4.2 Prefix Codes

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Example 4.10

x	$\mathcal{C}'(x)$
A	0
B	10
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- A D-ary tree is a graphical representation of a collection of finite sequences of D-ary symbols.
- A node is either an internal node or a leaf.
- The tree representation of a prefix code is called a code tree.

Instantaneous Decoding

- Using the code $\mathcal{C}'$ in Example 4.10:

$$BCDAC \cdots \rightarrow 1011011110110 \cdots$$

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Instantaneous Decoding

- Using the code $\mathcal{C}'$ in Example 4.10:

$$BCDAC\cdots \rightarrow \underline{10}11011110110\cdots$$

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- Using the code $\mathcal{C}'$ in Example 4.10:

$$B\underline{C}DAC \cdots \to 1011011110110 \cdots$$

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- Upon concatenating the codewords, their boundaries are no longer explicit.

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Instantaneous Decoding

- Using the code $\mathcal{C}'$ in Example 4.10:

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- Upon concatenating the codewords, their boundaries are no longer explicit.
- The stream of coded symbols are then transmitted to the receiver.

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Instantaneous Decoding

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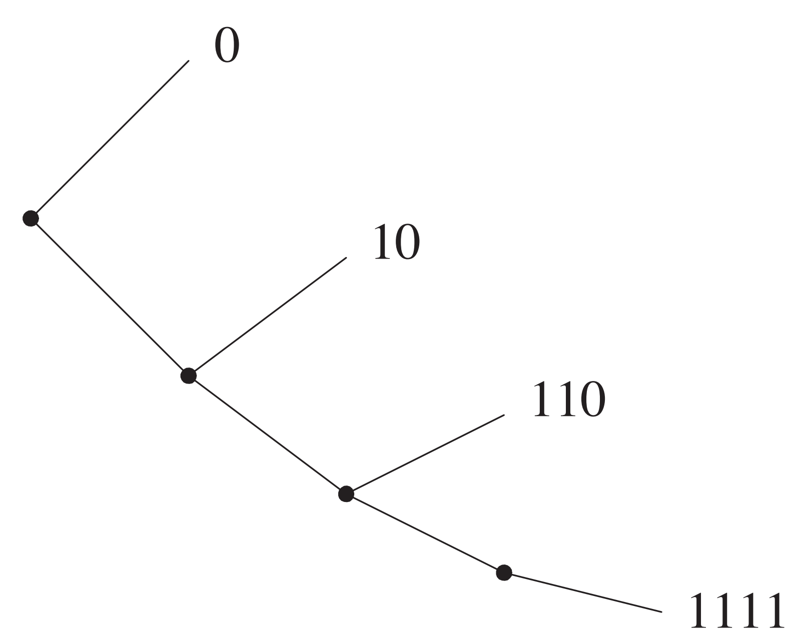

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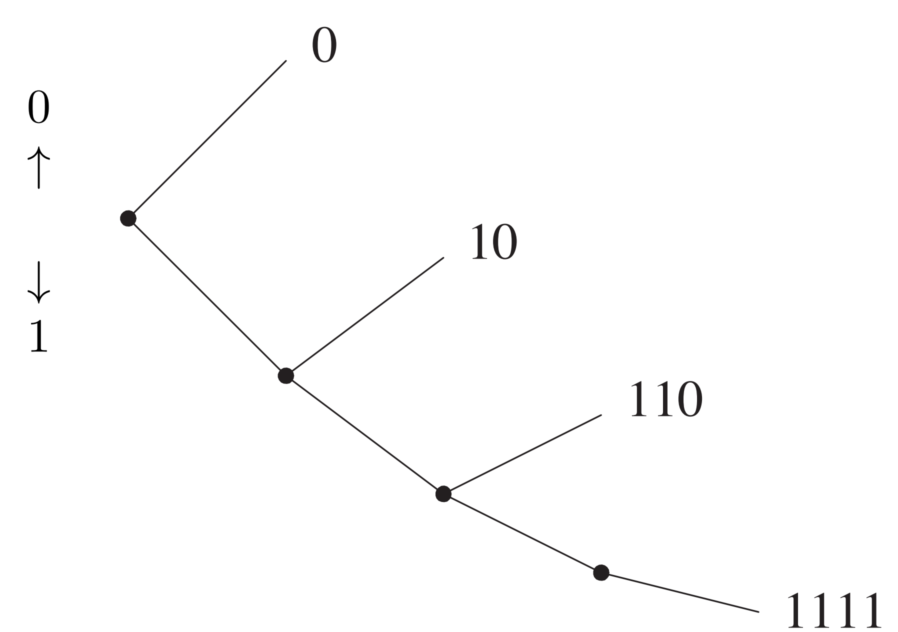

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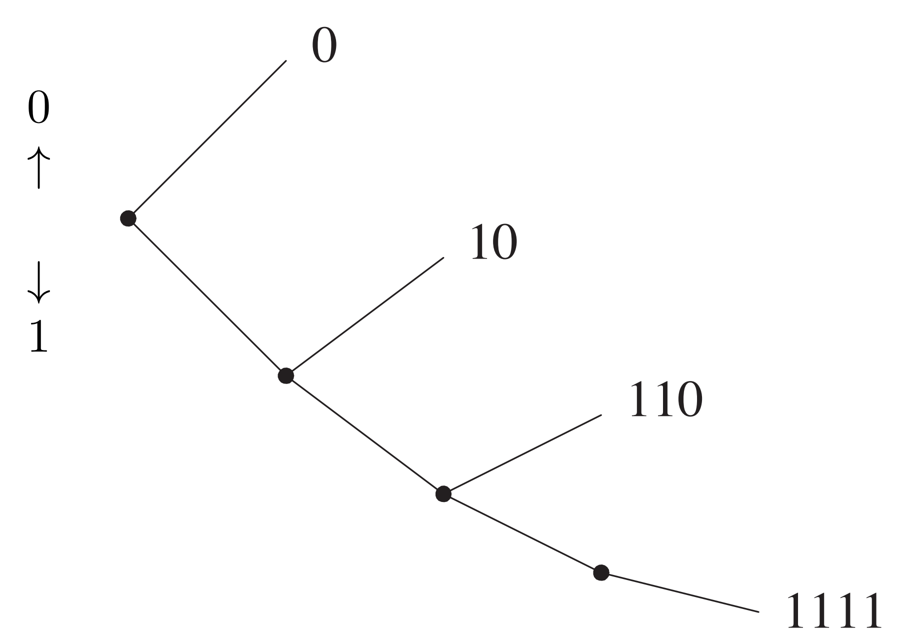

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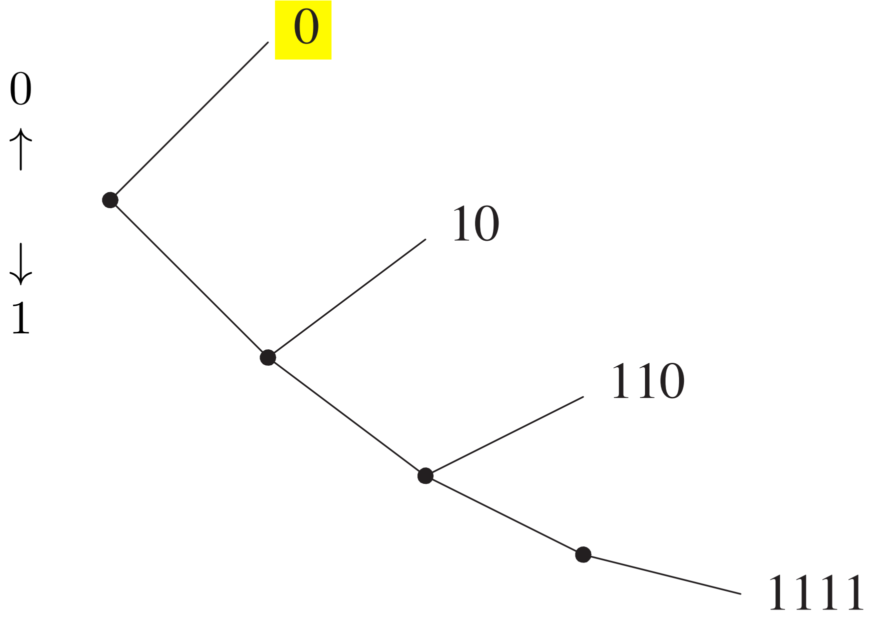

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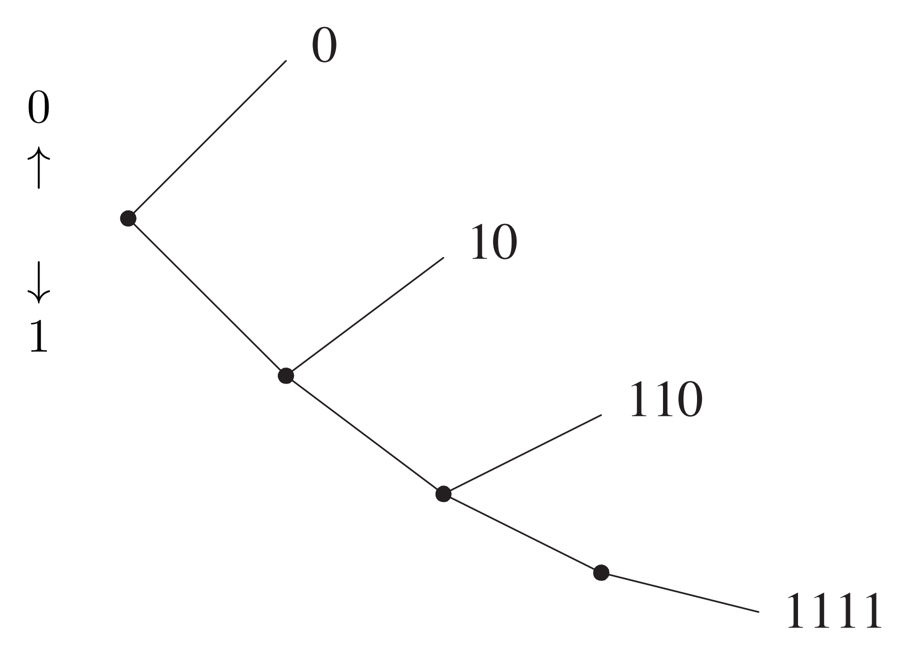

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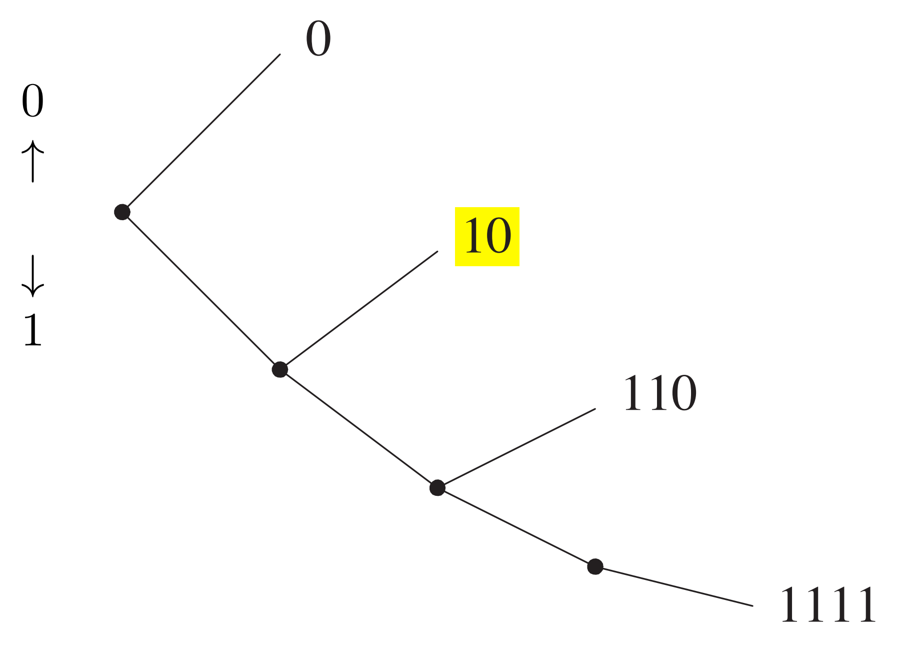

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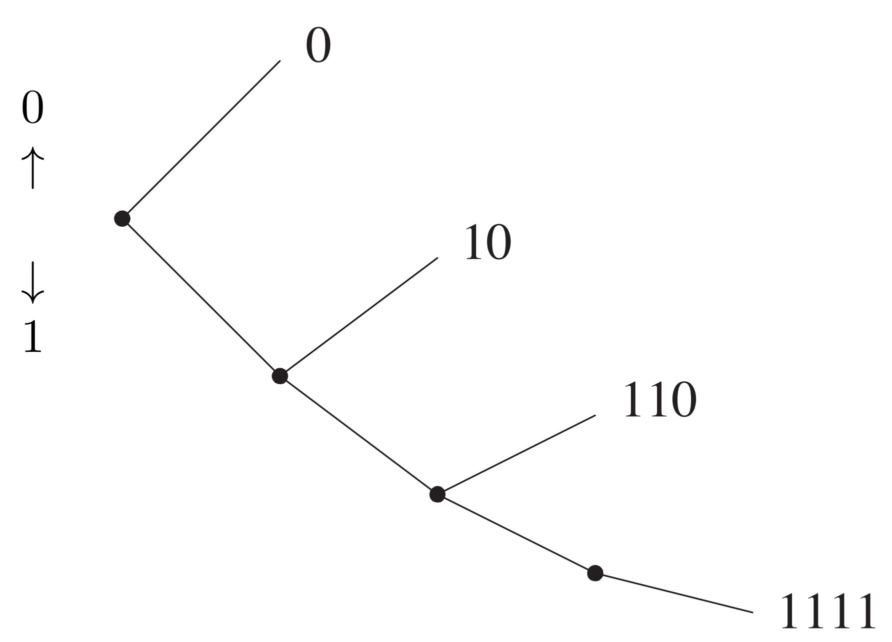

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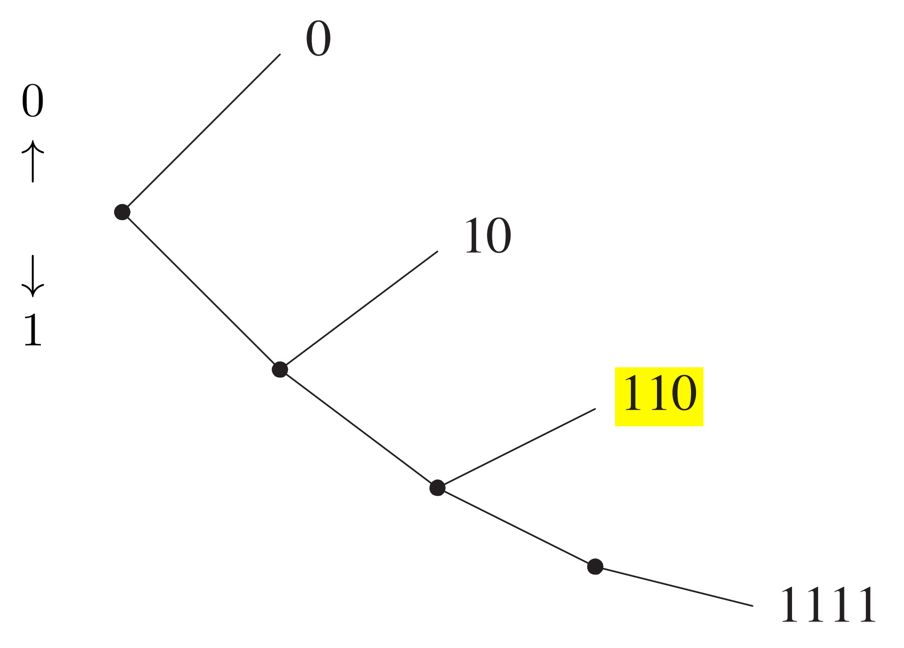

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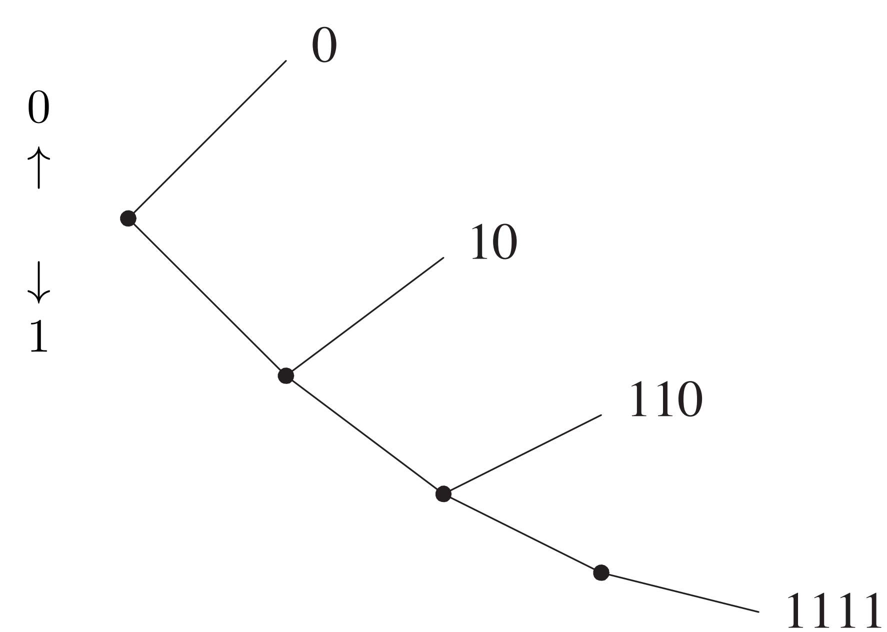

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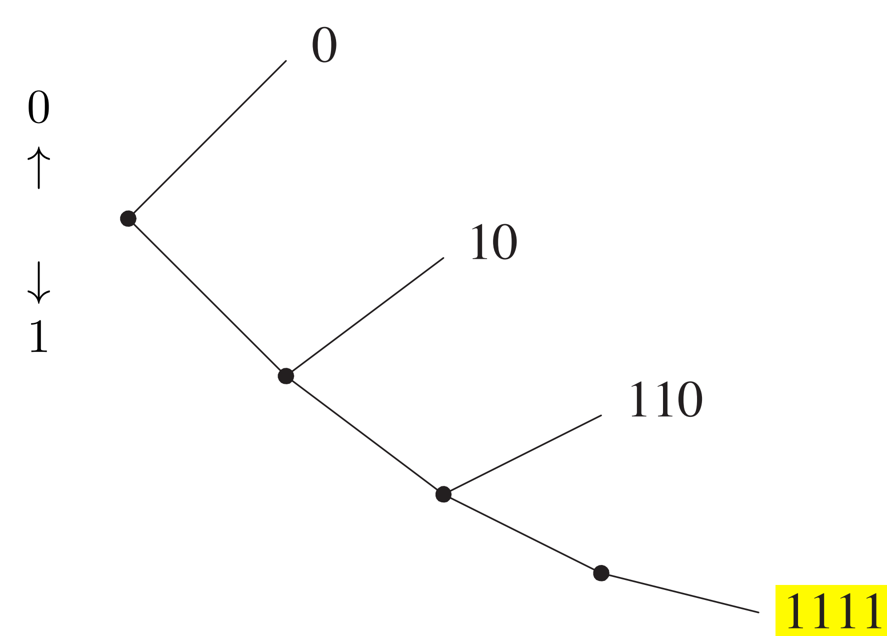

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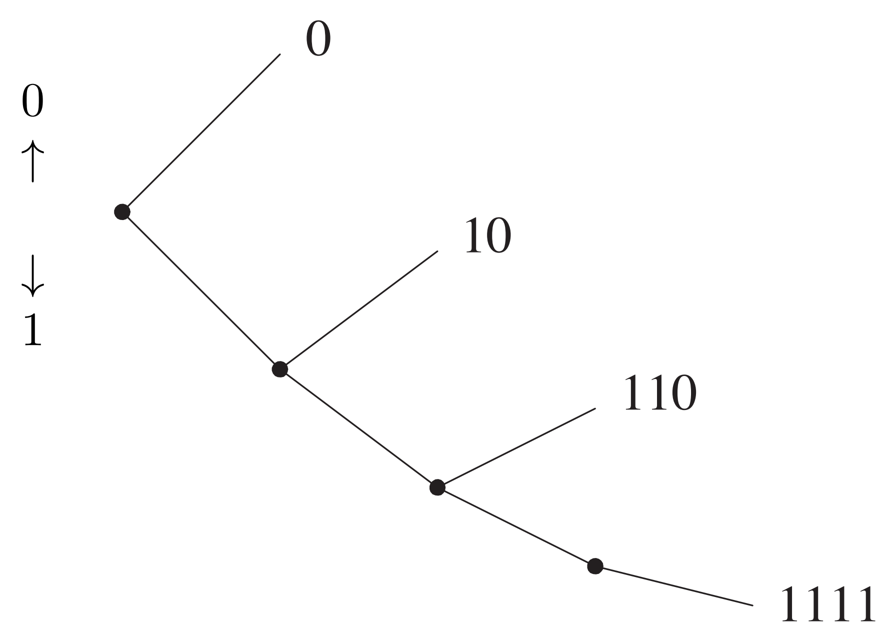

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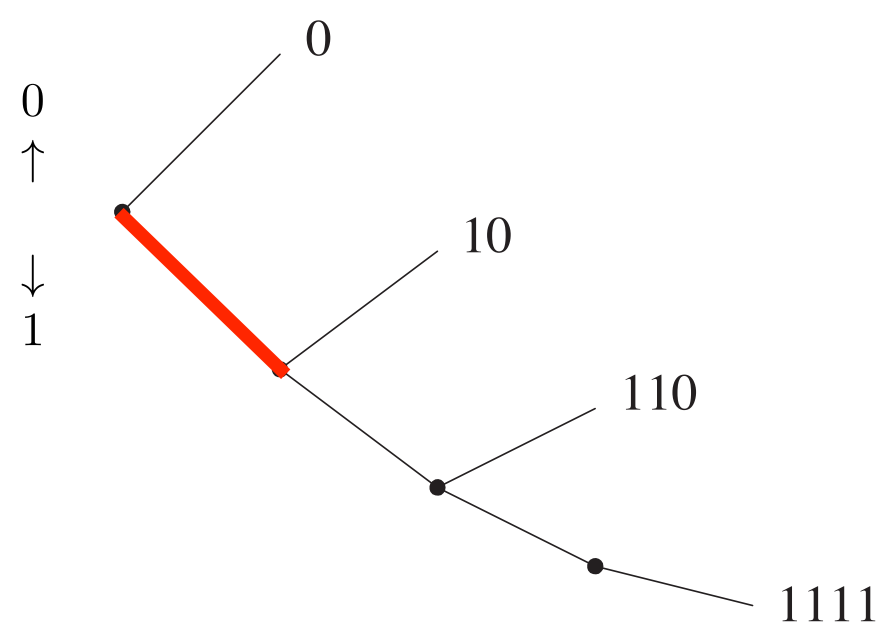

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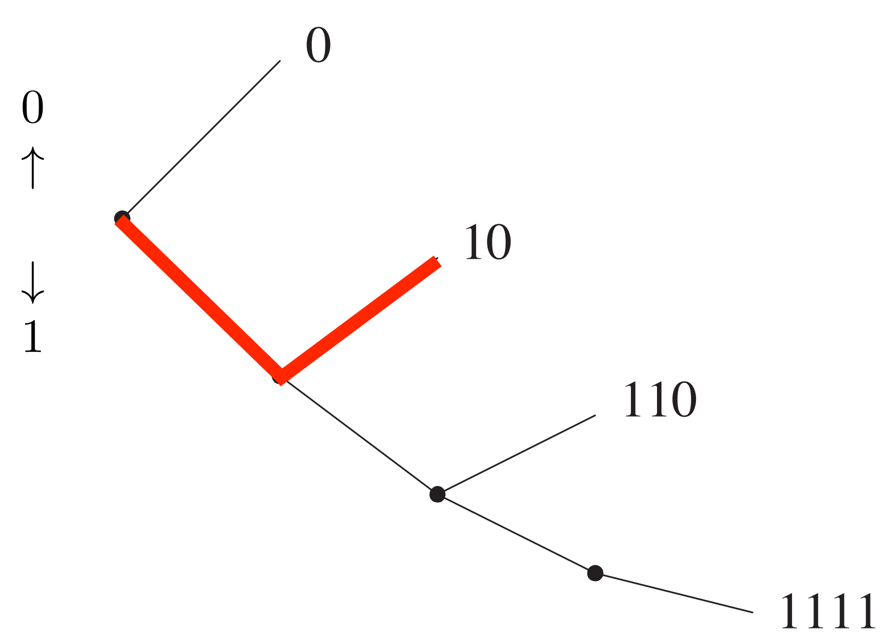

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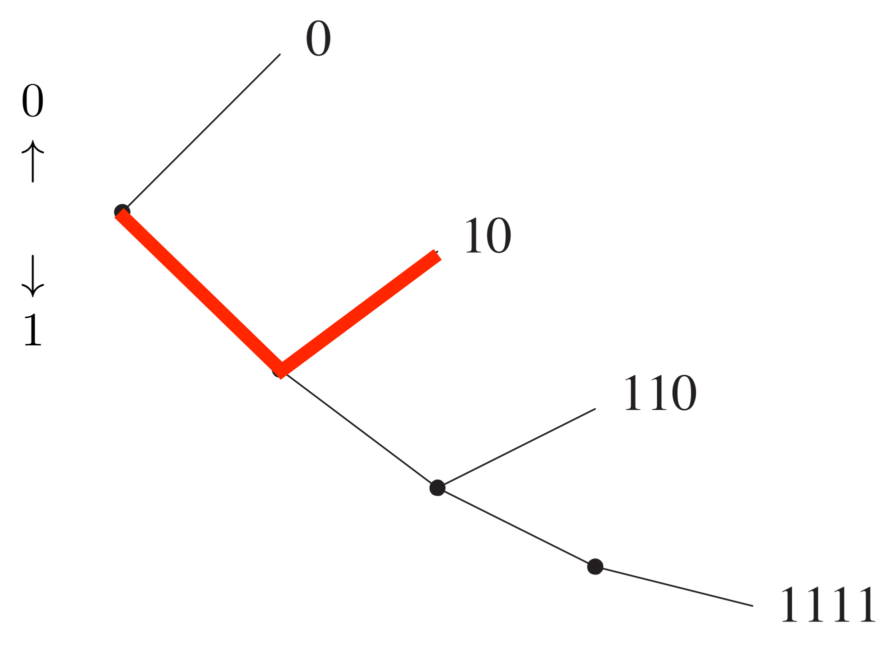

