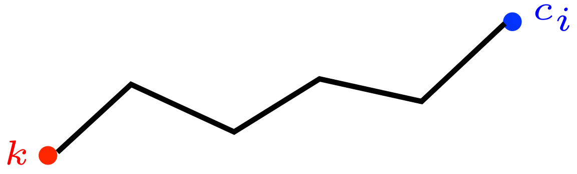

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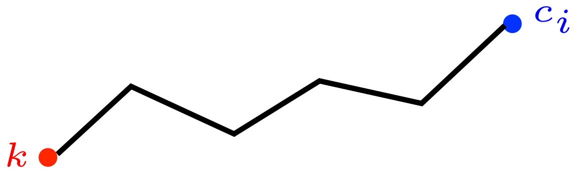

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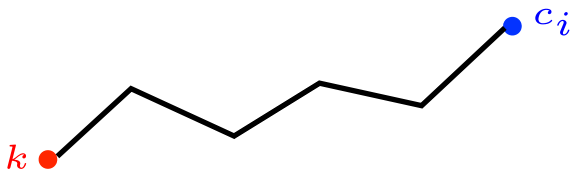

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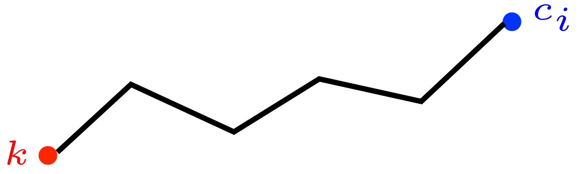

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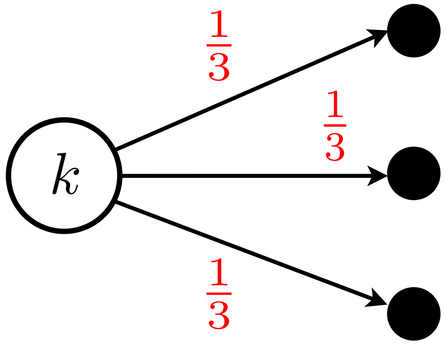

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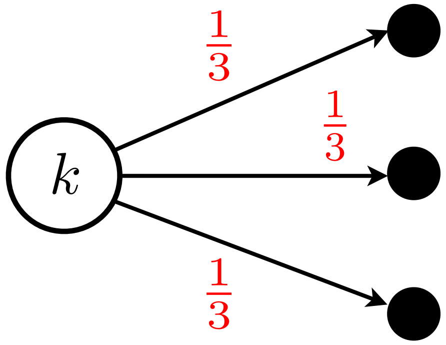

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Theorem 4.21 (Local Redundancy Theorem) Let R be the redundancy of a D-ary prefix code for a source random variable X. Then

$$R = \sum_{k \in \mathcal{I}} r_k.$$

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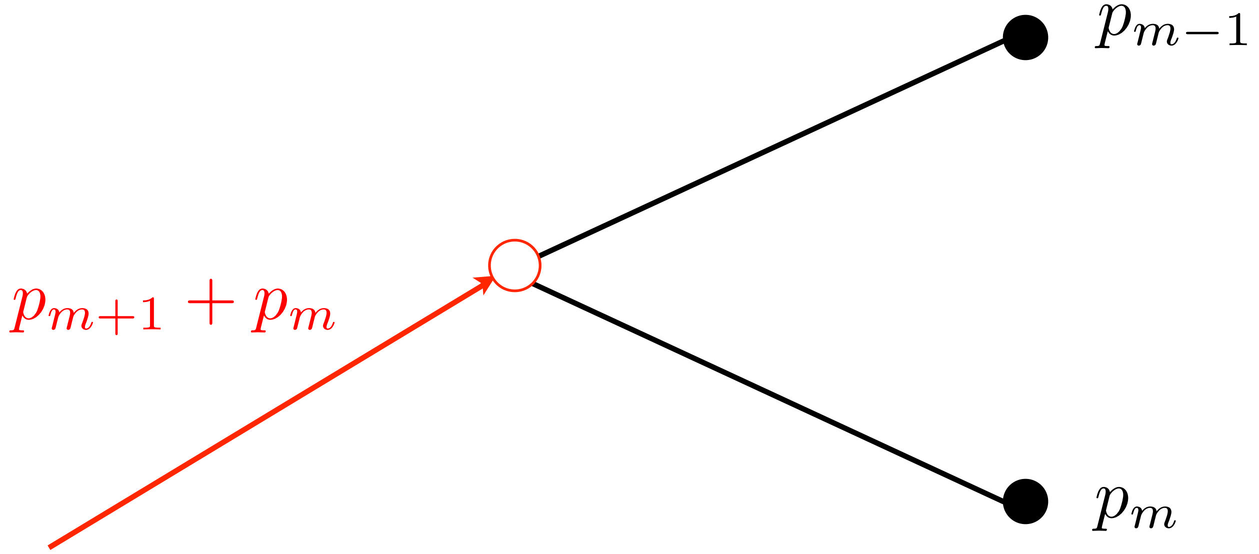

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- The local redundancy of this internal node is

$$(p_{m-1}+p_m)\left[1 - H_2\left(\left\{\frac{p_{m-1}}{p_{m-1}+p_m}, \frac{p_m}{p_{m-1}+p_m}\right\}\right)\right],$$

  which is a lower bound on R.