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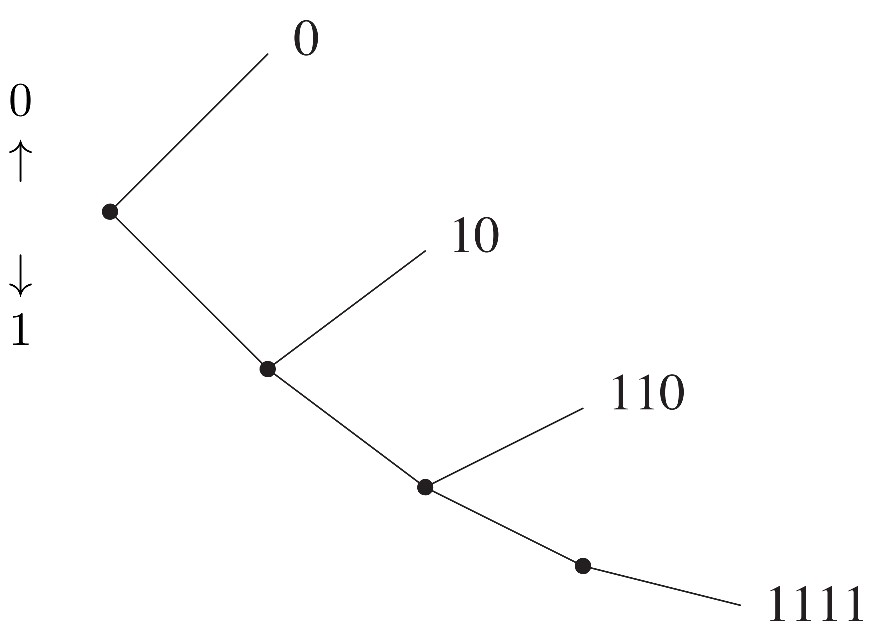

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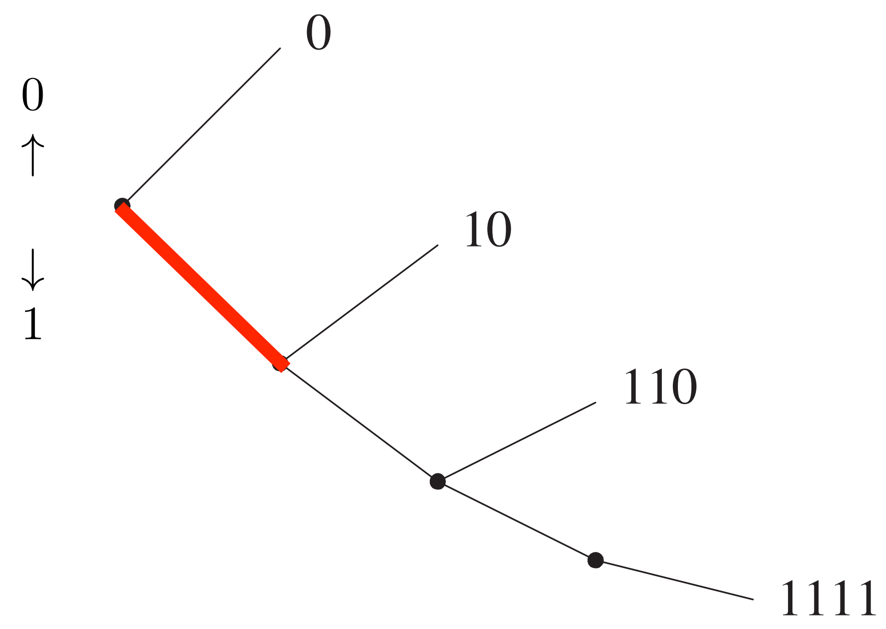

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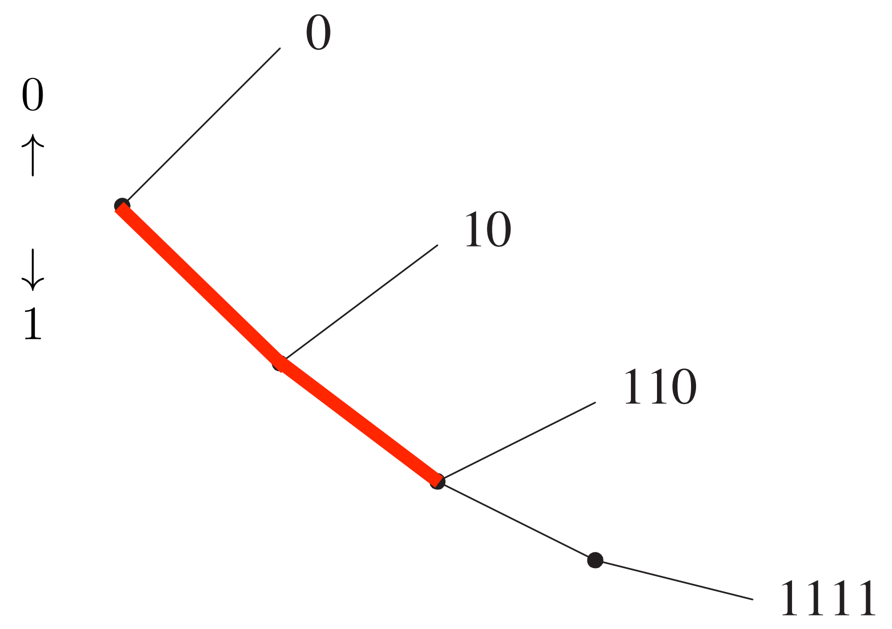

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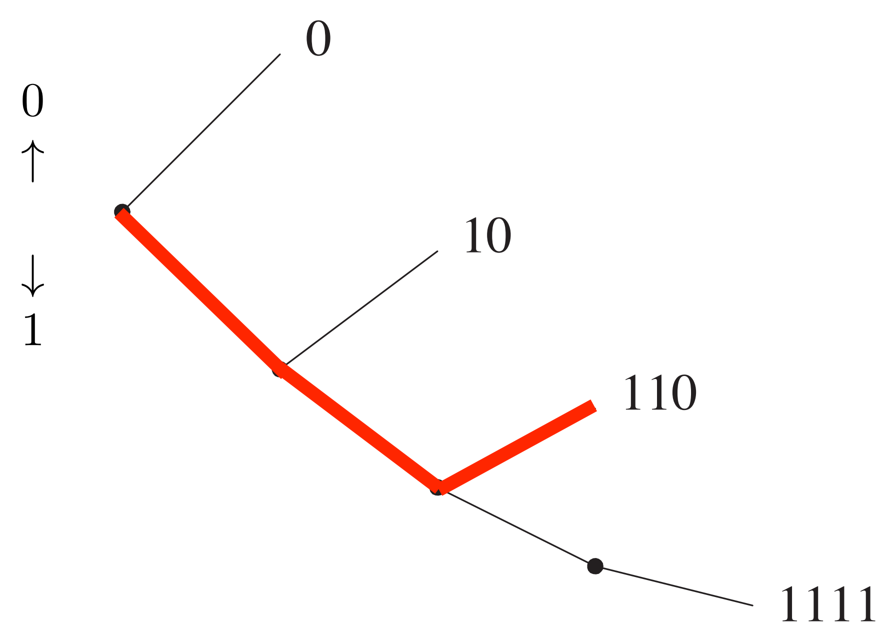

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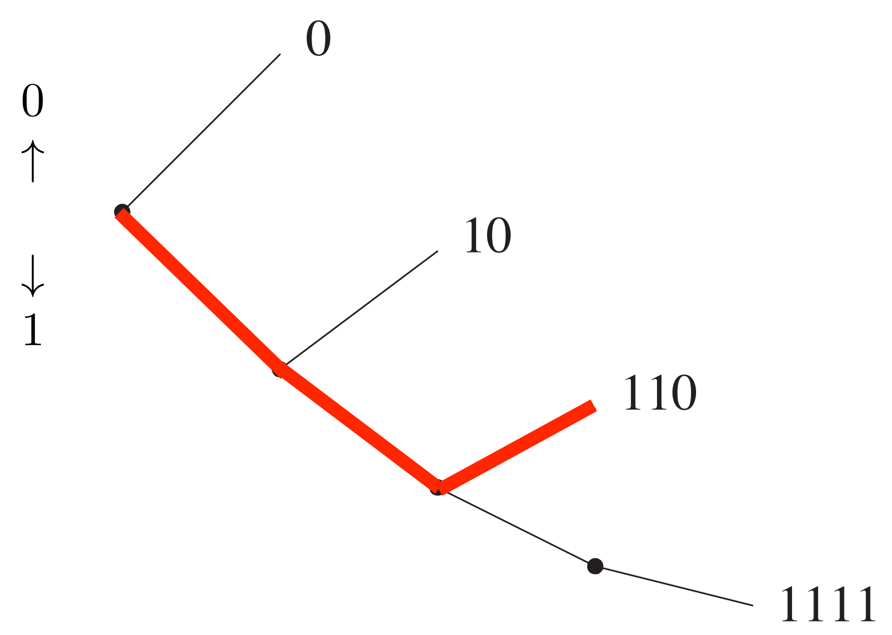

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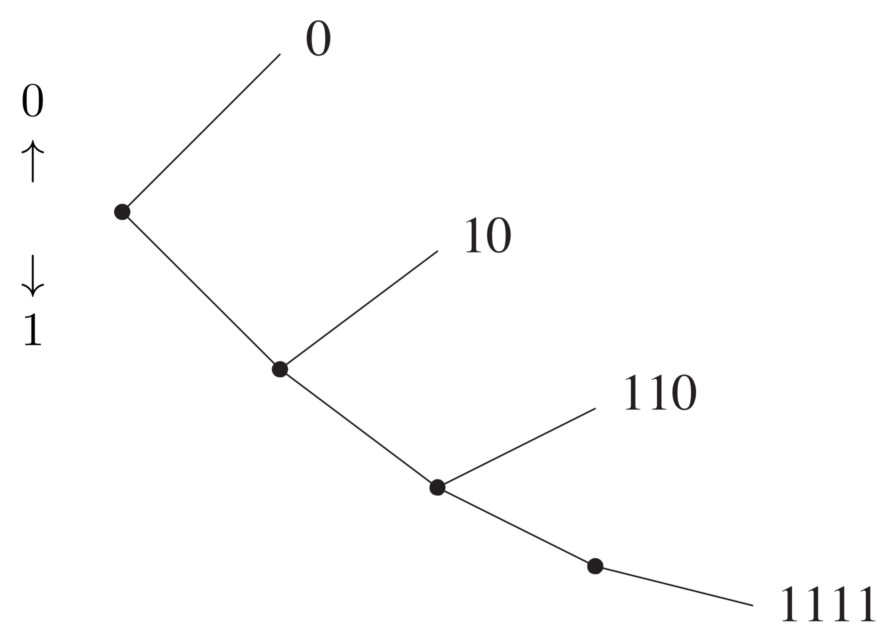

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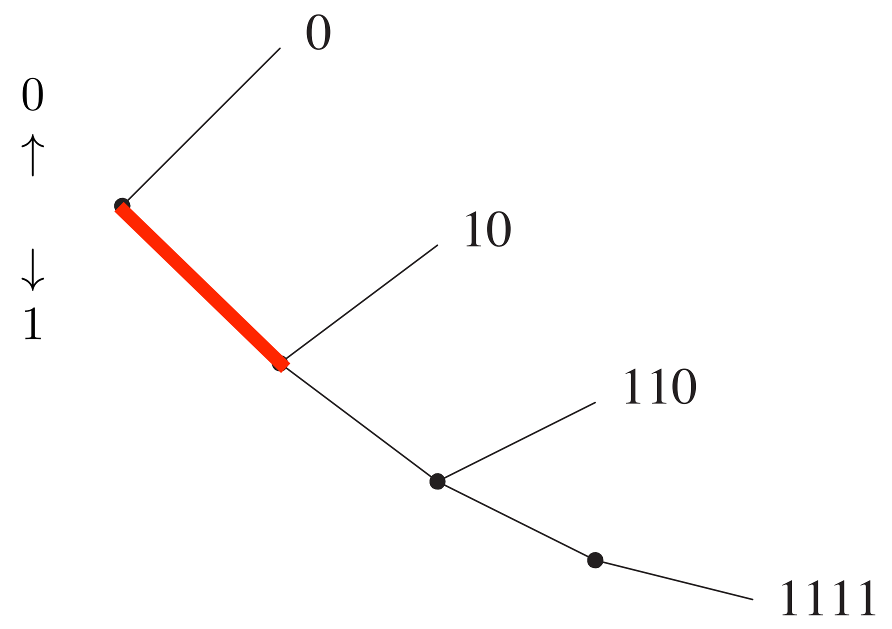

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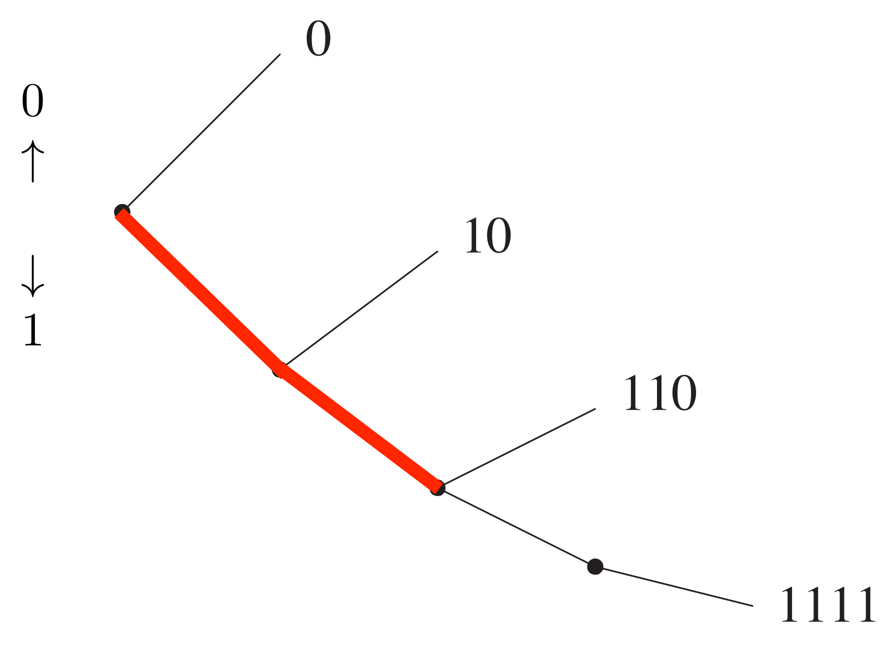

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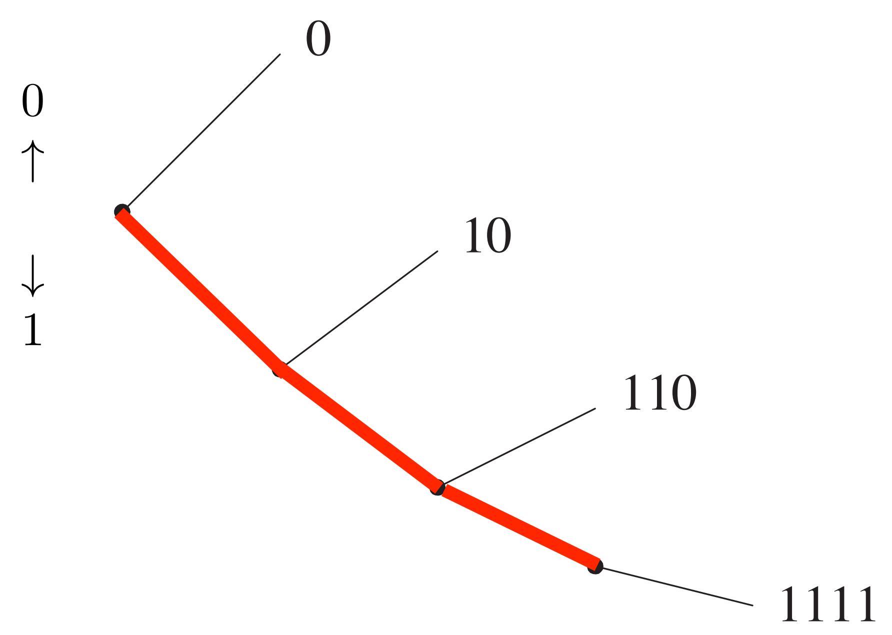

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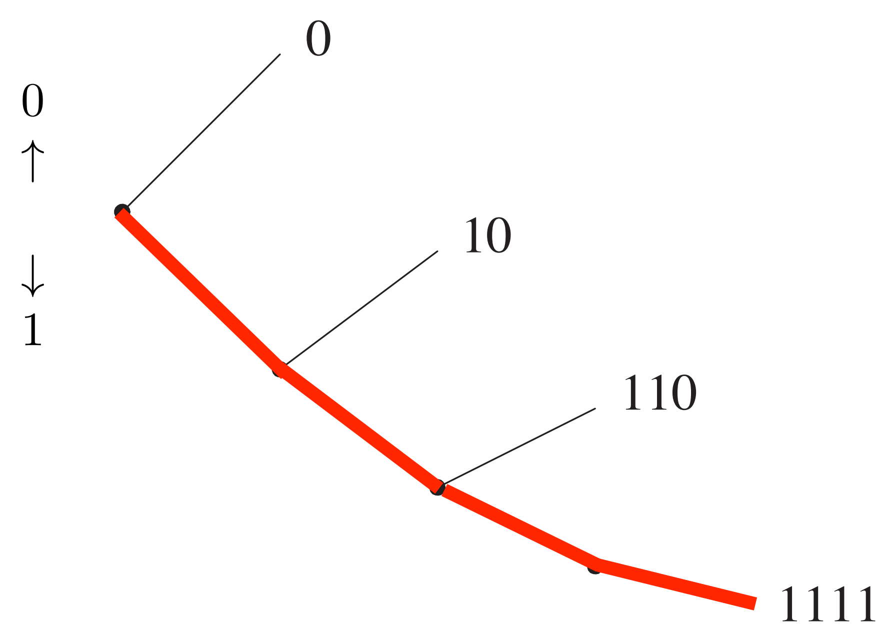

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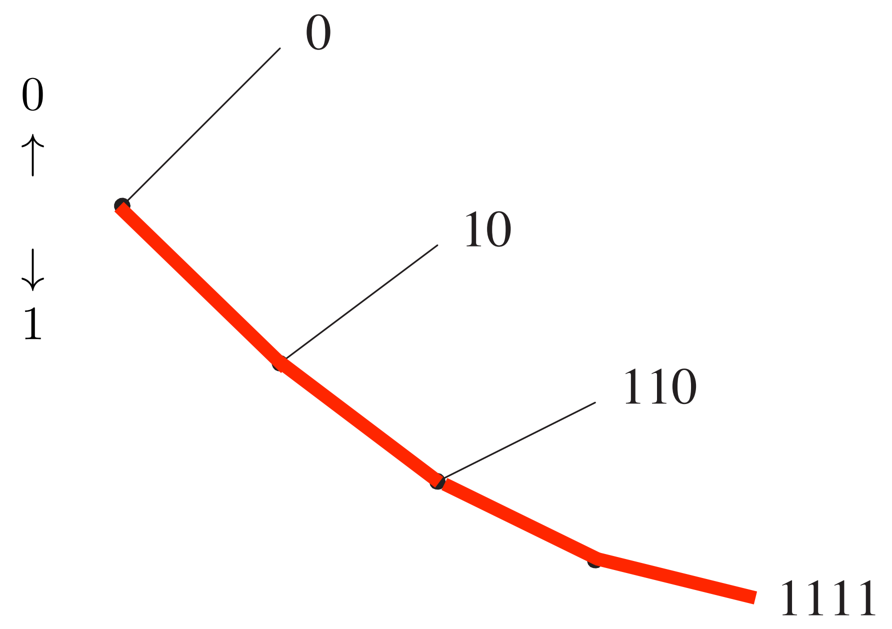

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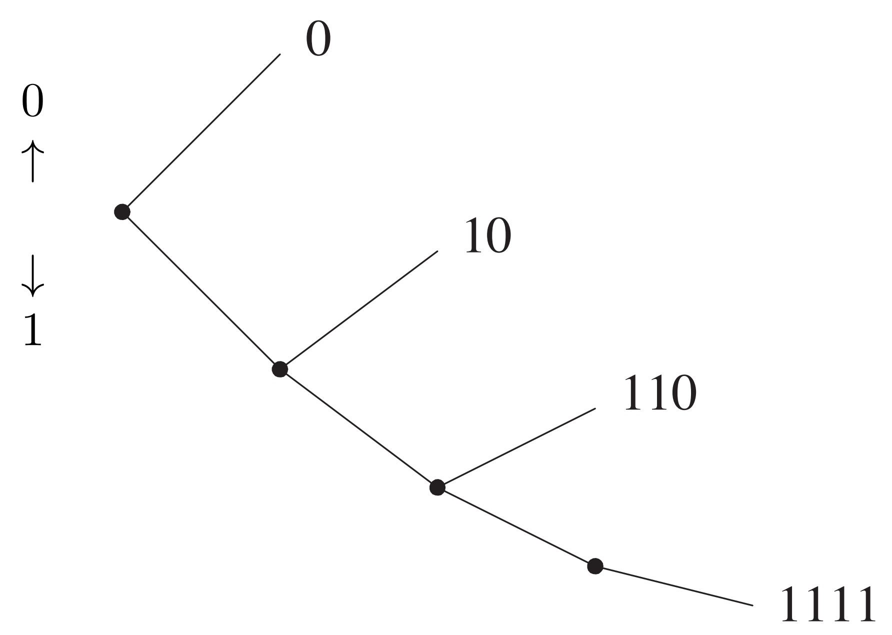

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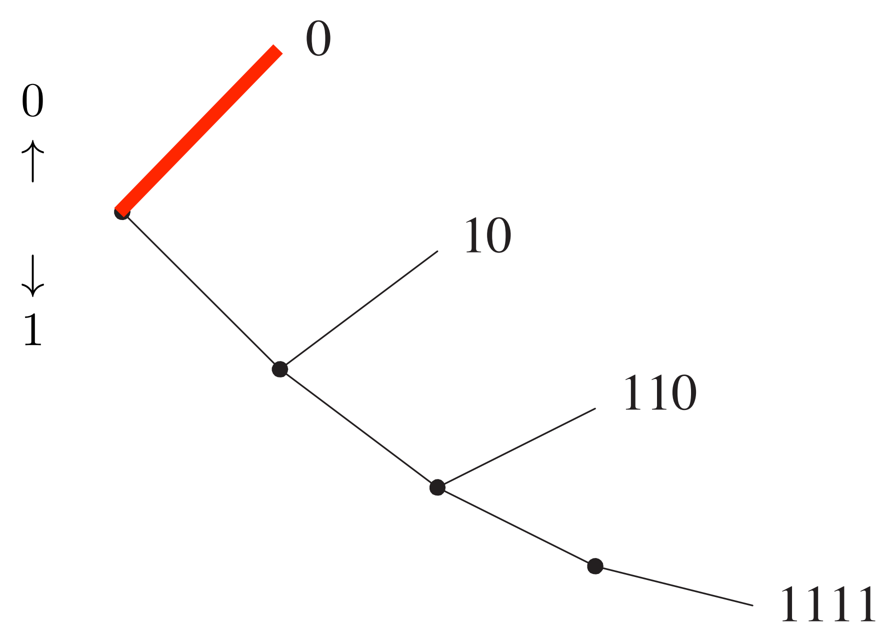

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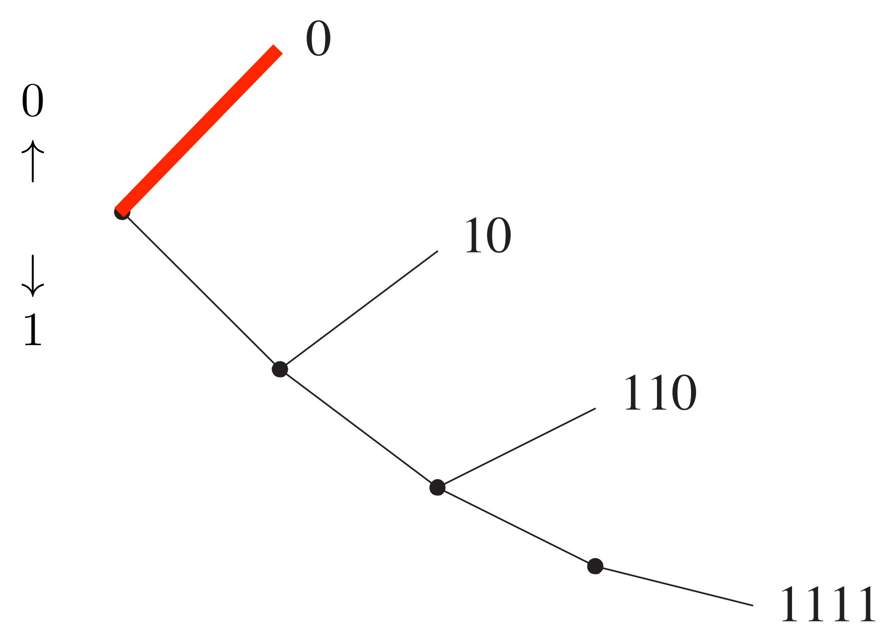

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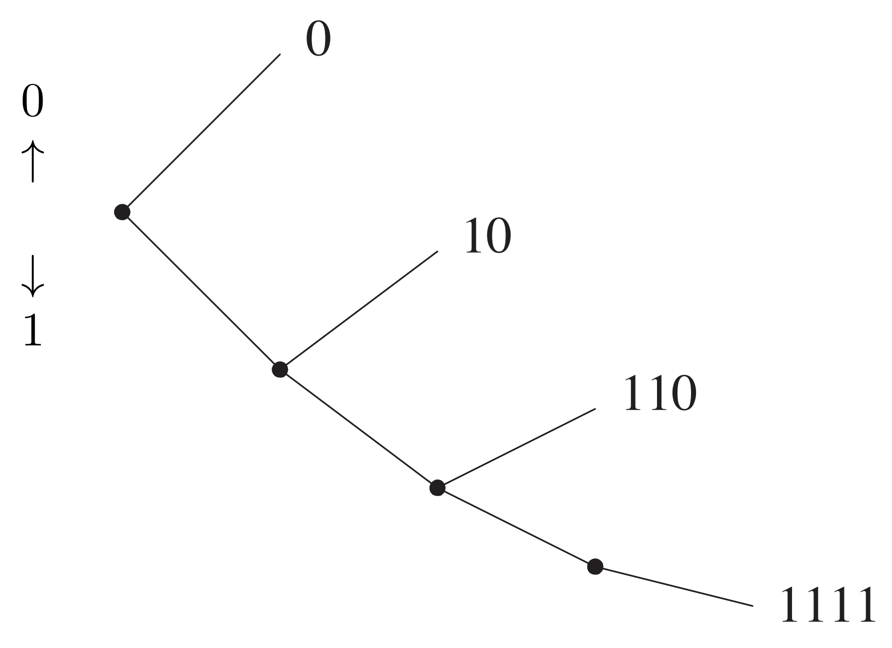

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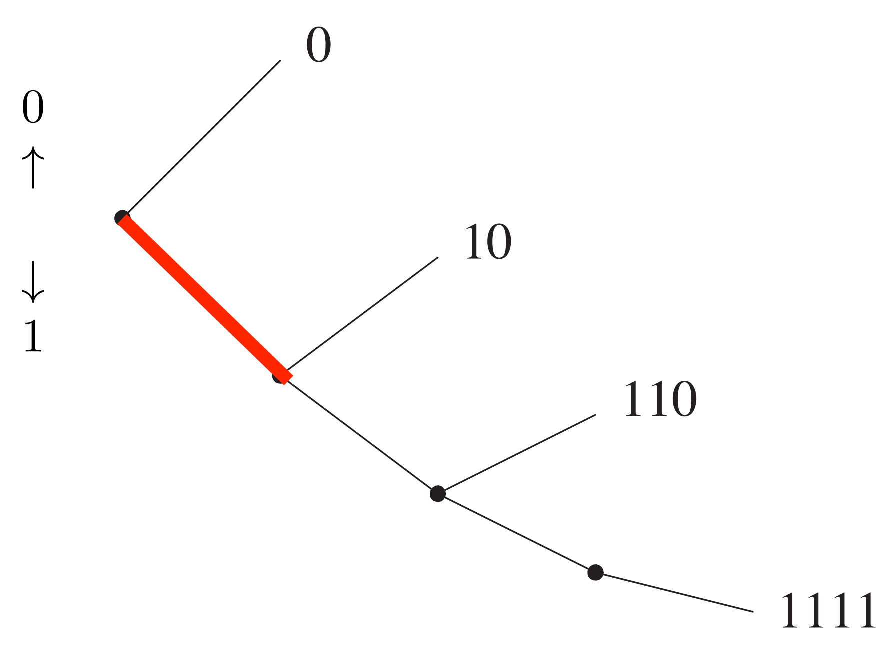

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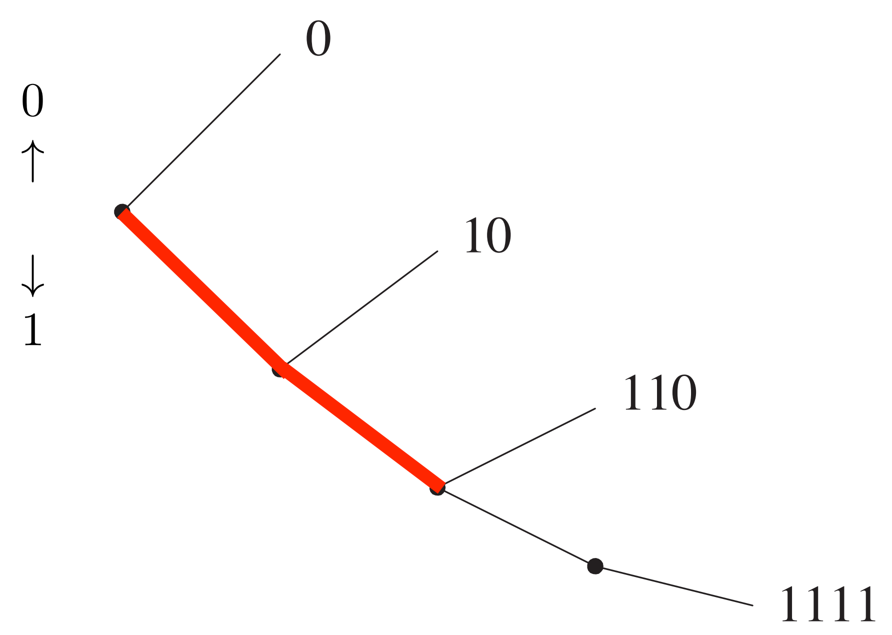

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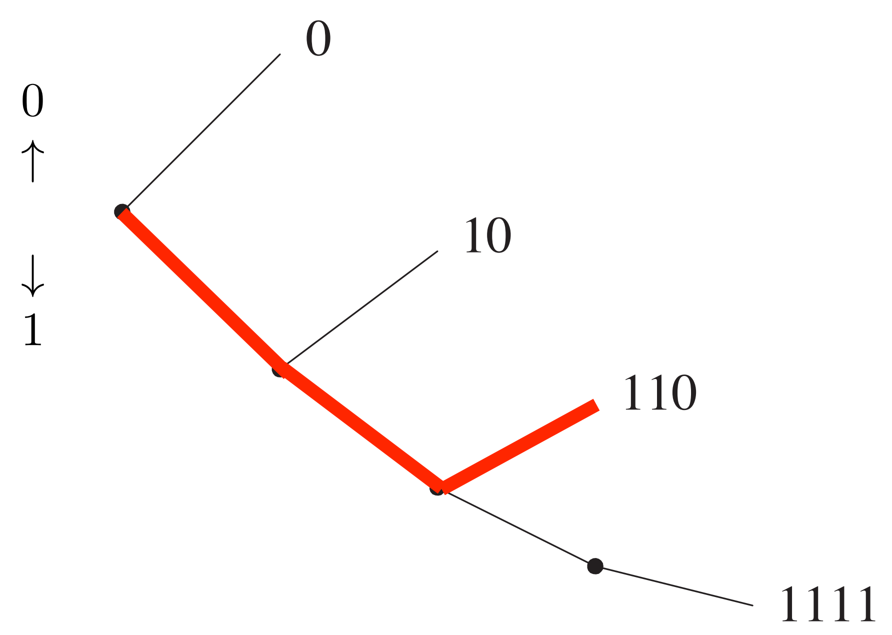

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- Instantaneous decoding at the receiver can be done by tracing the code tree from the root:

$$1011011110110\cdots \rightarrow 10, 110, 1111, 0, \underline{110}, \cdots$$

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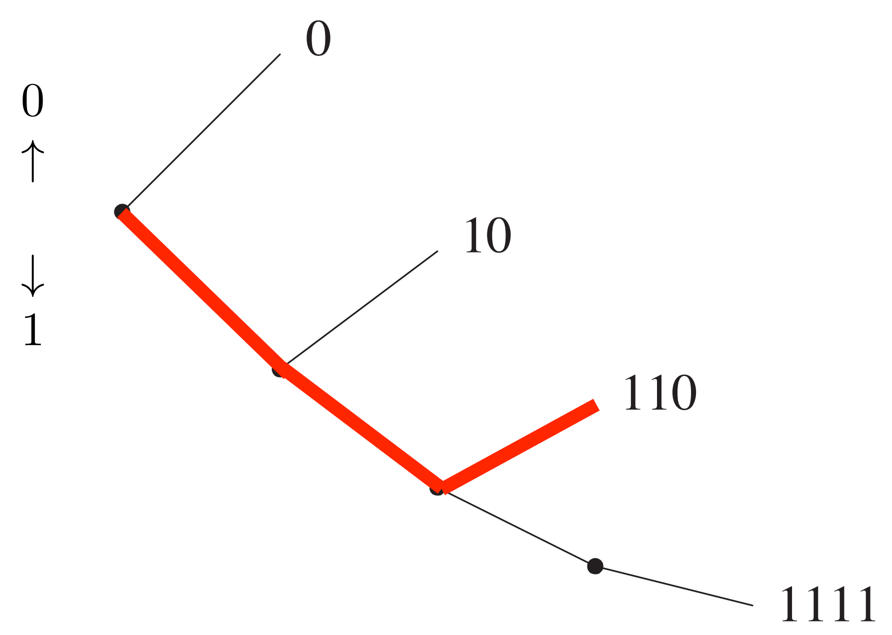

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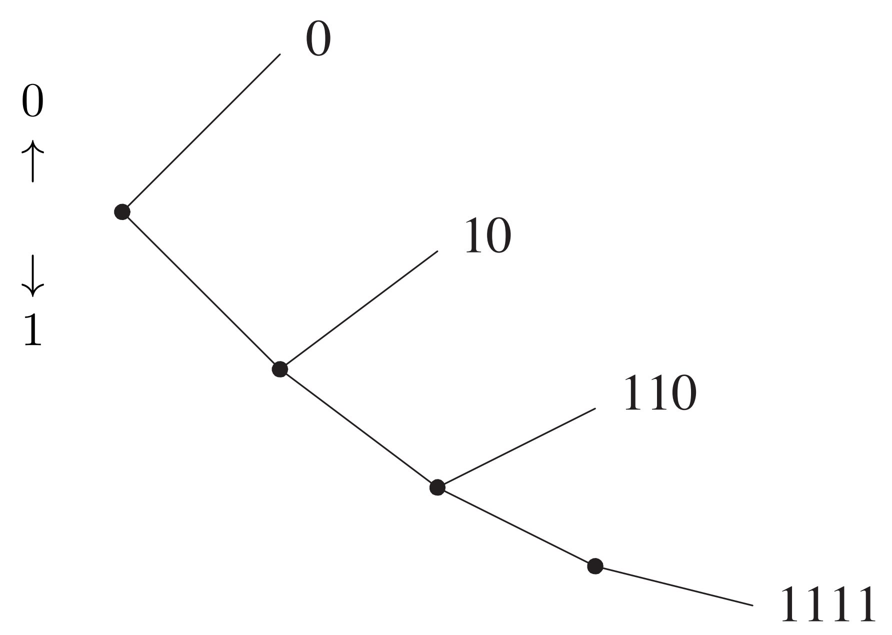

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- The boundaries of the codewords can thus be recovered. In this sense, a prefix code is said to be self-punctuating.

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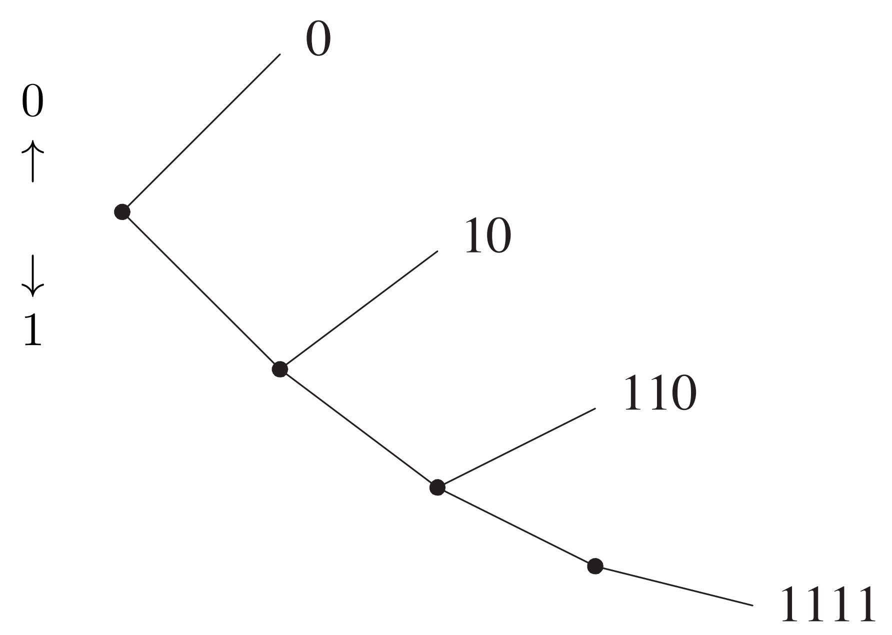

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Proof Direct part follows because a prefix code is uniquely decodable and hence satisfies Kraft's inequality.

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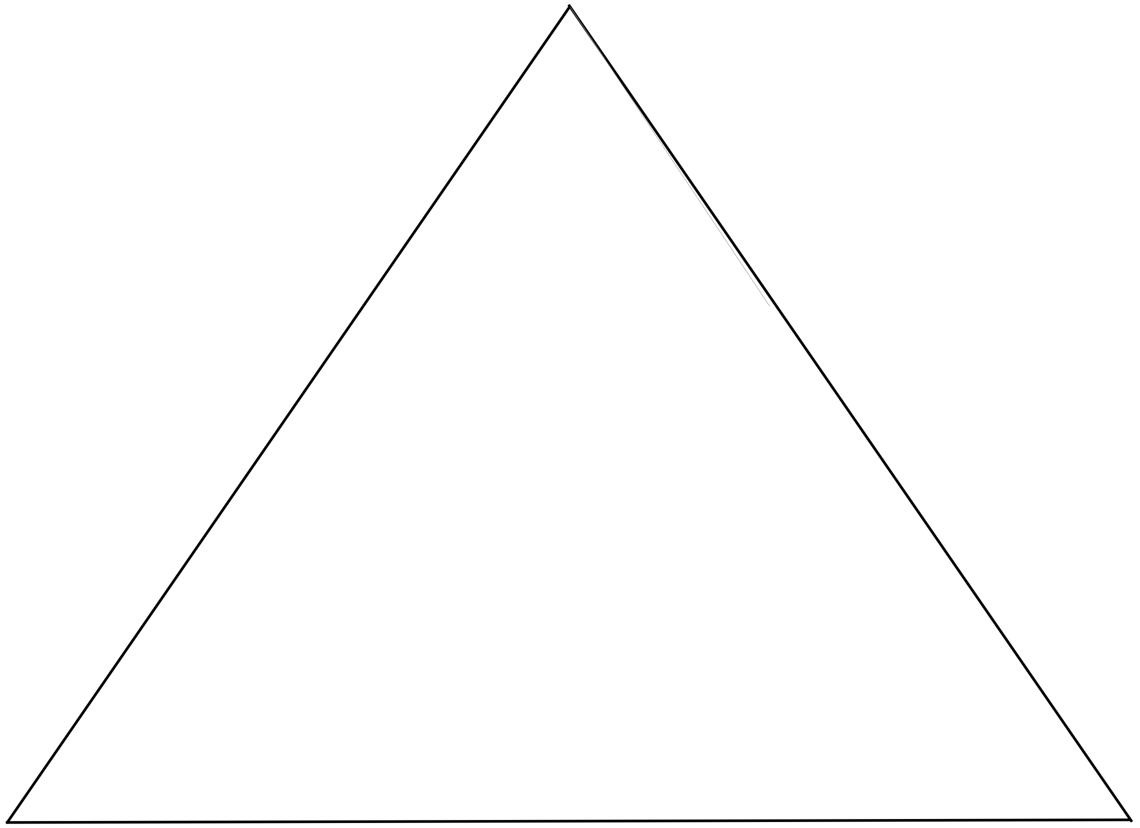

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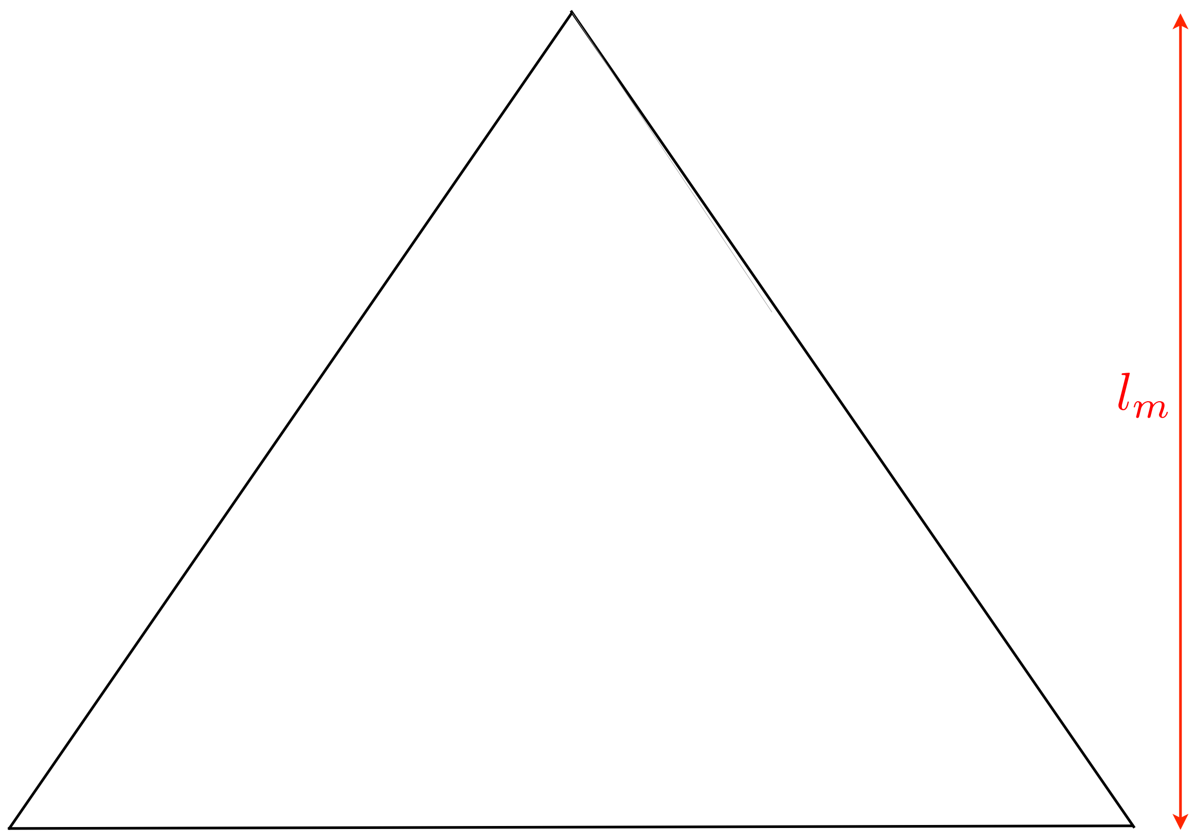

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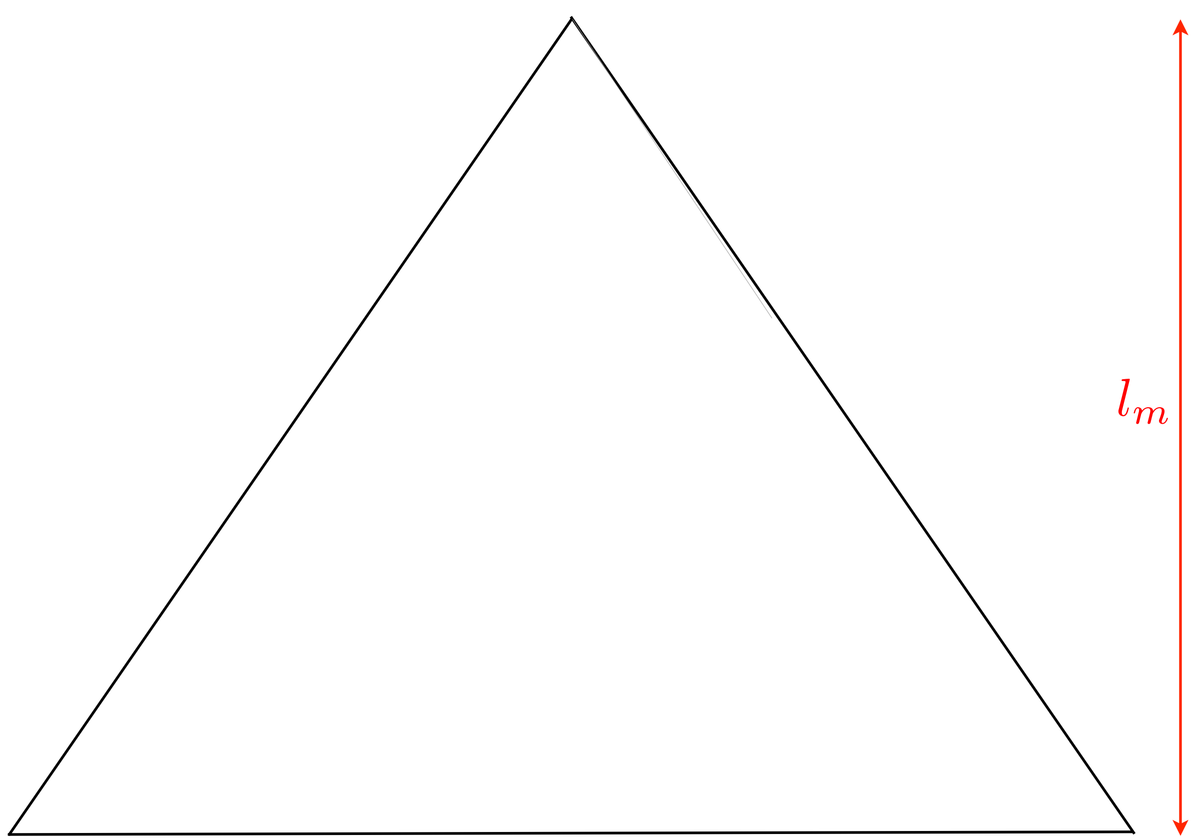

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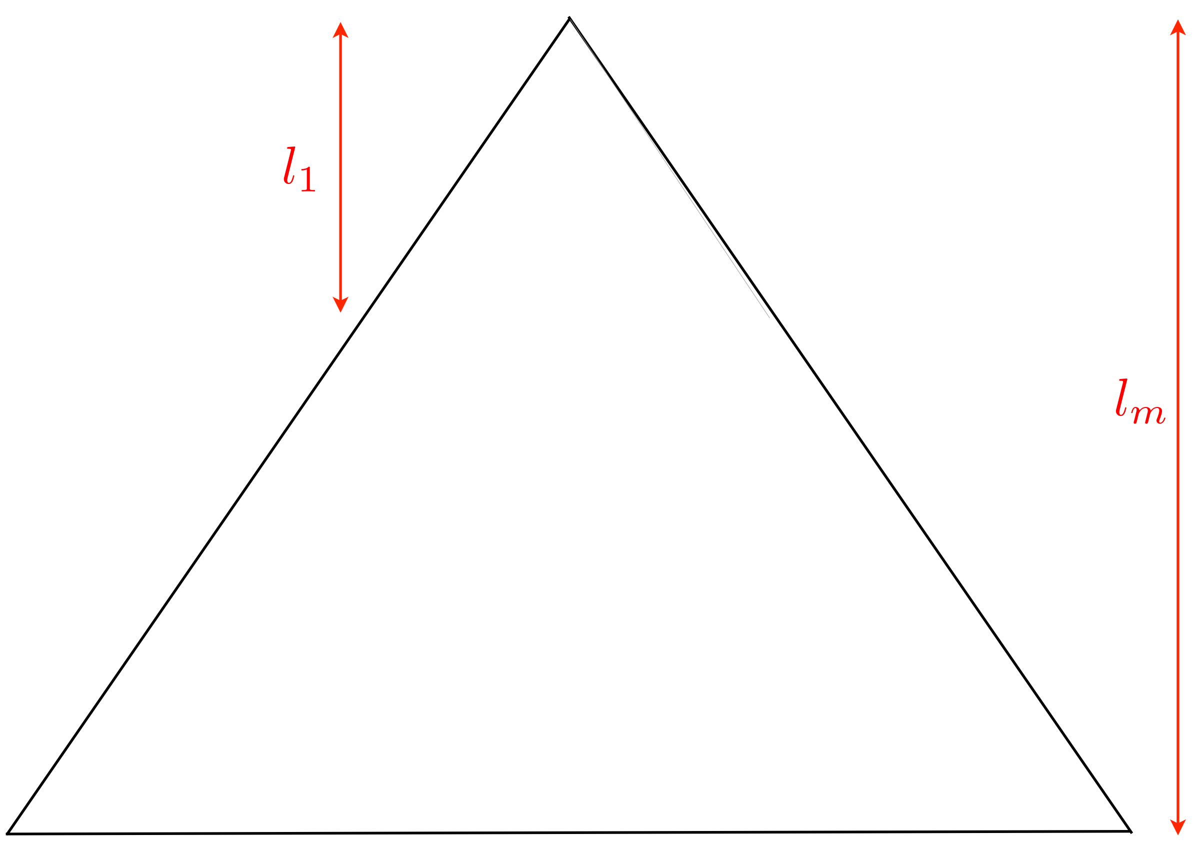

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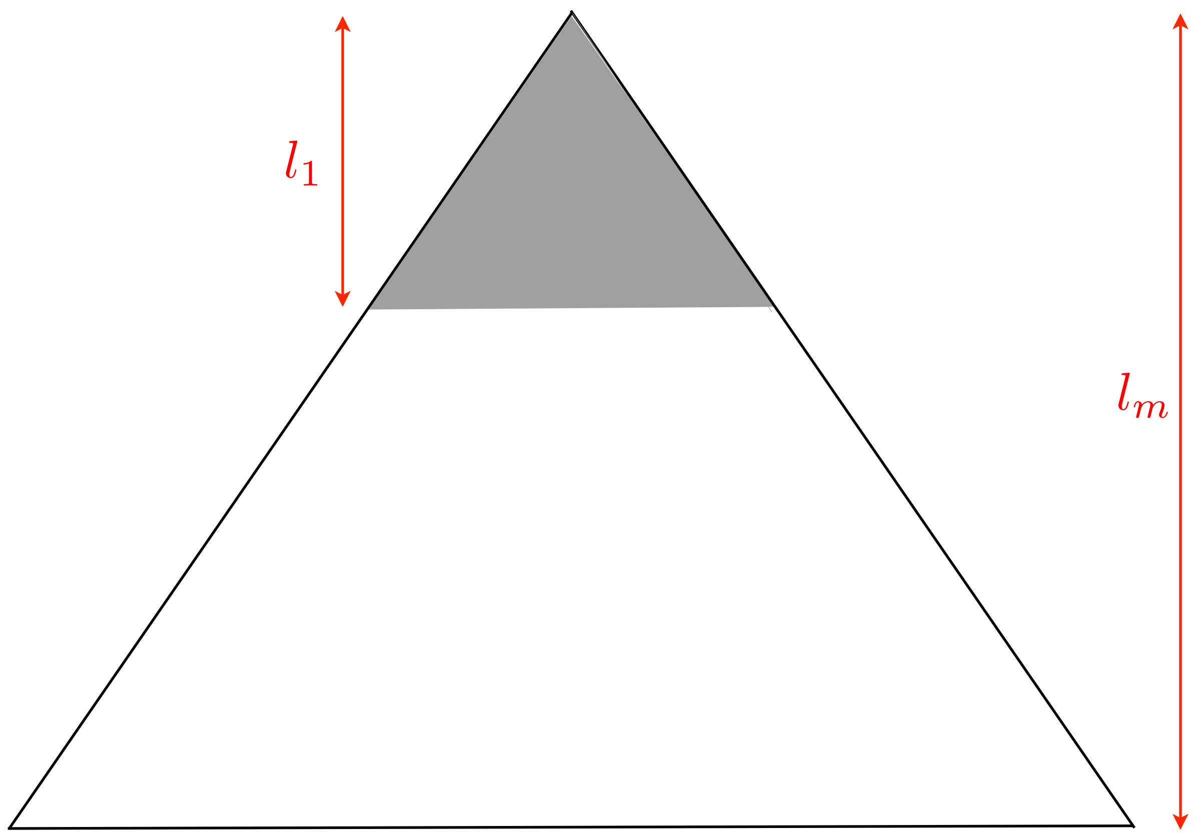

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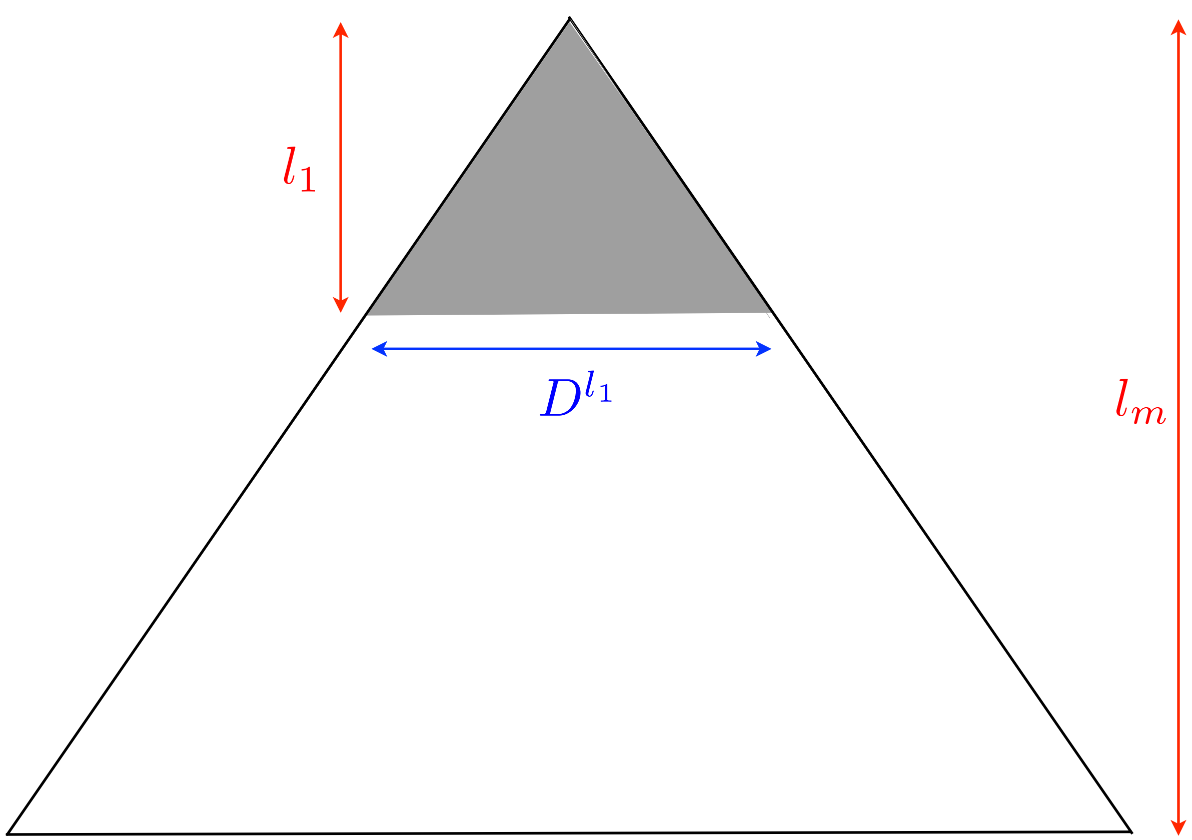

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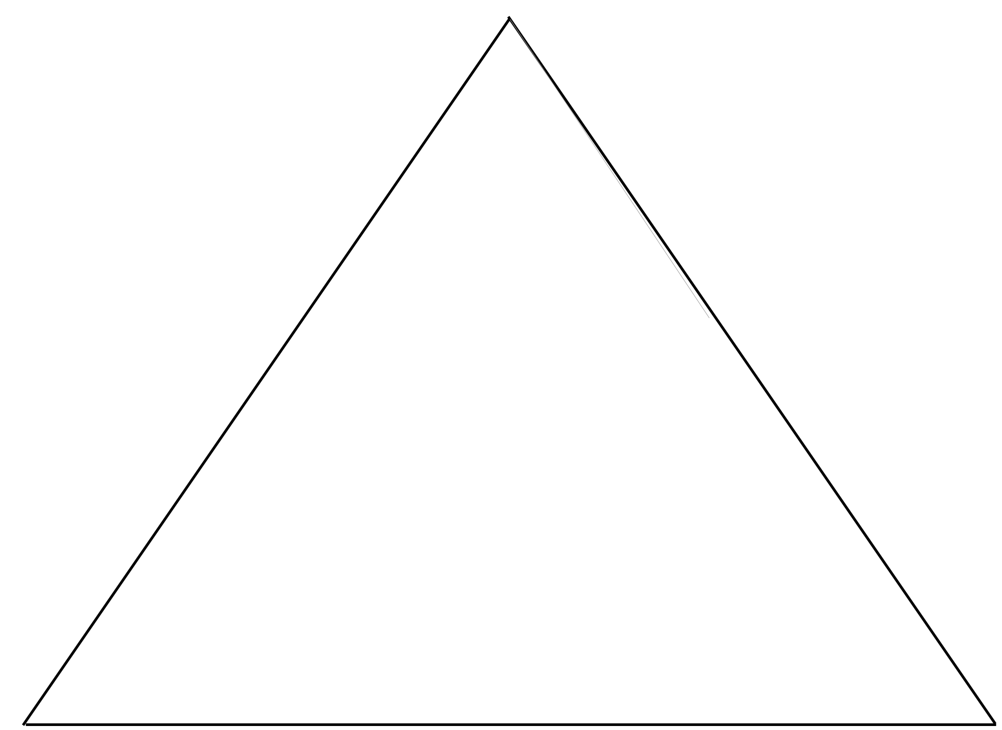

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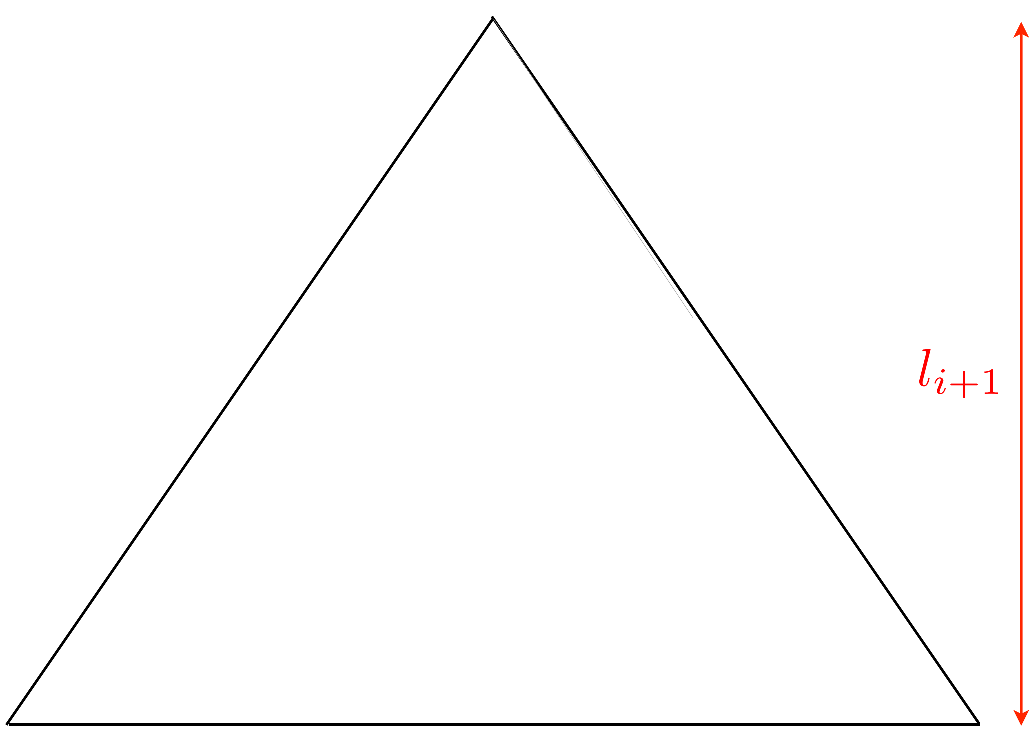

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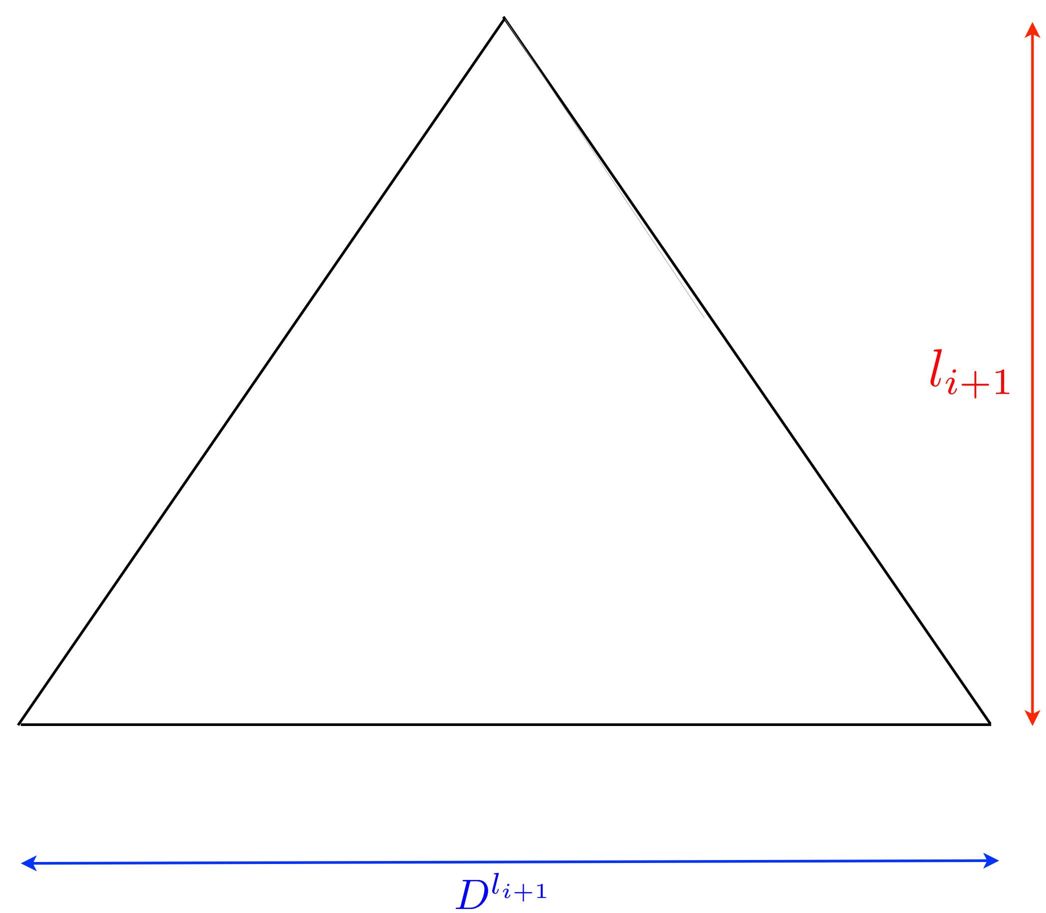

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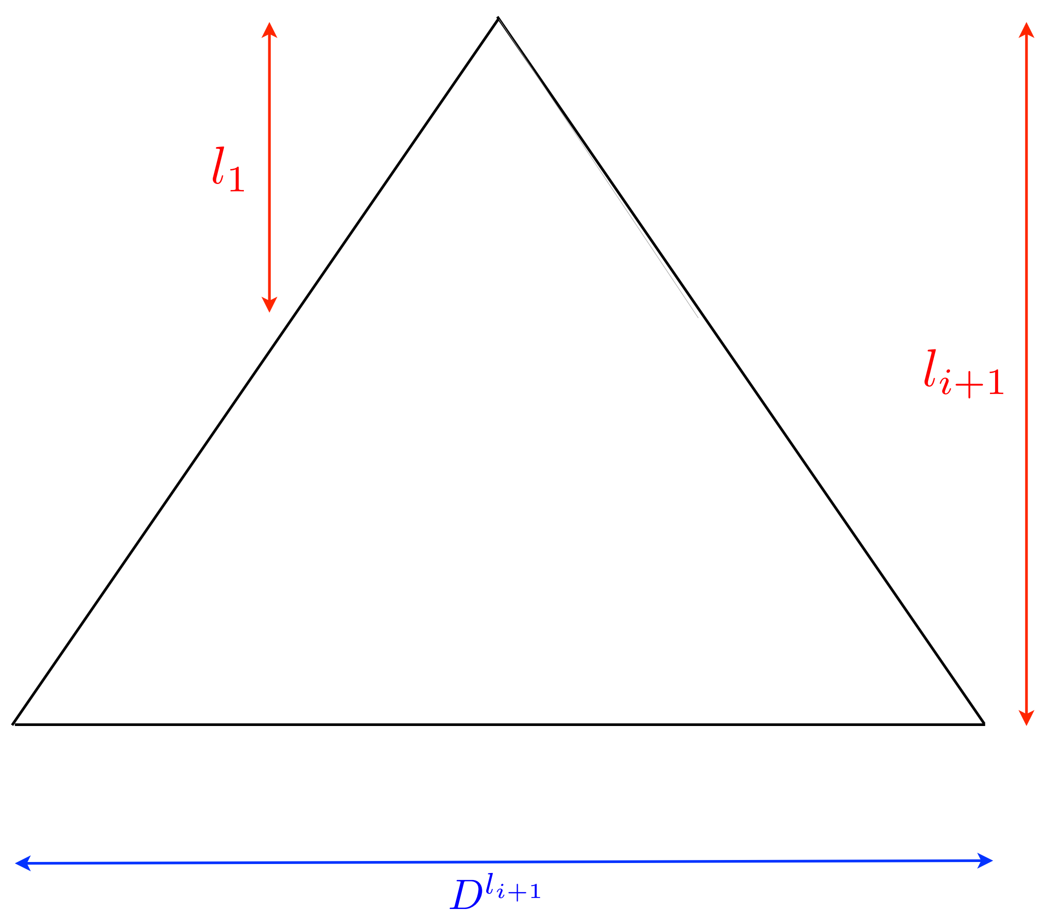

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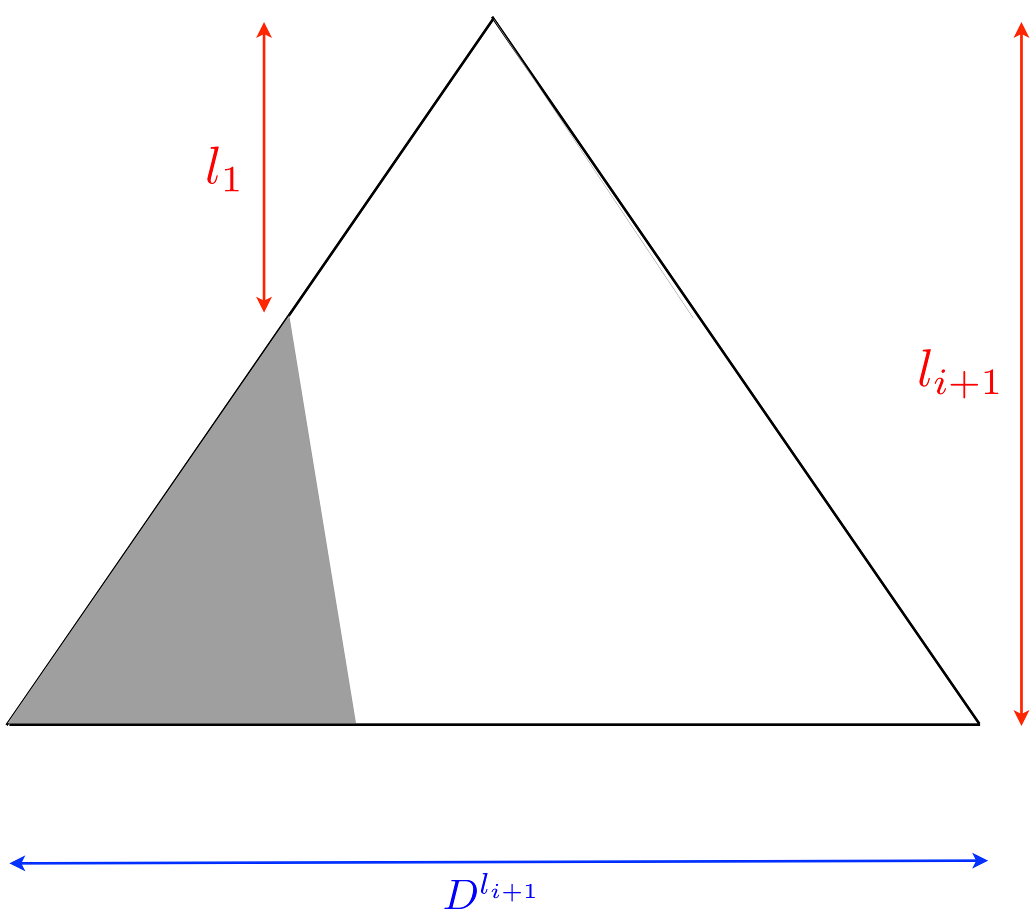

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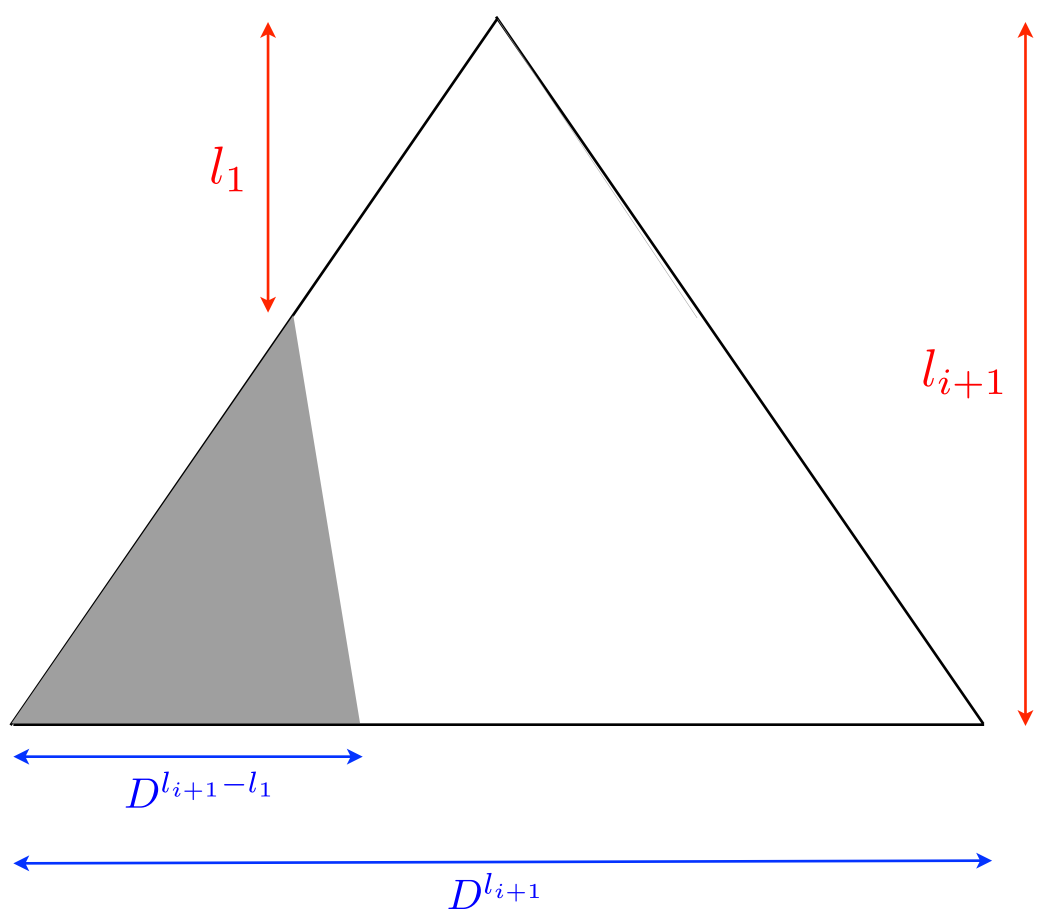

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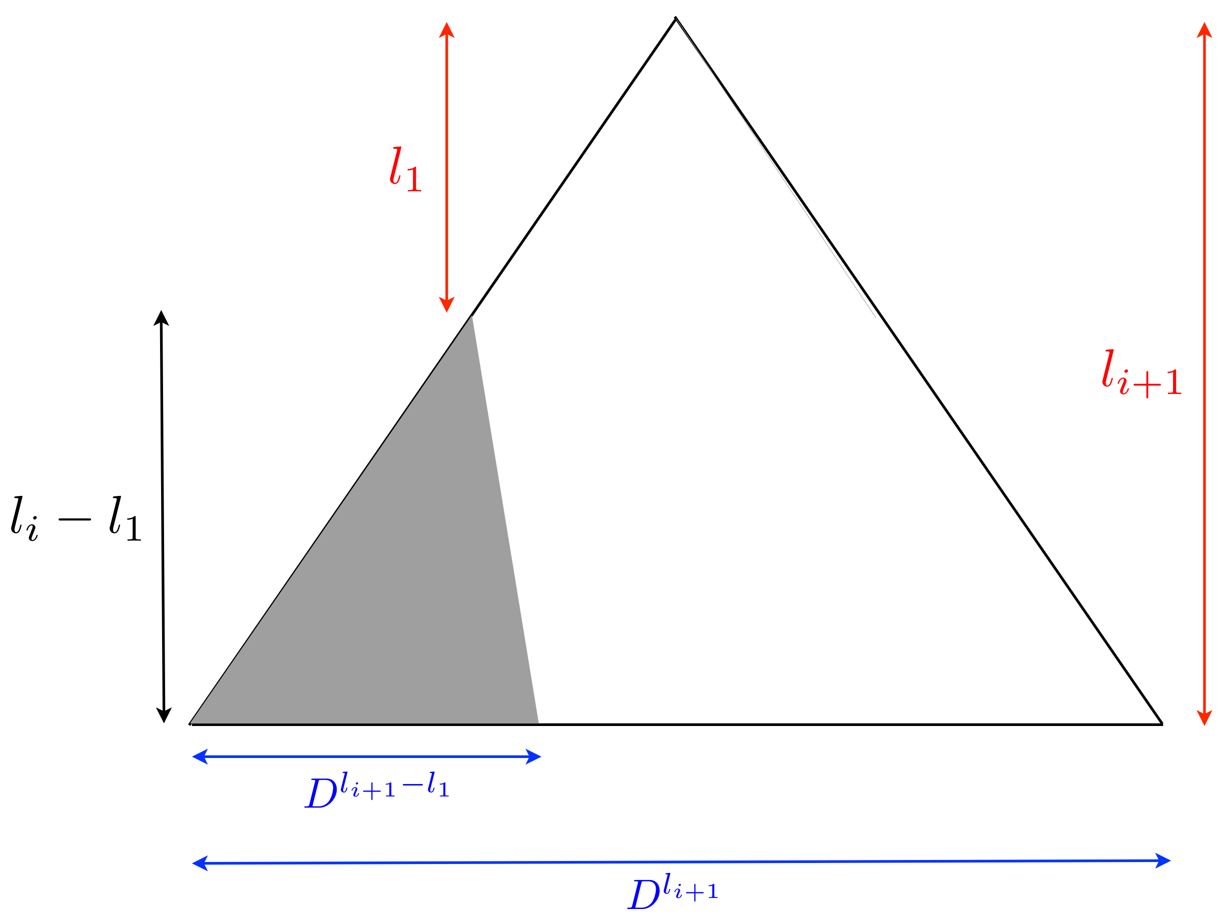

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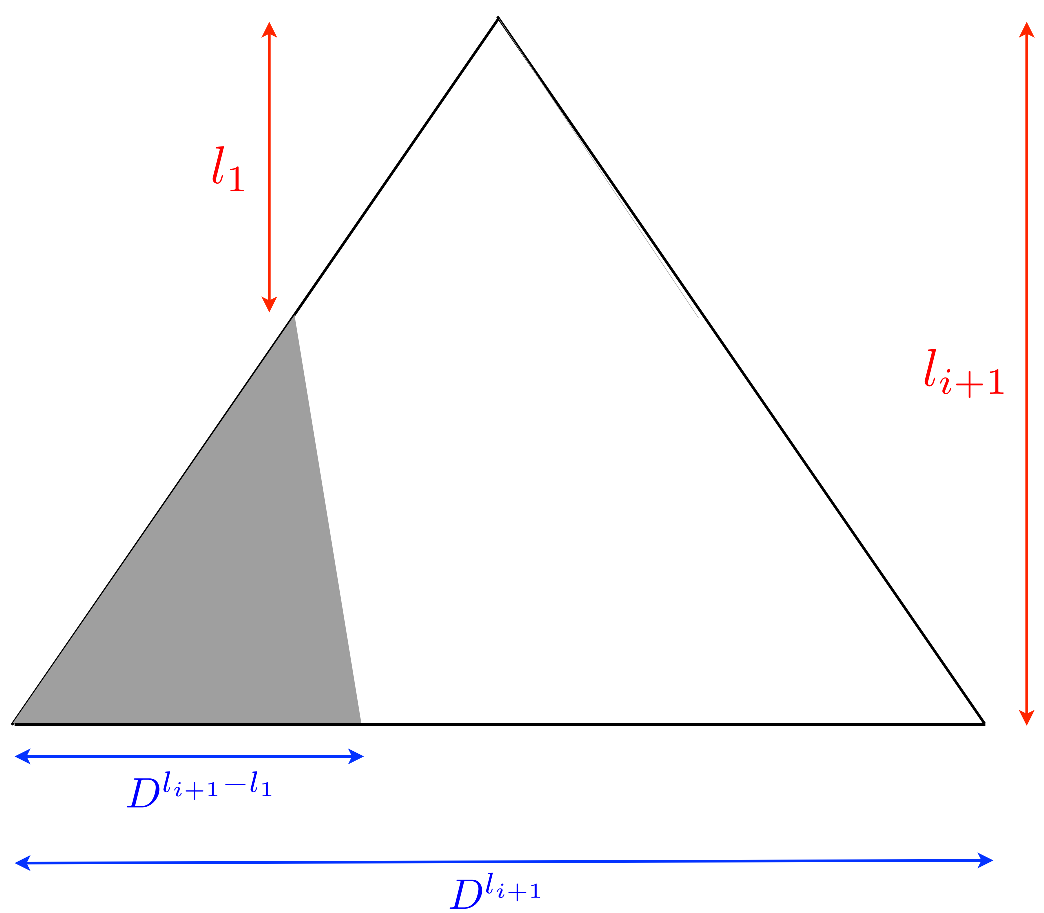

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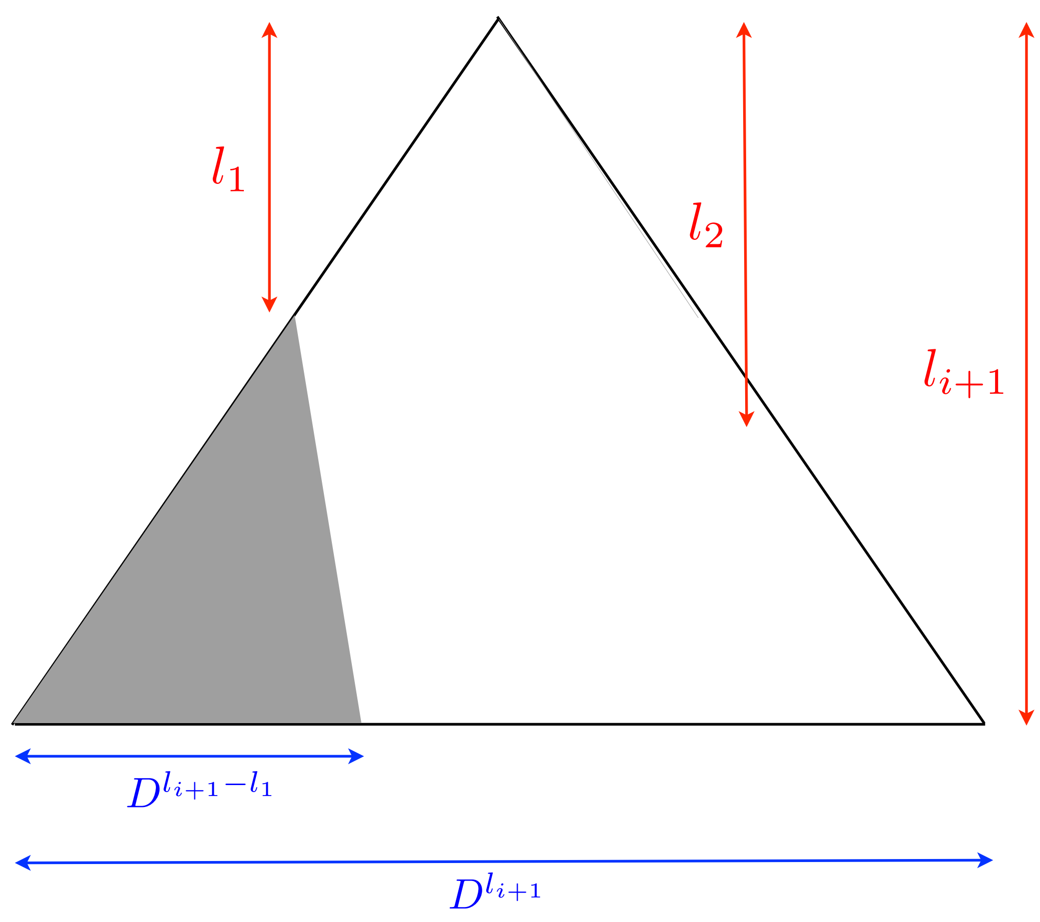

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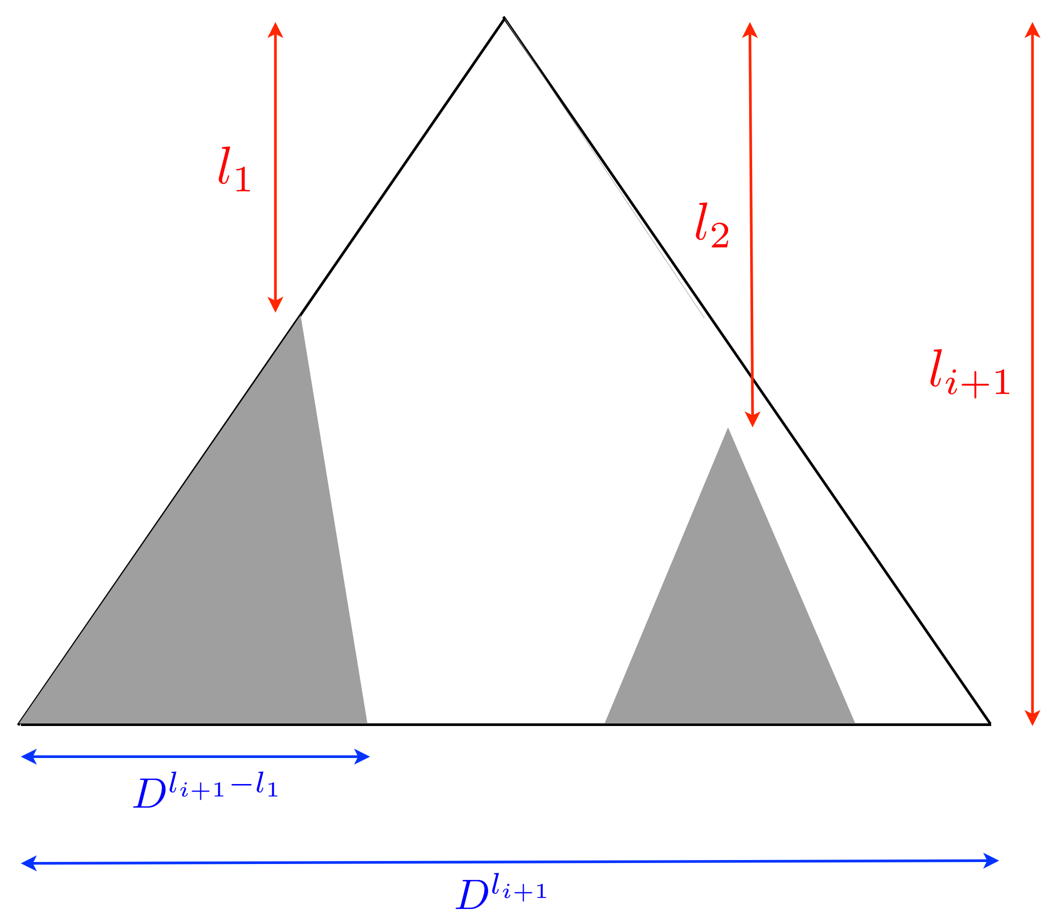

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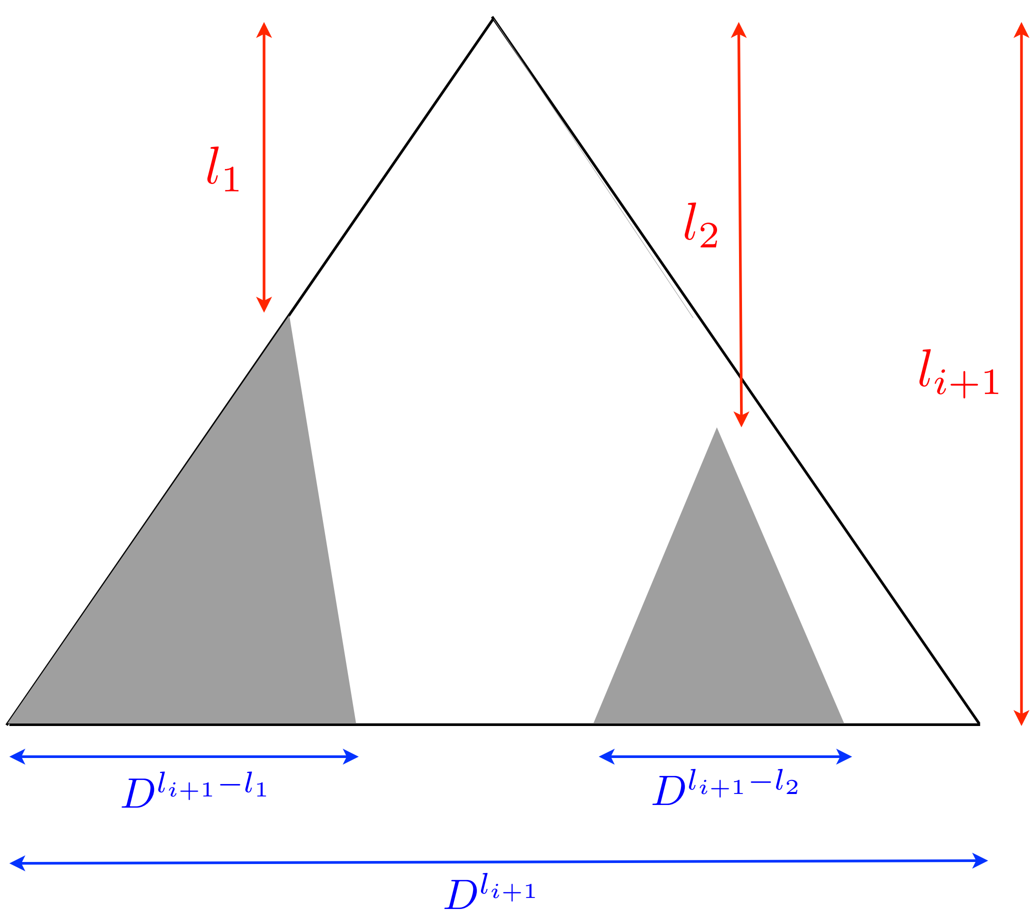

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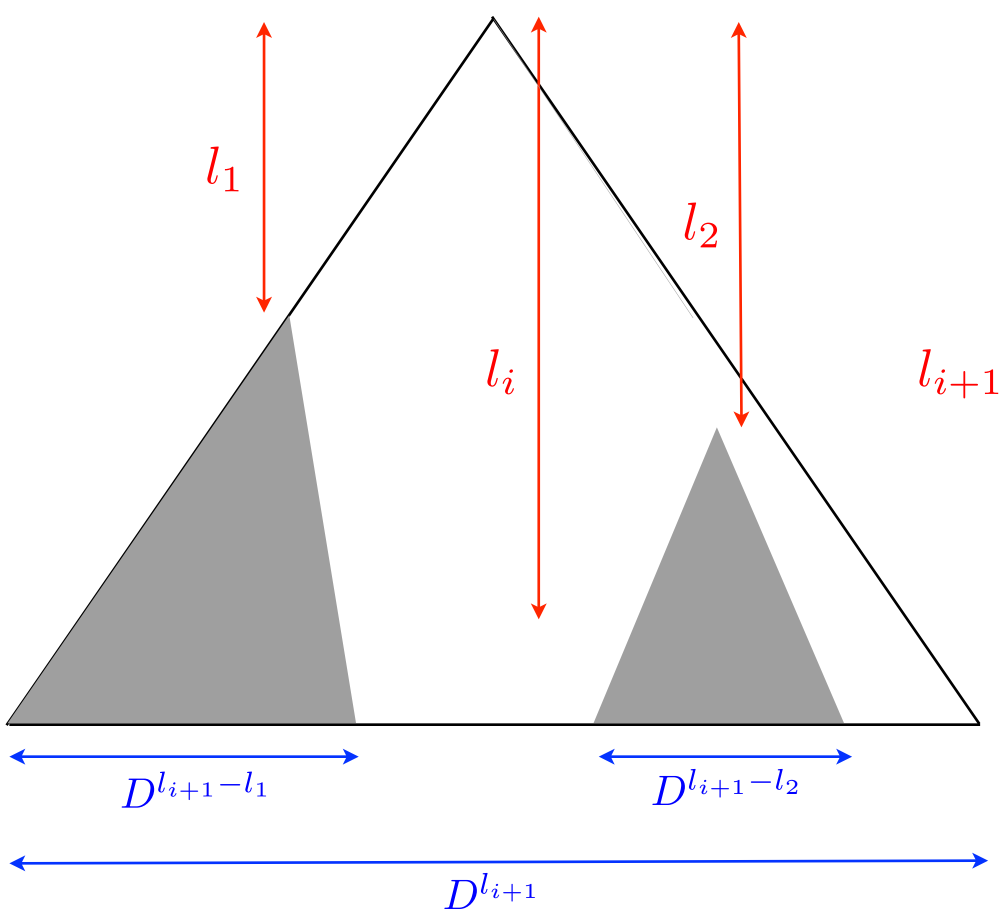

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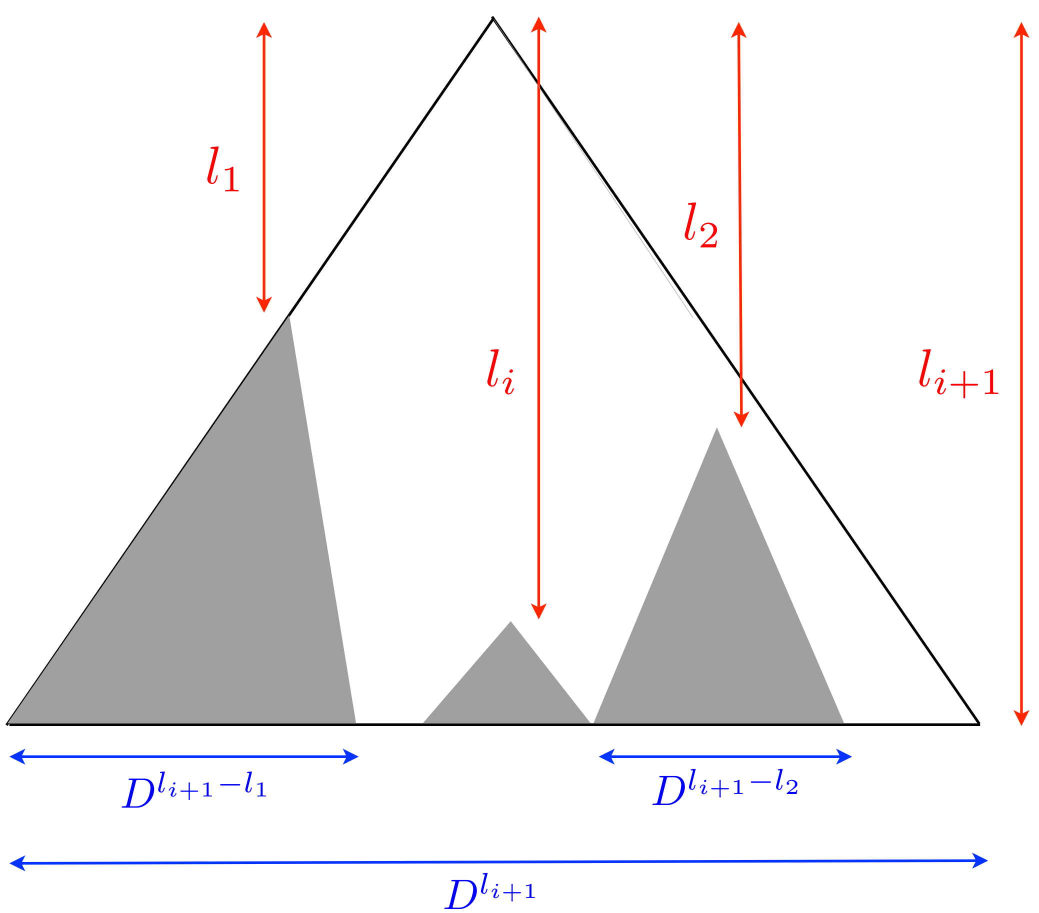

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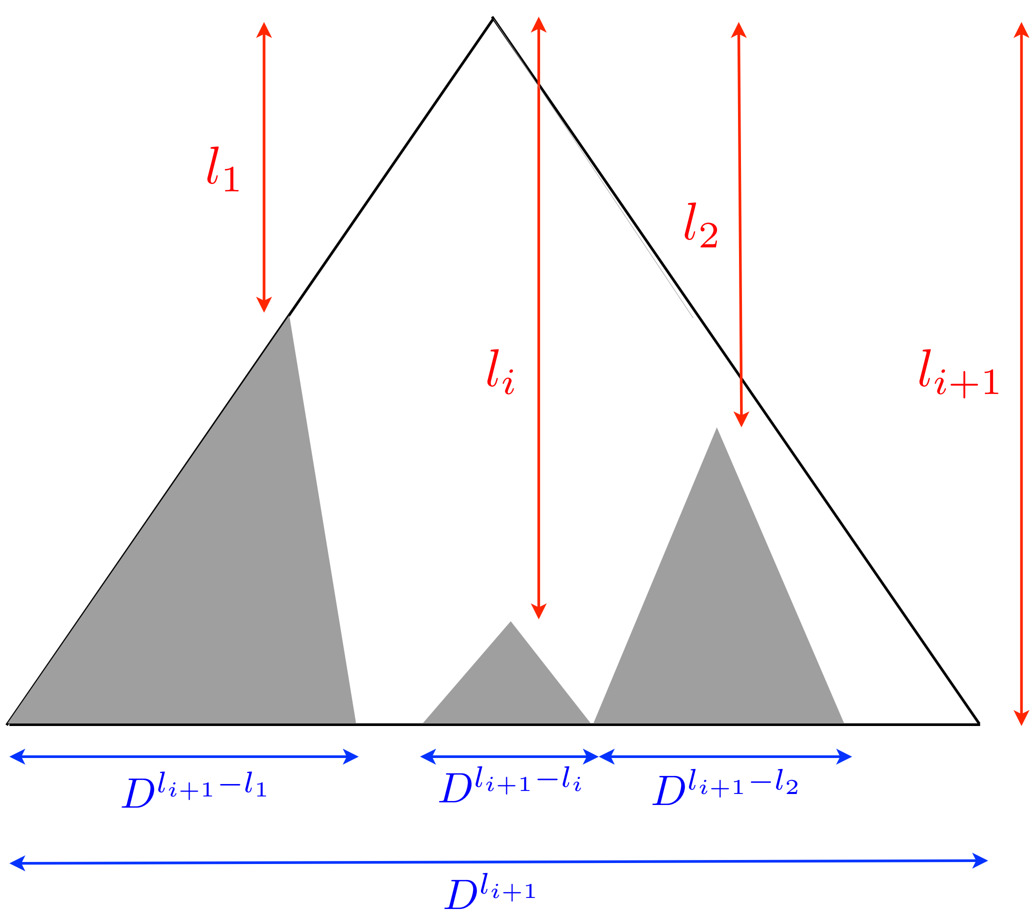

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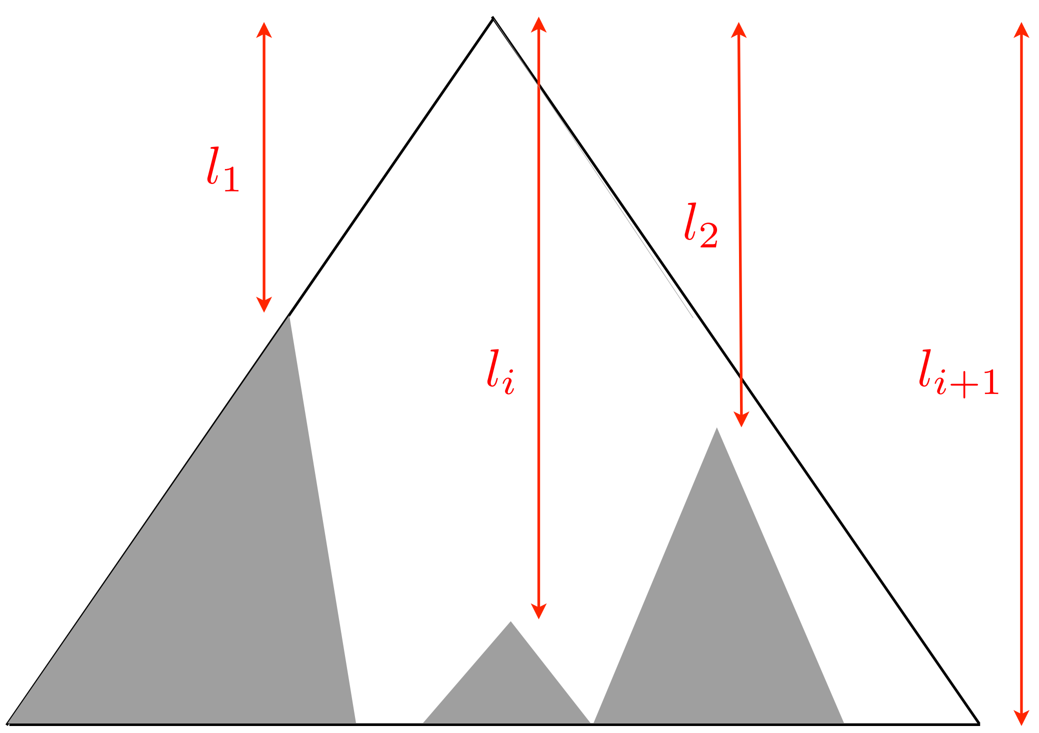

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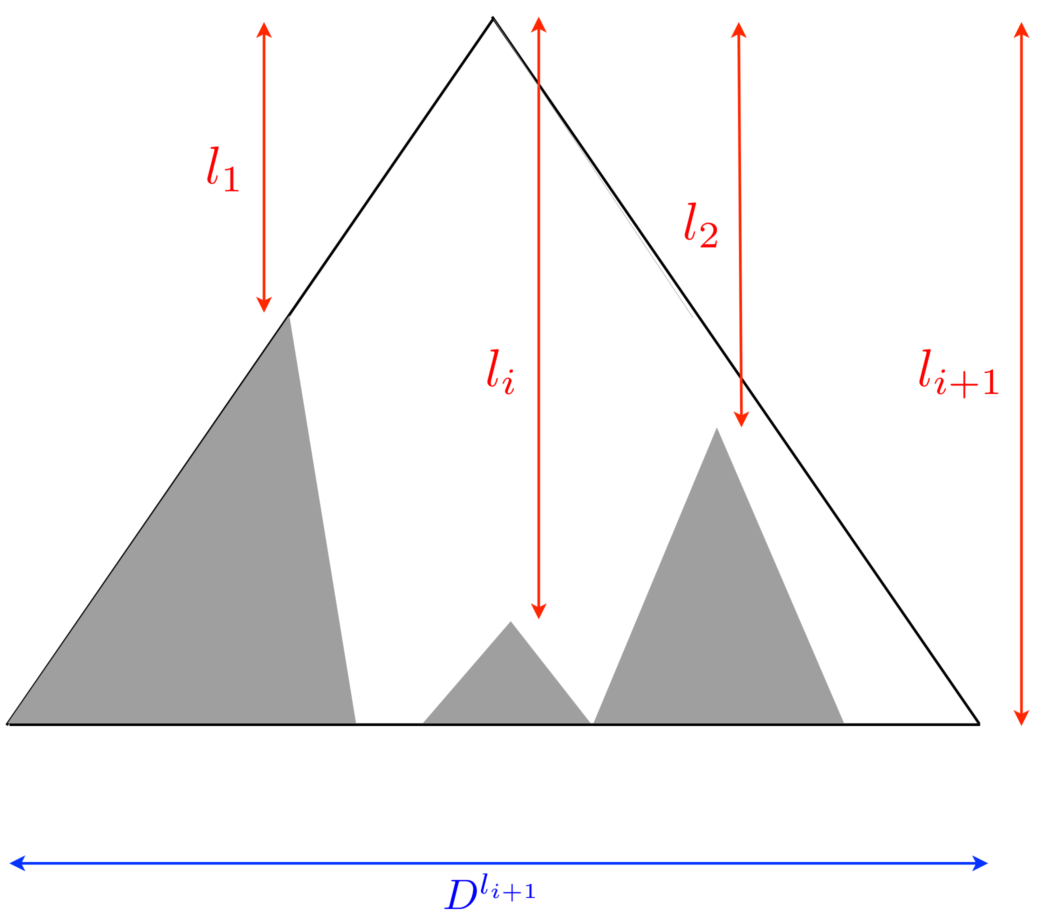

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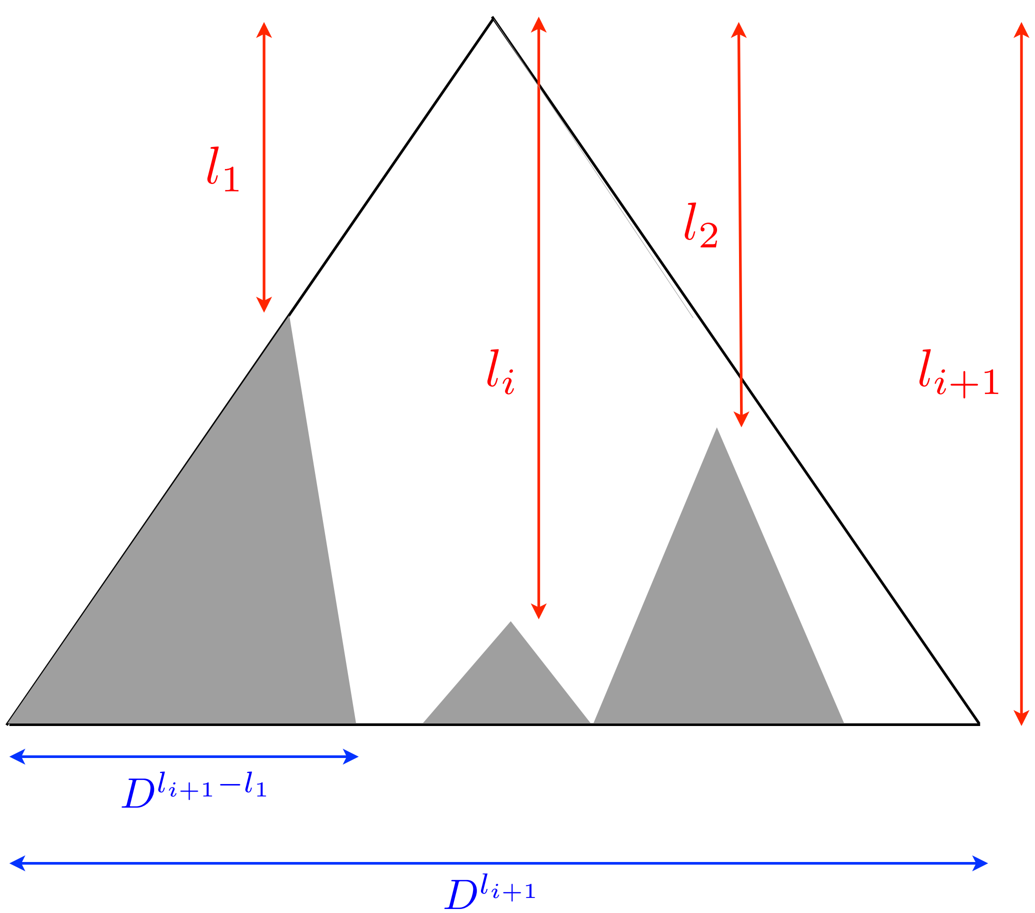

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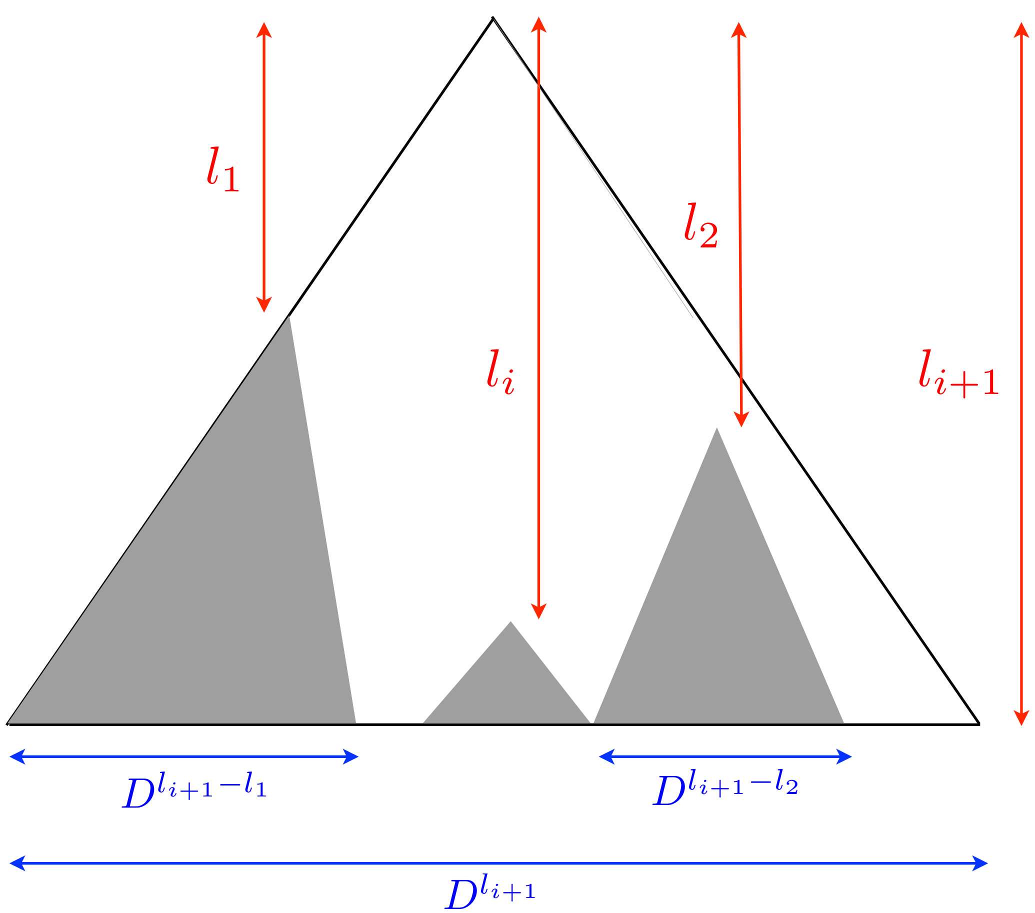

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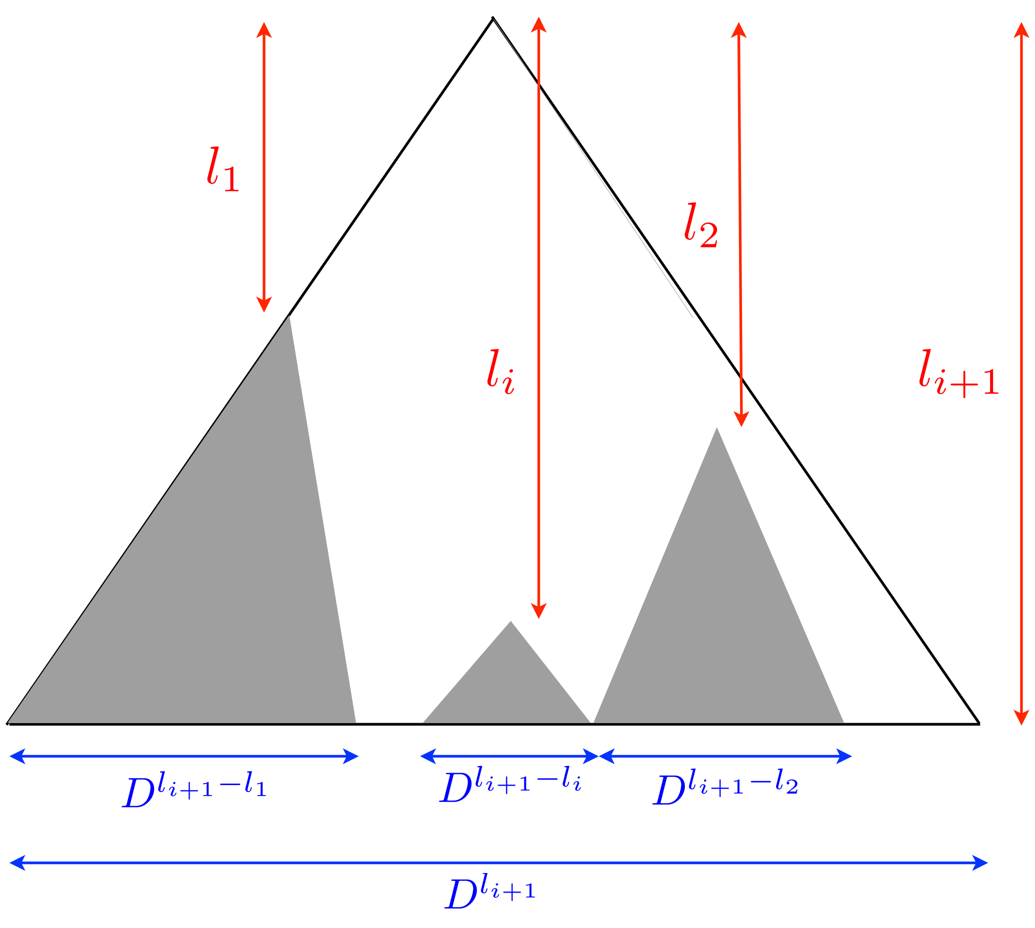

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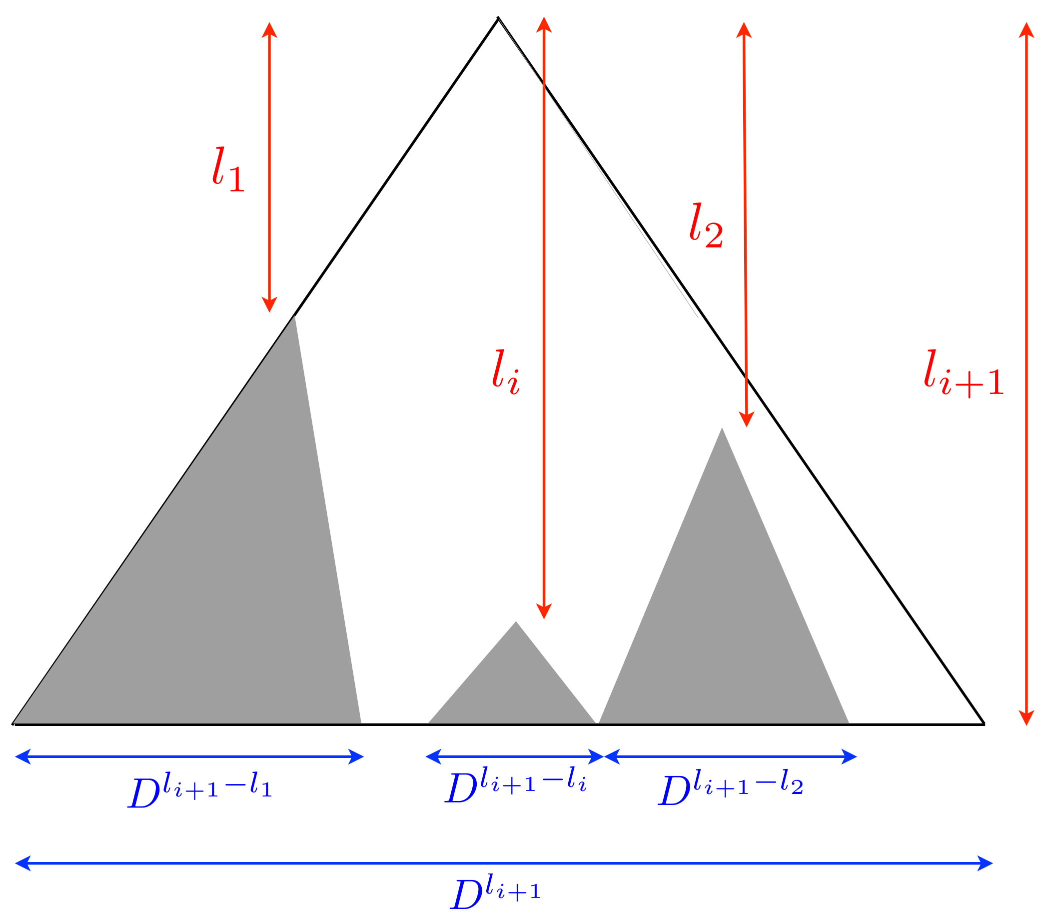

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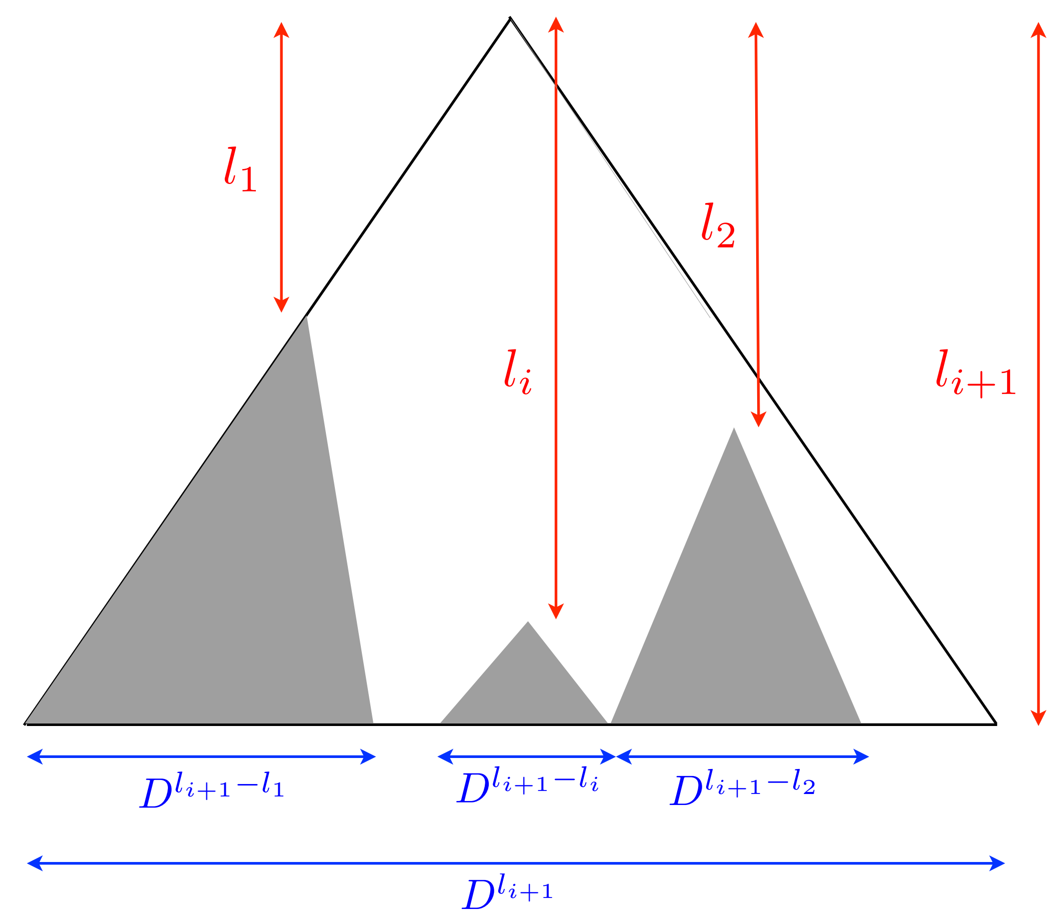

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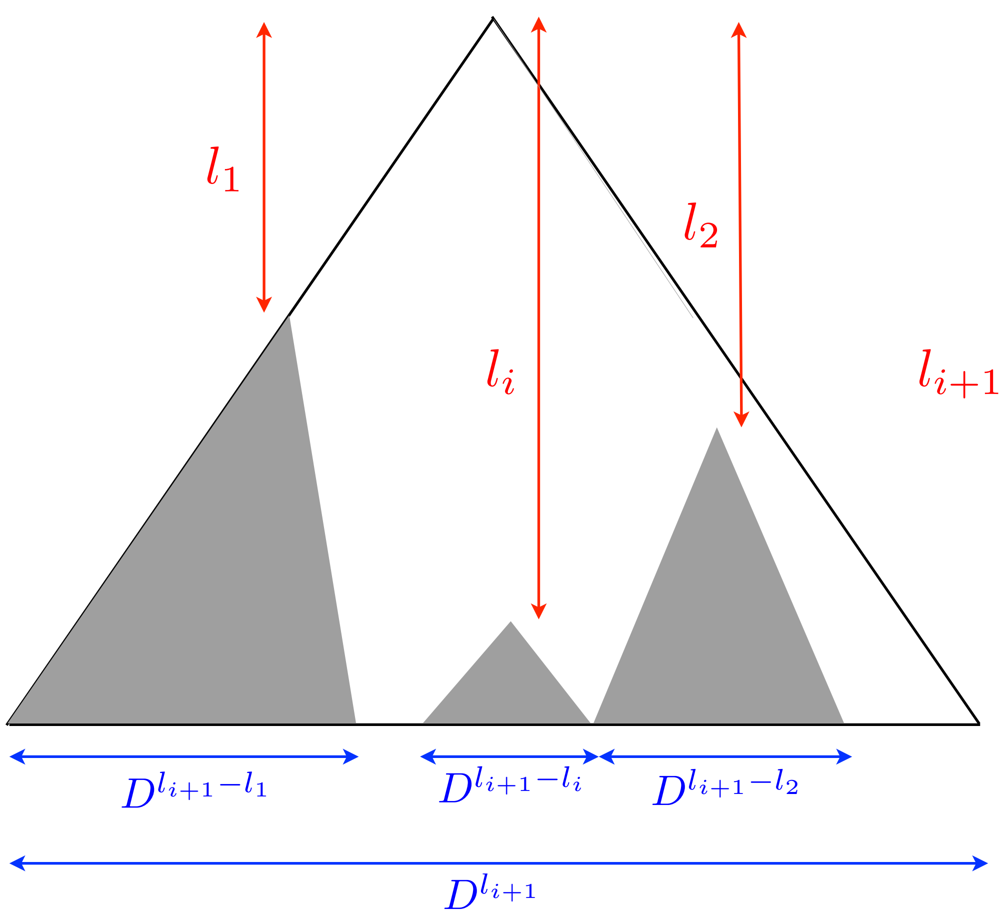

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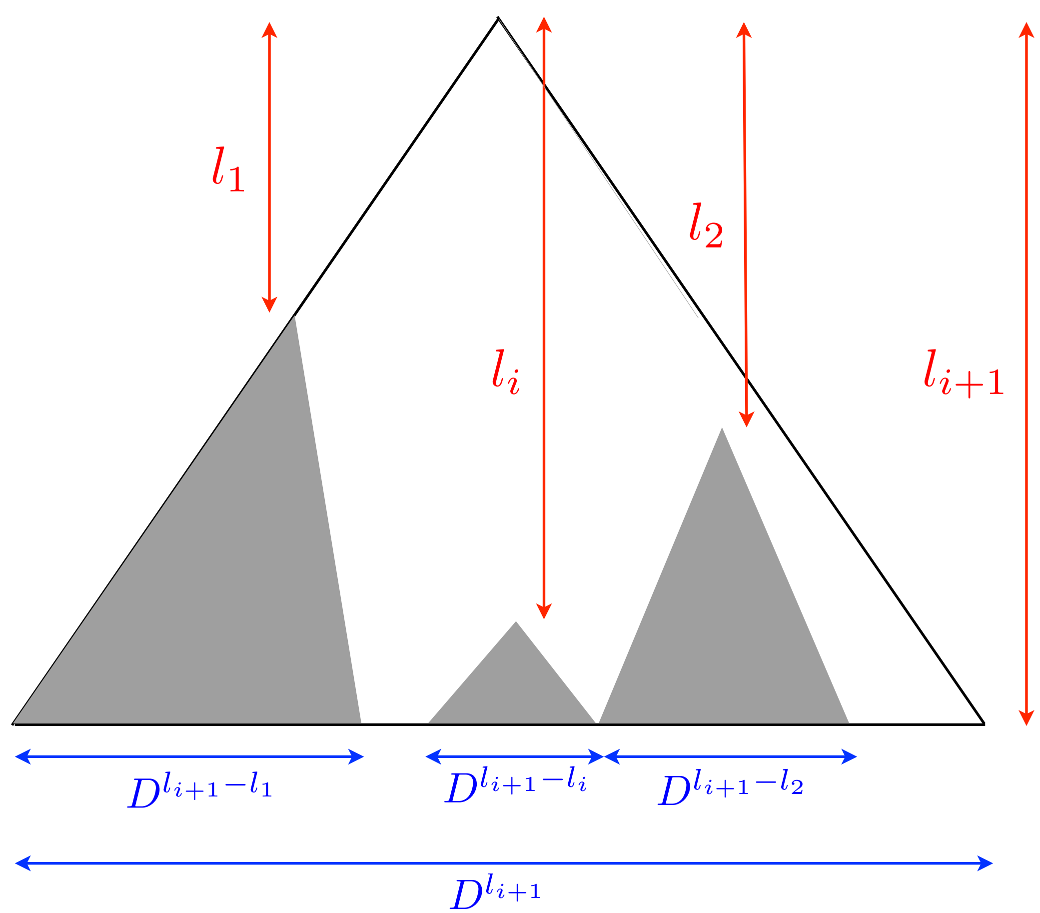

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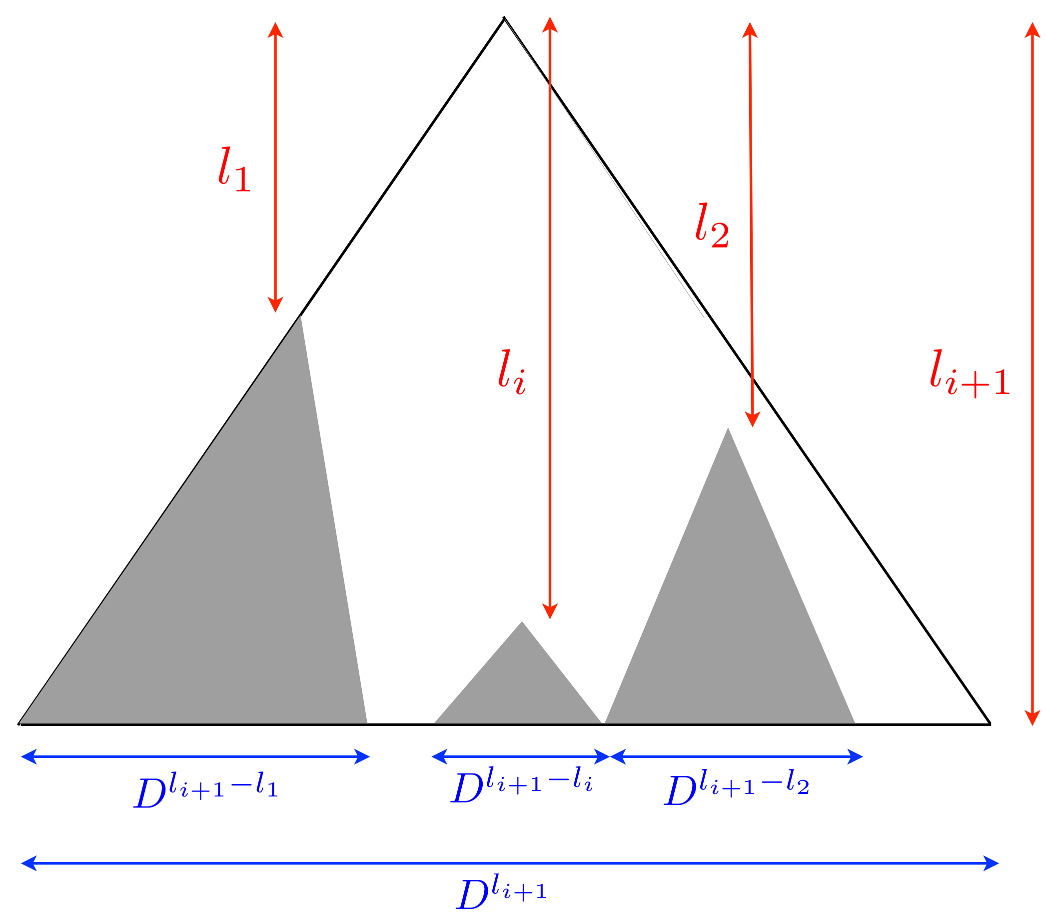

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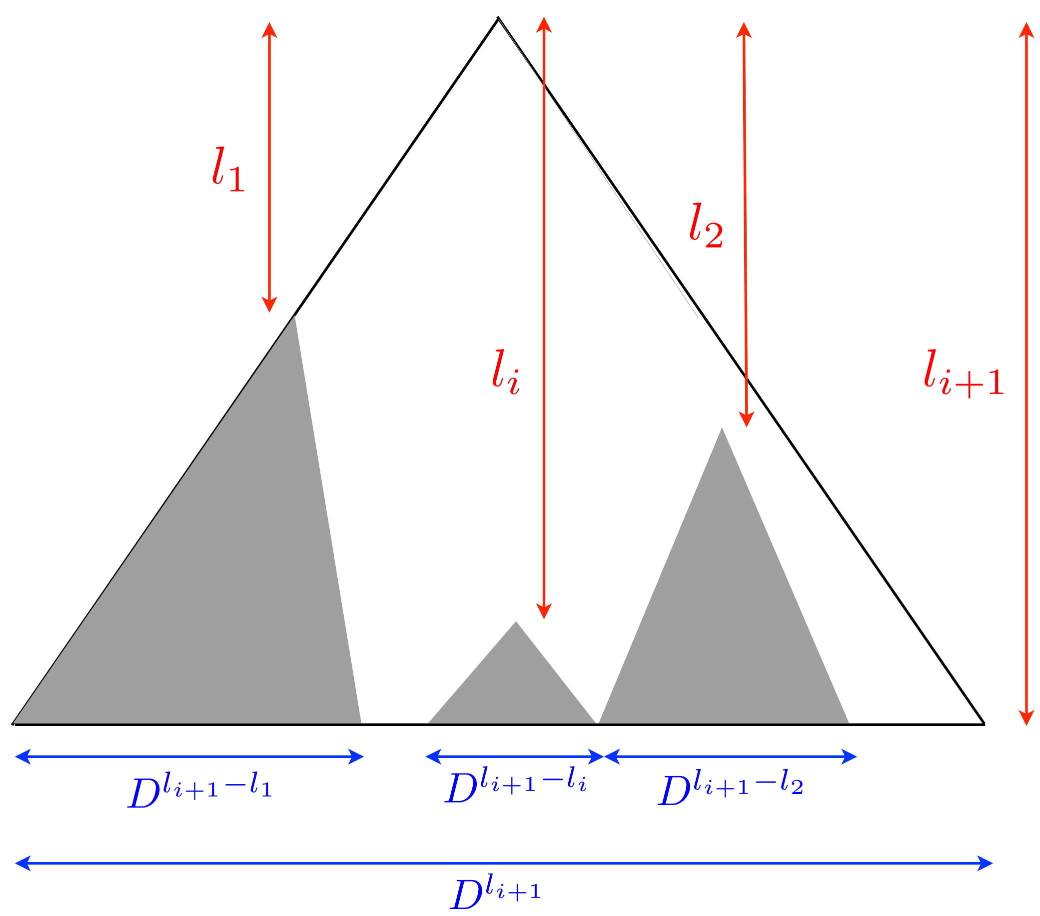

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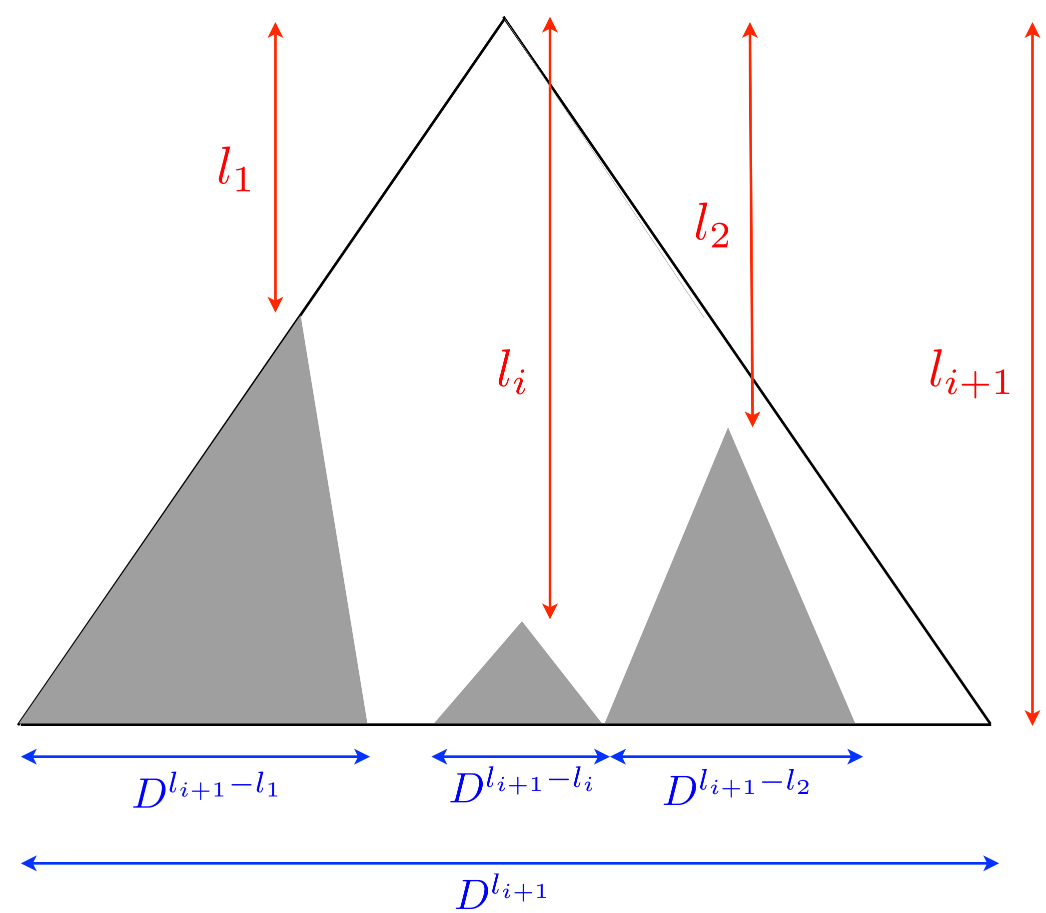

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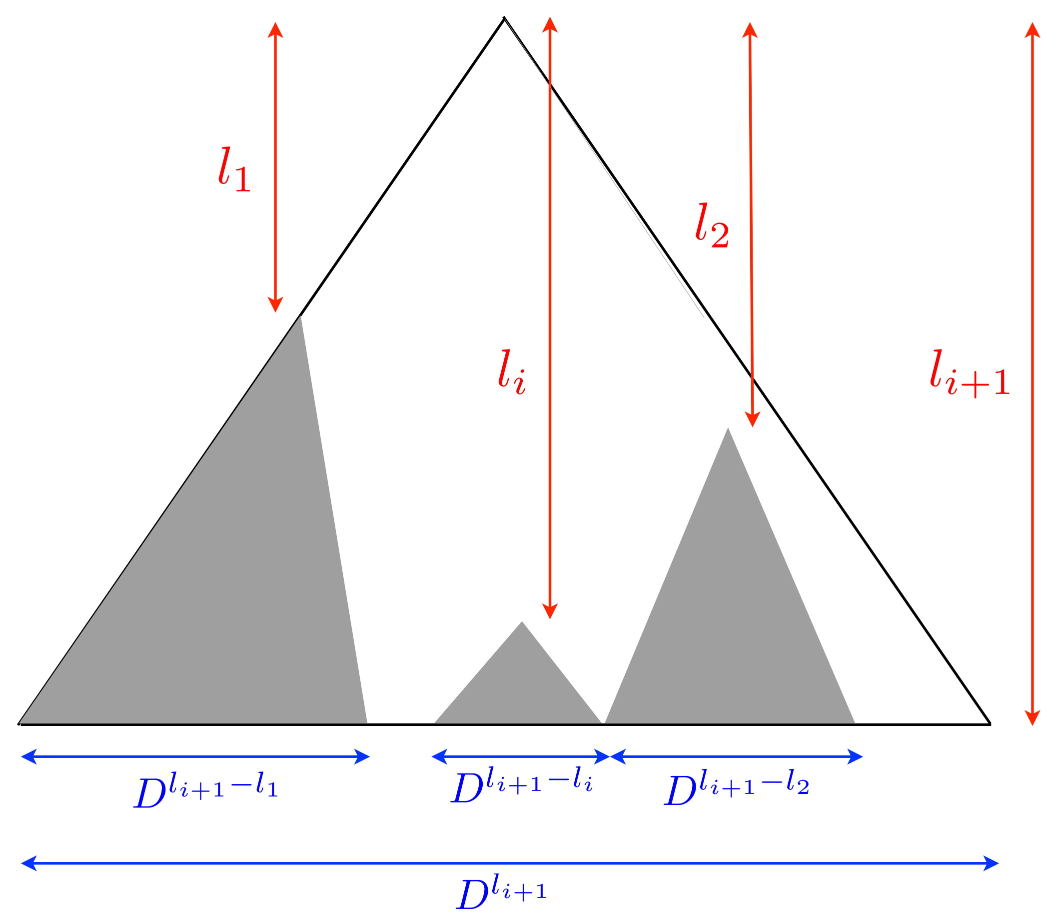

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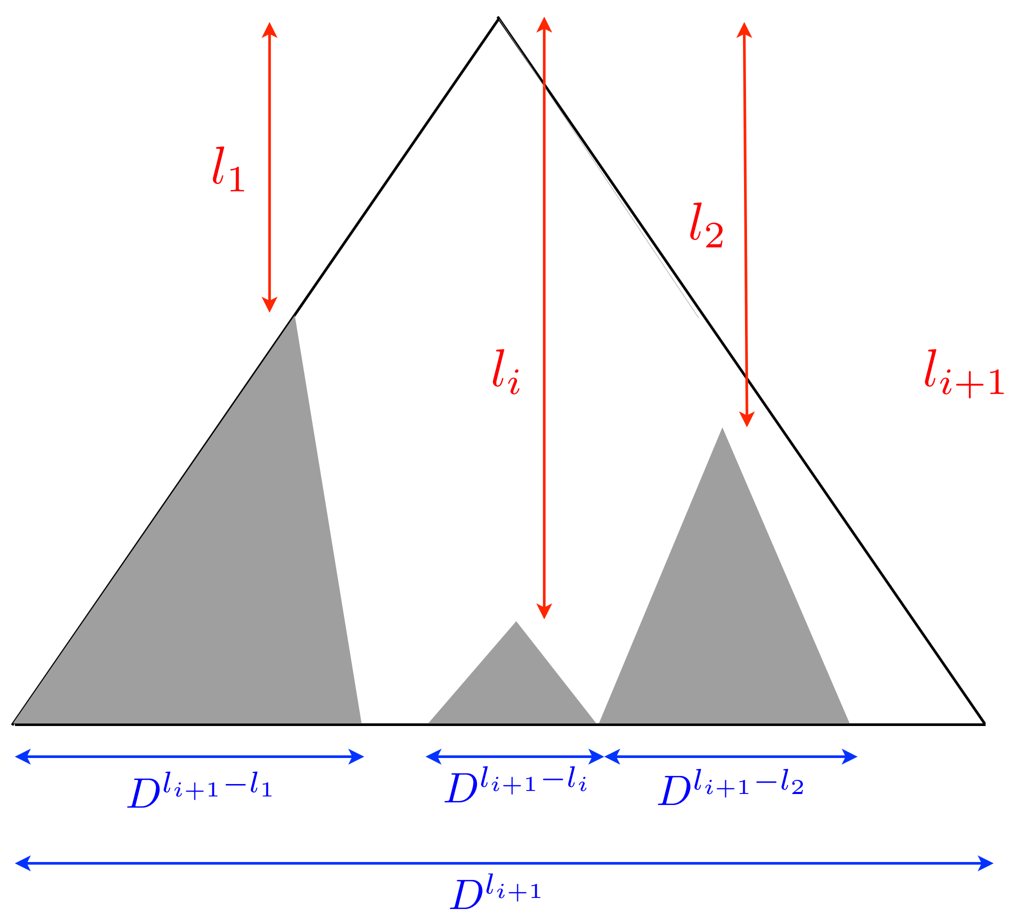

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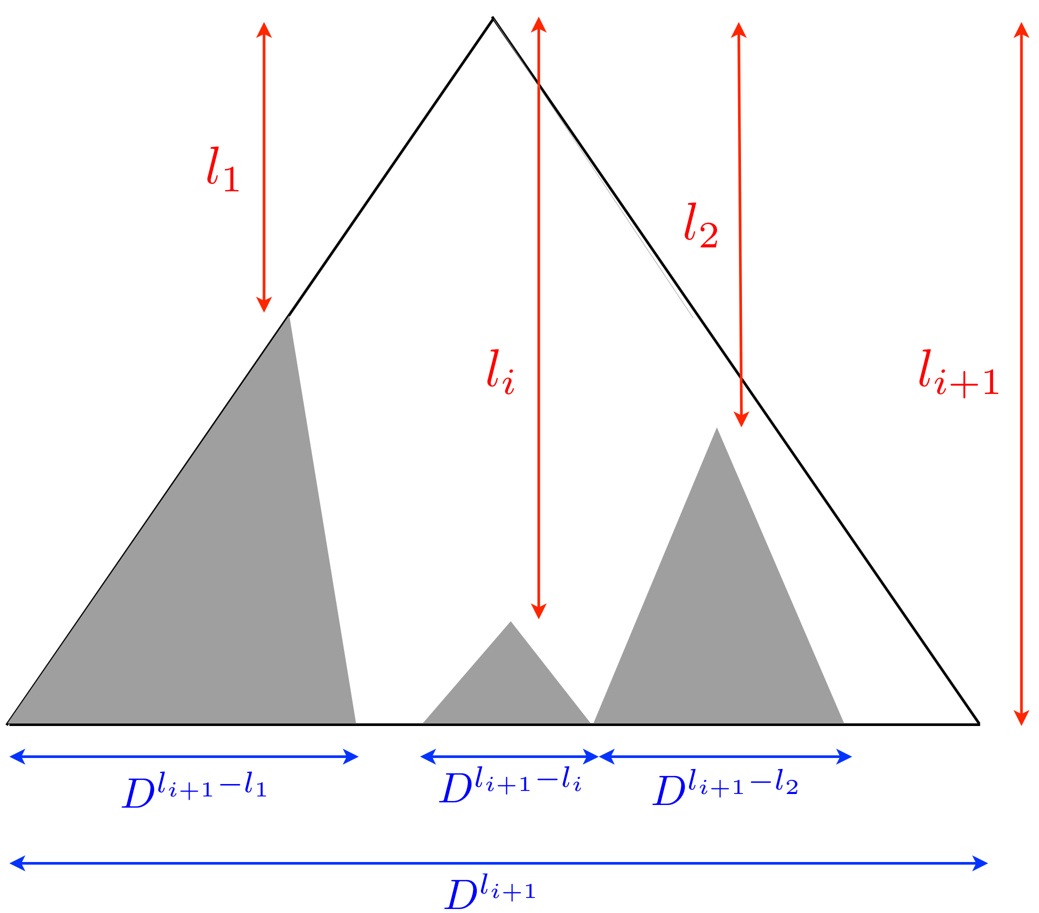

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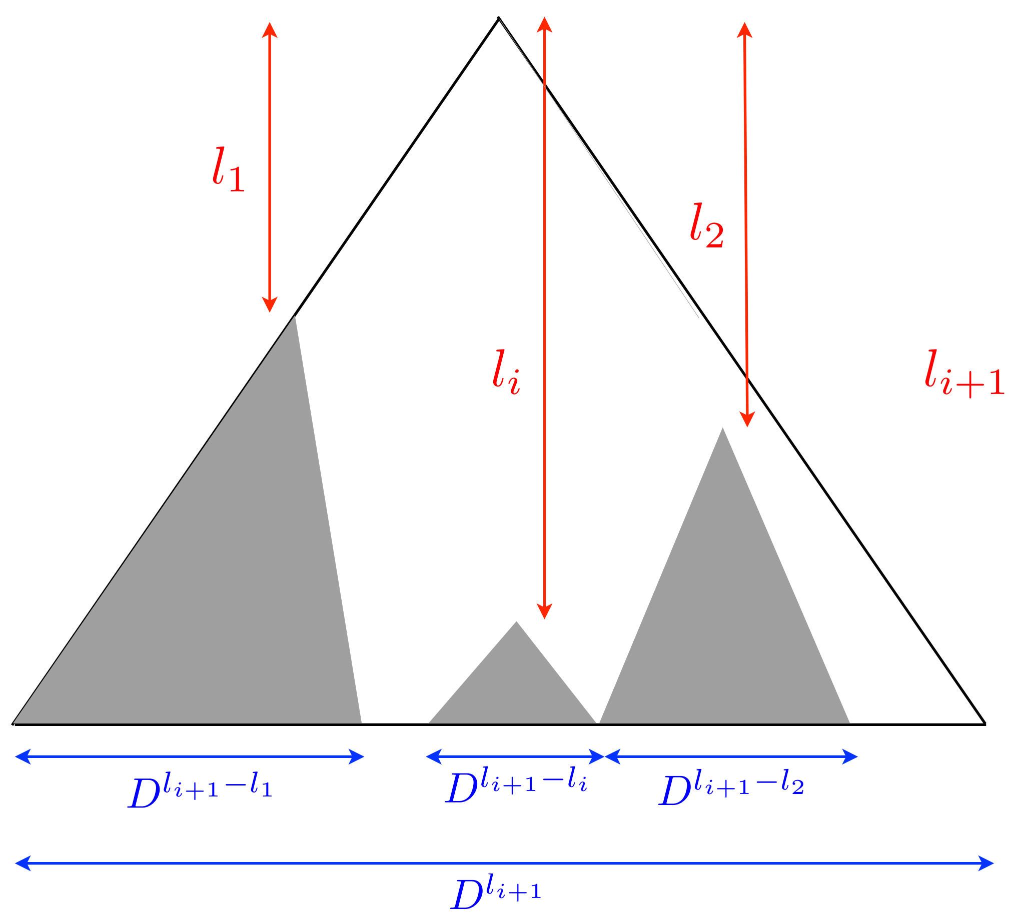

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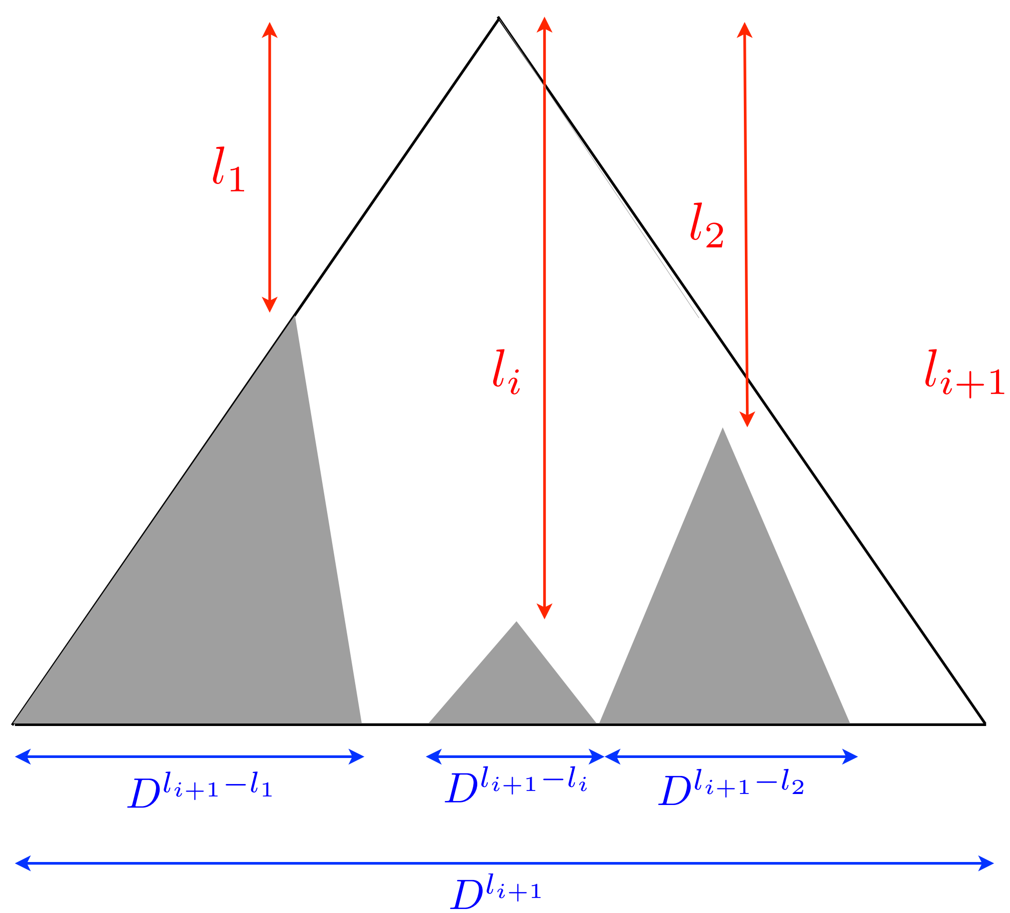

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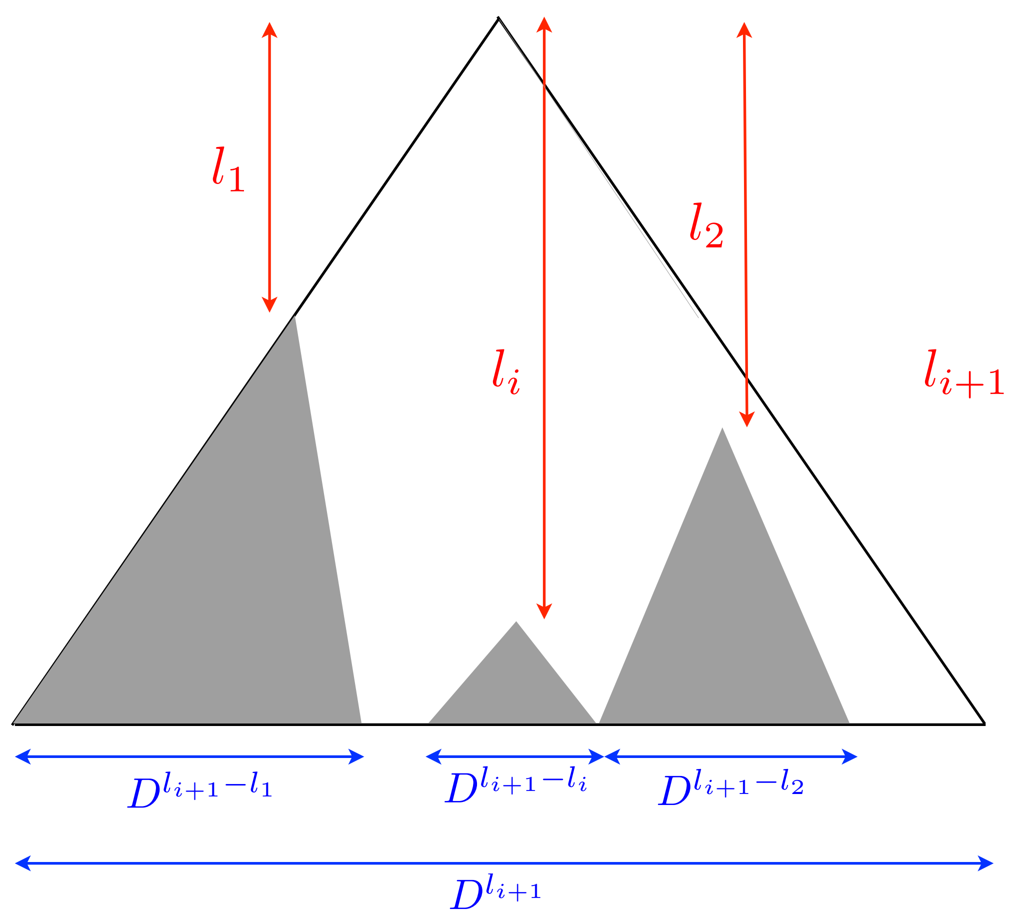

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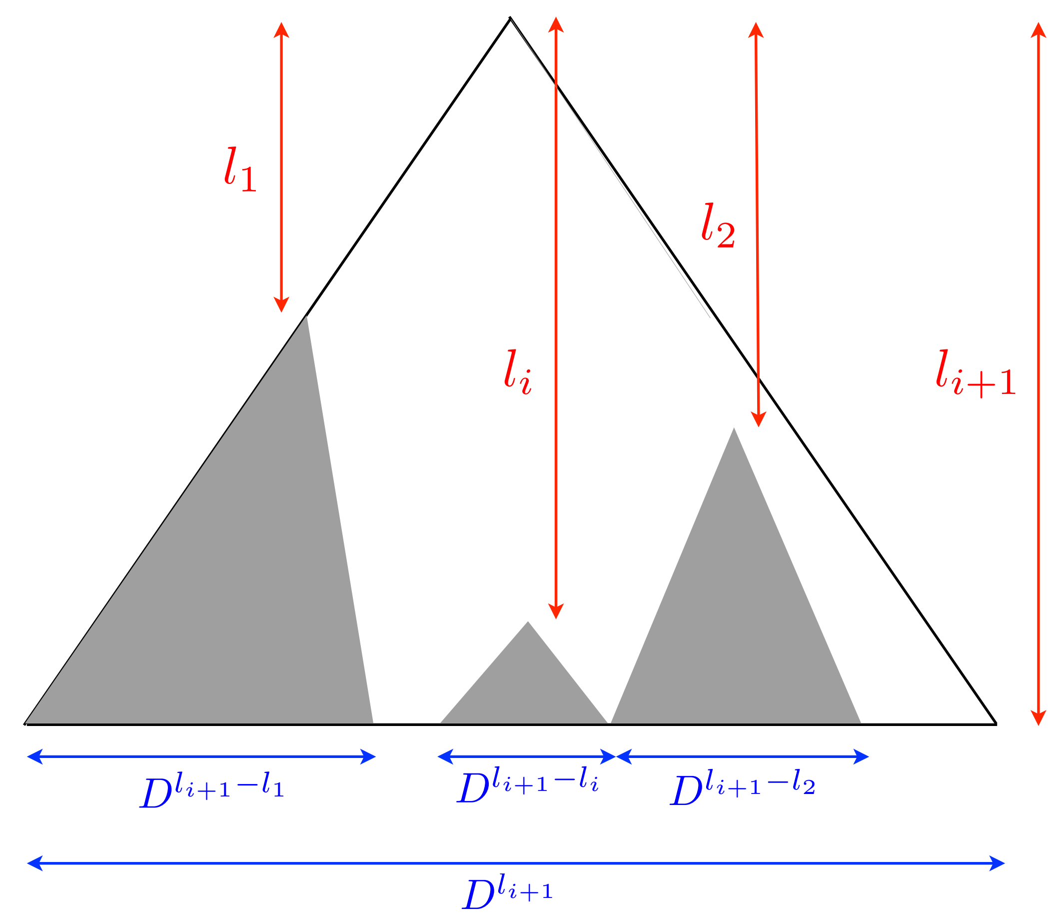

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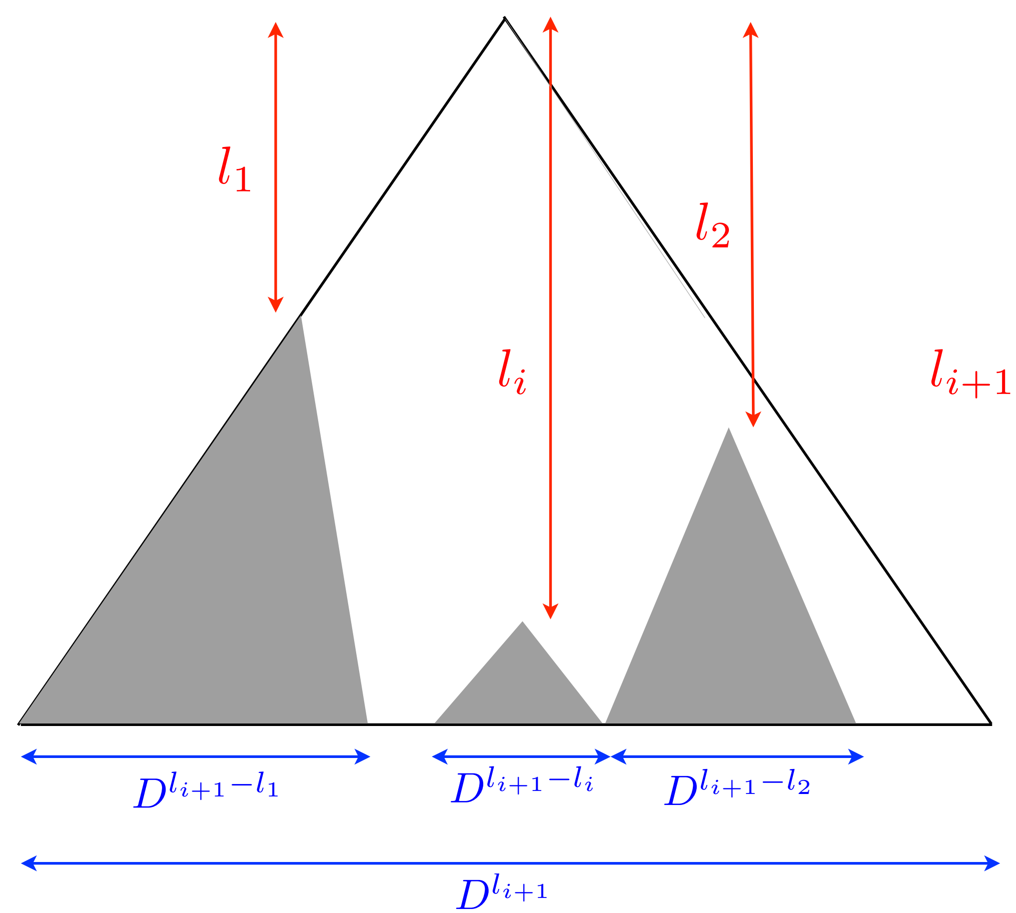

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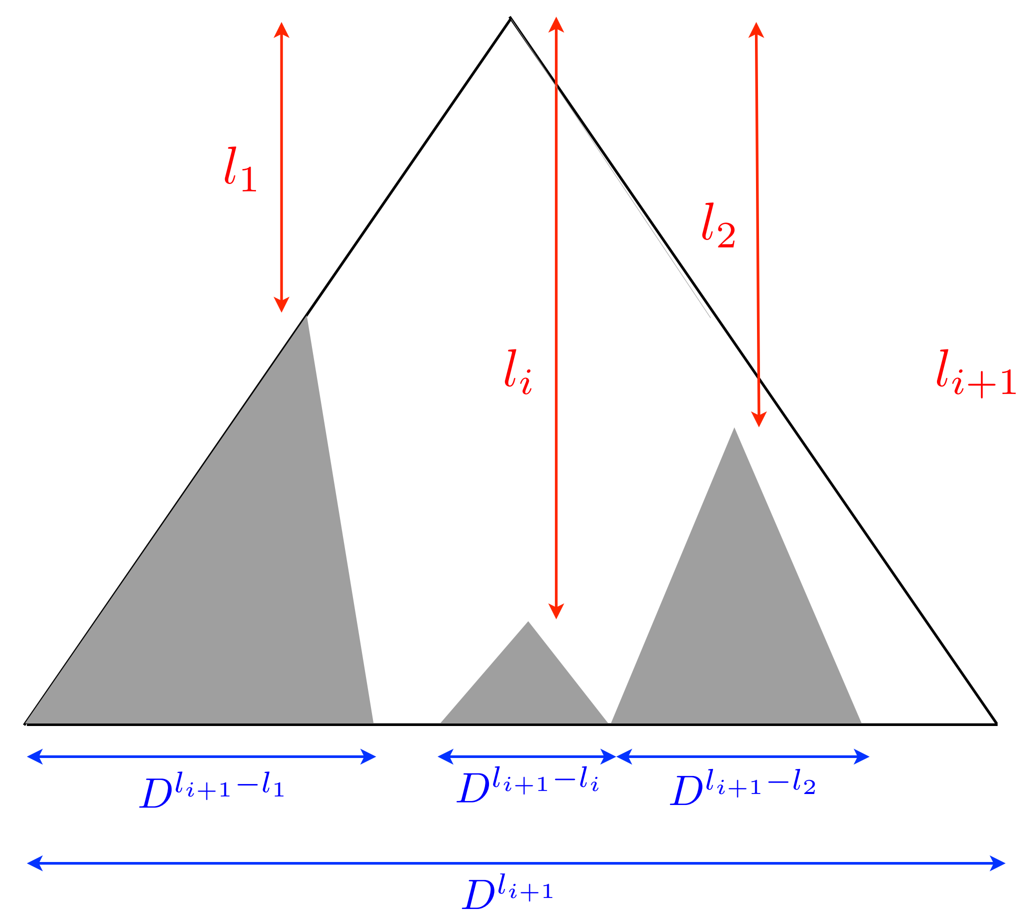

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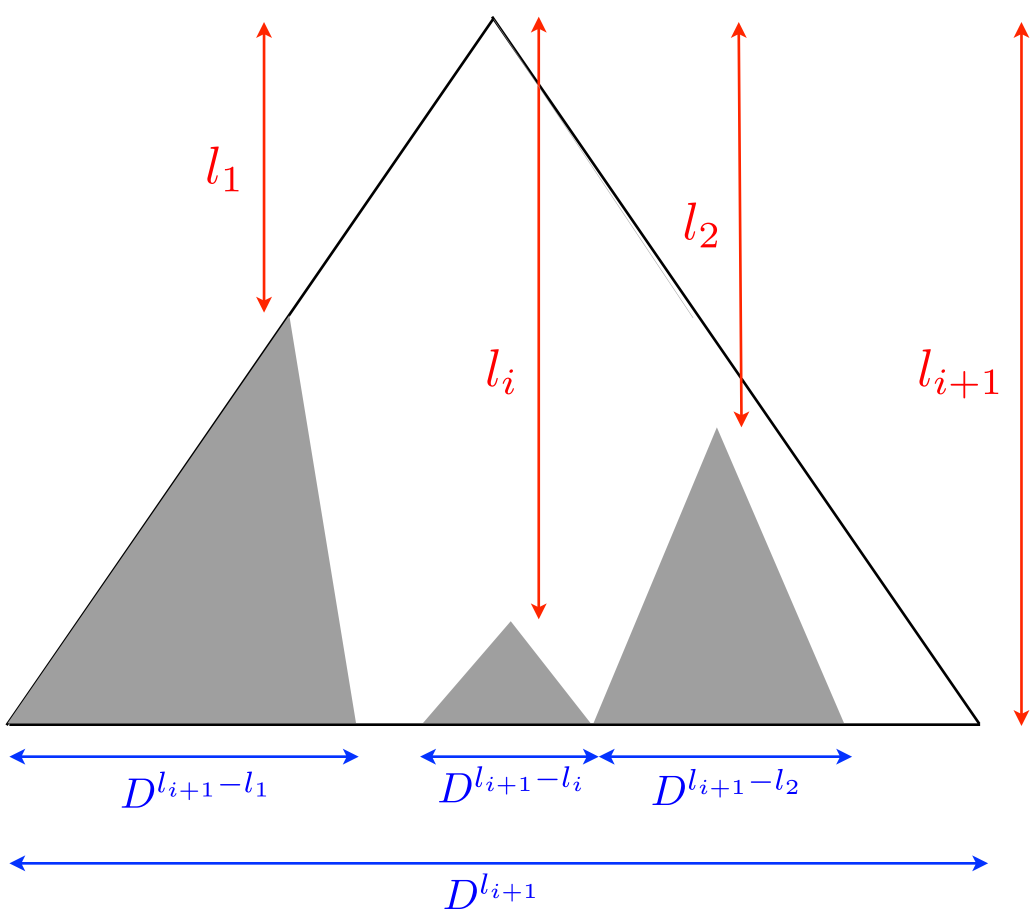

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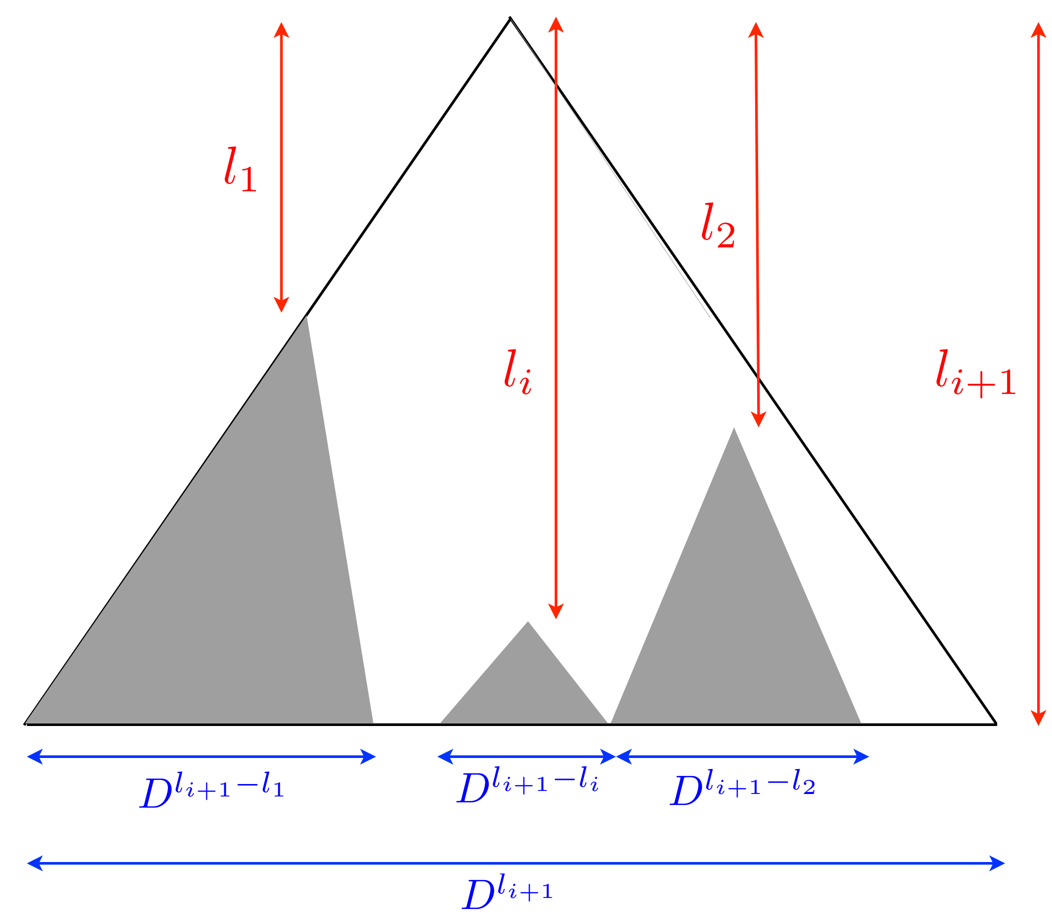

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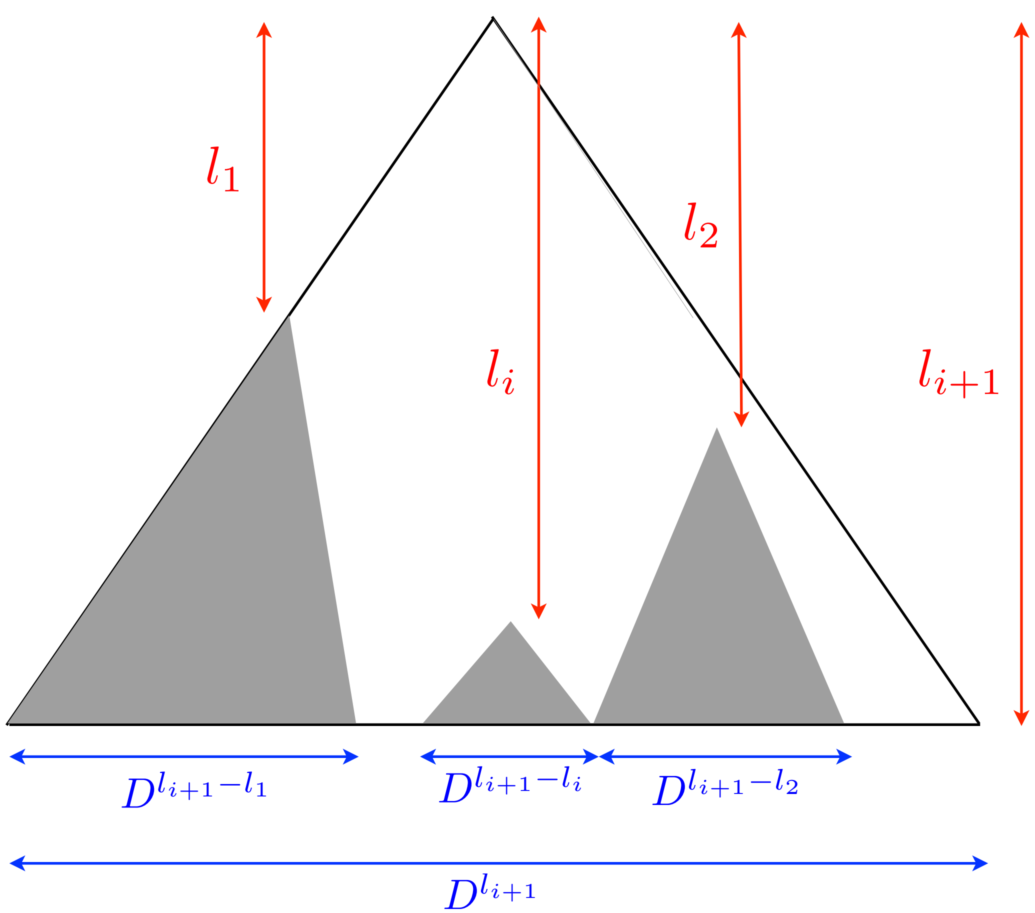

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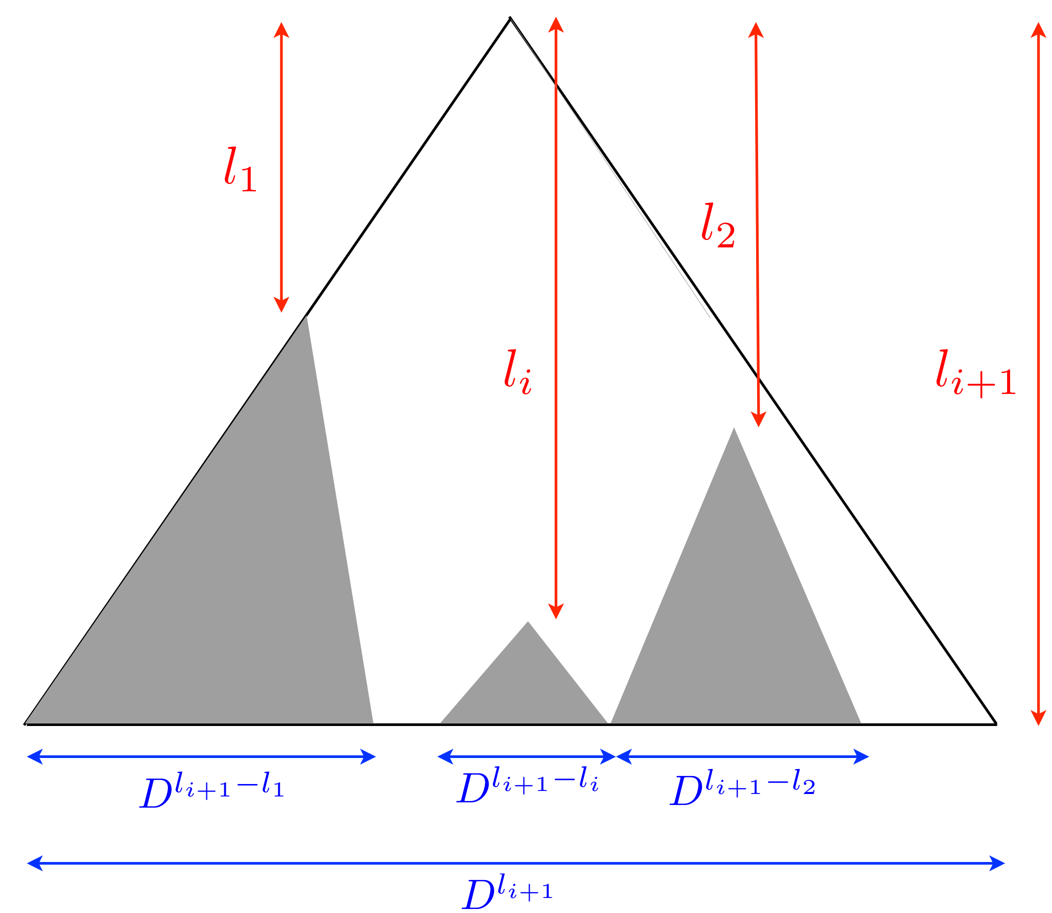

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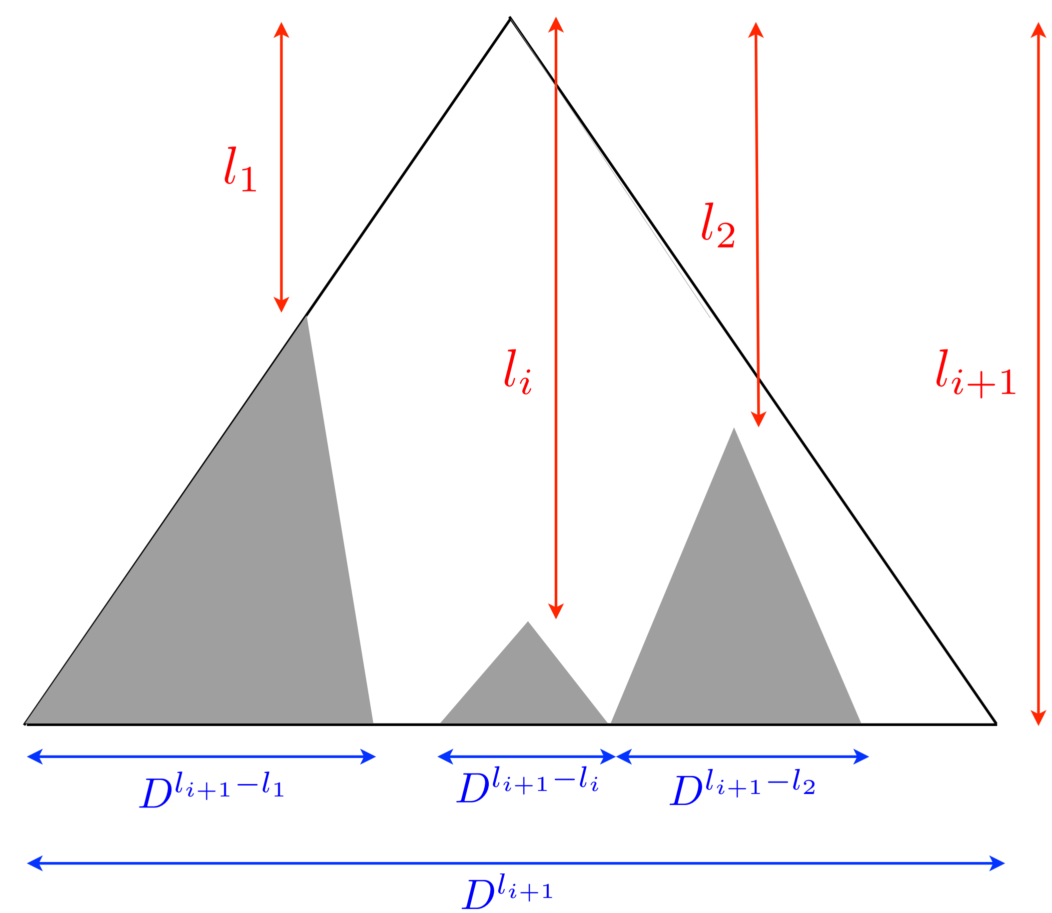
Thus we have shown by induction the existence of a prefix code with codeword lengths $l_1, l_2, \cdots, l_m$, completing the proof.



D-adic Distributions

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$$L \geq H_D(X)$$

with equality if and only if $l_i = -\log_D p_i$ for all i.

Corollary 4.12 There exists a D-ary prefix code which achieves the entropy bound for a distribution $\{p_i\}$ if and only if $\{p_i\}$ is D-adic.

Proof

A. 'Only if'

1. Consider a D-ary prefix code which achieves the entropy bound for a distribution $\{p_i\}$.

2. Let l_i be the length of the codeword assigned to the probability p_i. By Theorem 4.6, for all i,

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$$p_i = D^{-l_i}.$$

Thus $\{p_i\}$ is D-adic.

B. 'If'

1. Suppose $\{p_i\}$ is D-adic. Let $p_i = D^{-t_i}$ for all i, where t_i is an integer. Then

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2. Let $l_i = t_i$ for all i. Verify that $\{l_i\}$ satisfies the Kraft inequality:

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4. Since for all i,

$$l_i = t_i = -\log_D p_i,$$

we see from Theorem 4.6 that this code achieves the entropy bound.

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Huffman Codes

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- In general there can be more than one Huffman code.

Example 4.13:
Huffman Procedure

	p_i	codeword
	0.35	
	0.1	
0 ↑	0.15	
↓ 1	0.2	
	0.2	

Example 4.13:
Huffman Procedure

	p_i	codeword
	0.35	
0	0.1	
$\uparrow$		
	0.15	
$\downarrow$		
1	0.2	
	0.2	

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Huffman Procedure

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	0.35	
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0 ↑		
	0.15	
↓ 1		
	0.2	
	0.2	

Example 4.13:
Huffman Procedure

	p_i	codeword
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	0.1	
0.25		
	0.15	
	0.2	
	0.2	

0 ↑ ↓ 1

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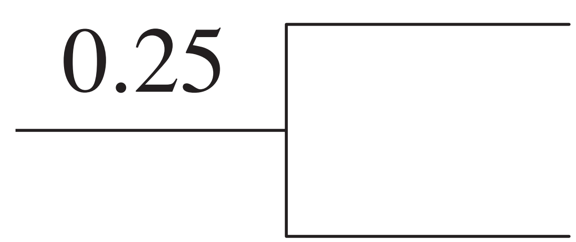
p_i	codeword
0.35	
0.1	
0.15	
0.2	
0.2	



0 ↑

↓ 1

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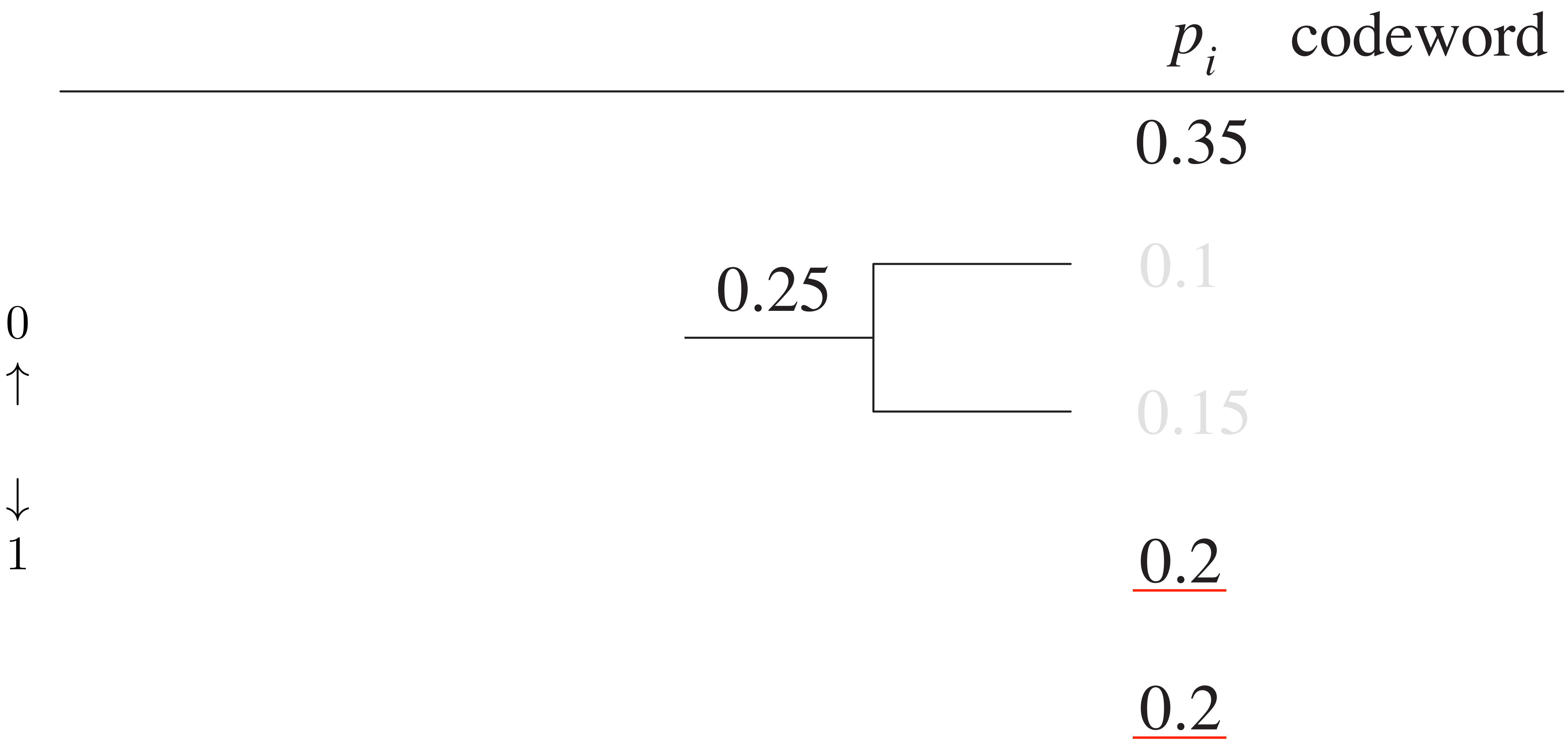

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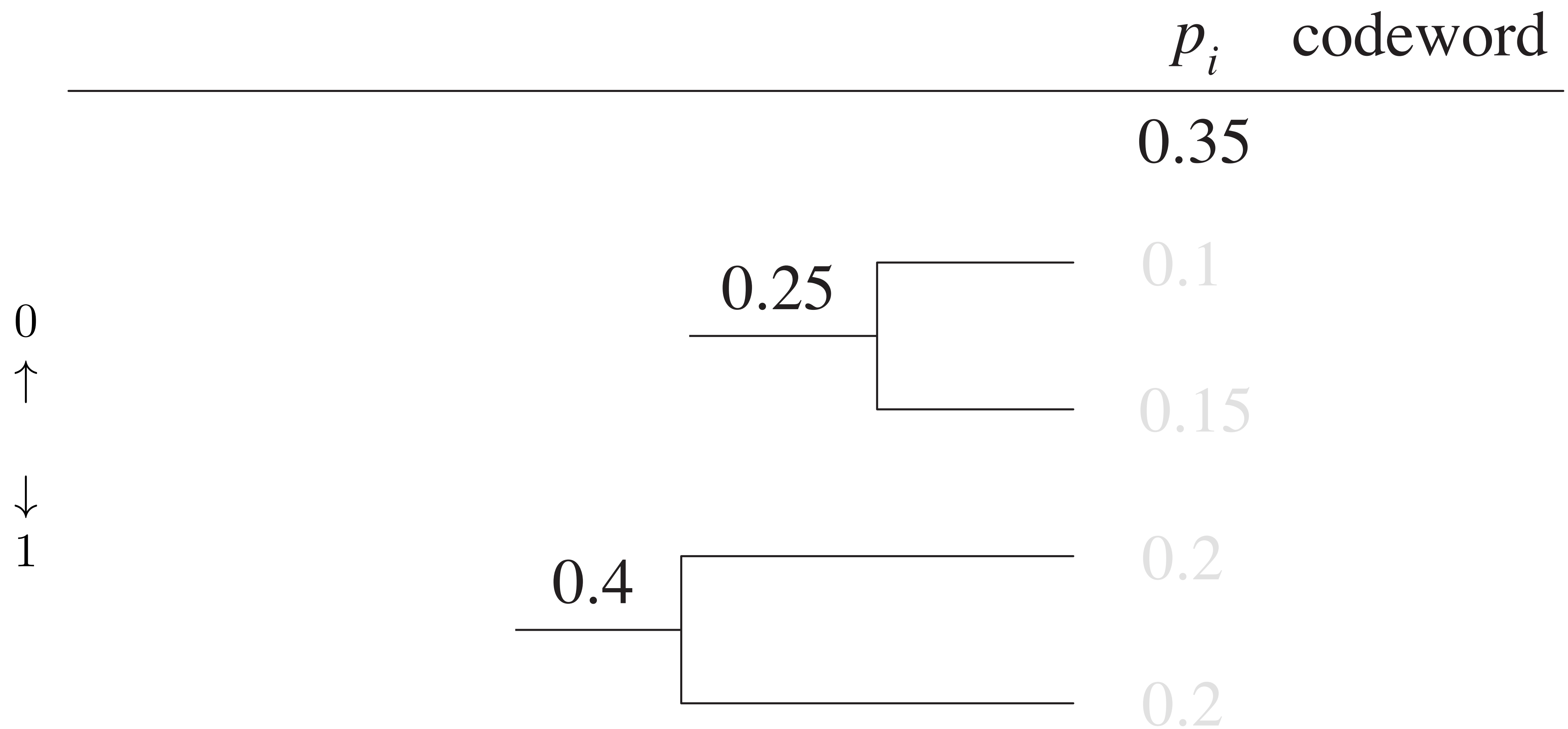

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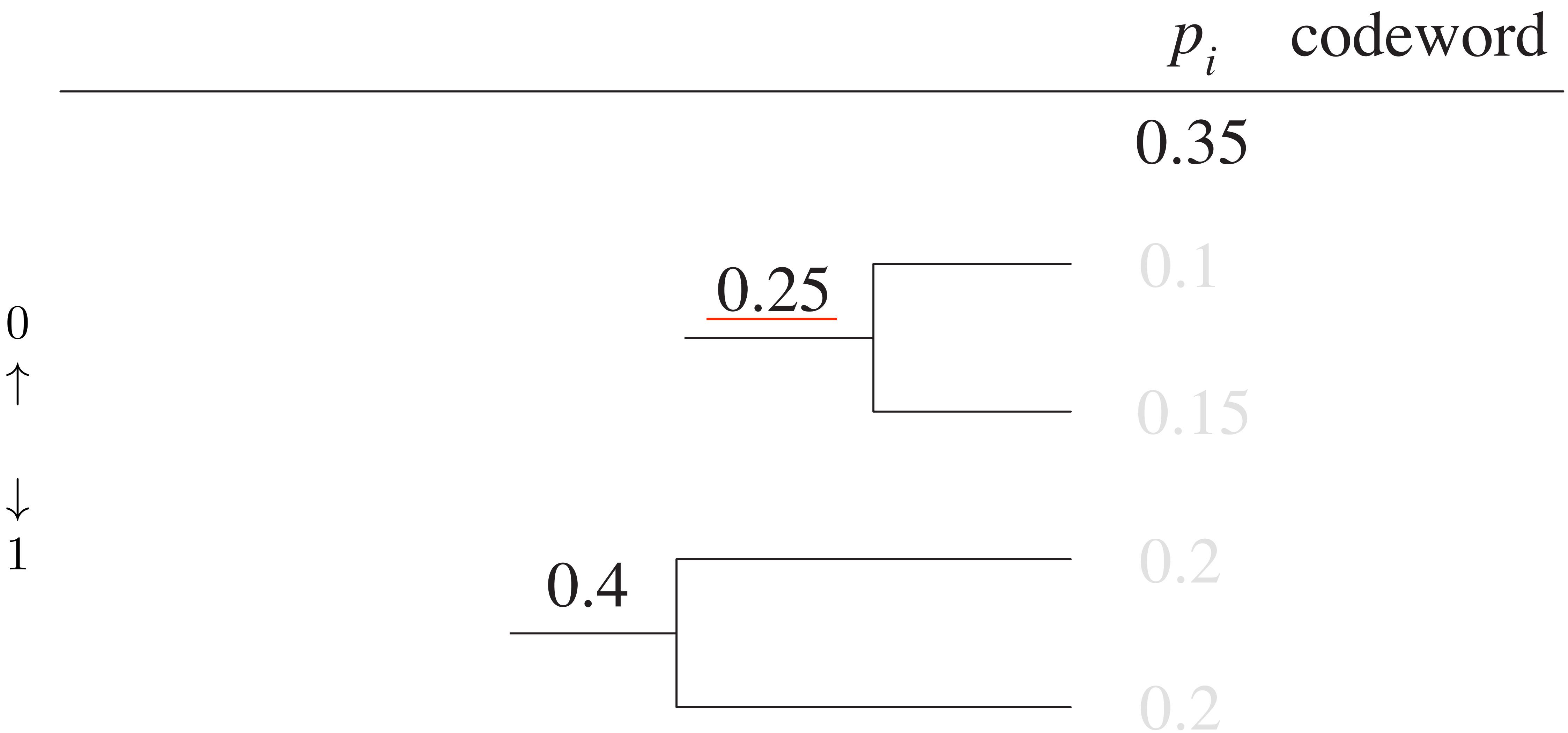

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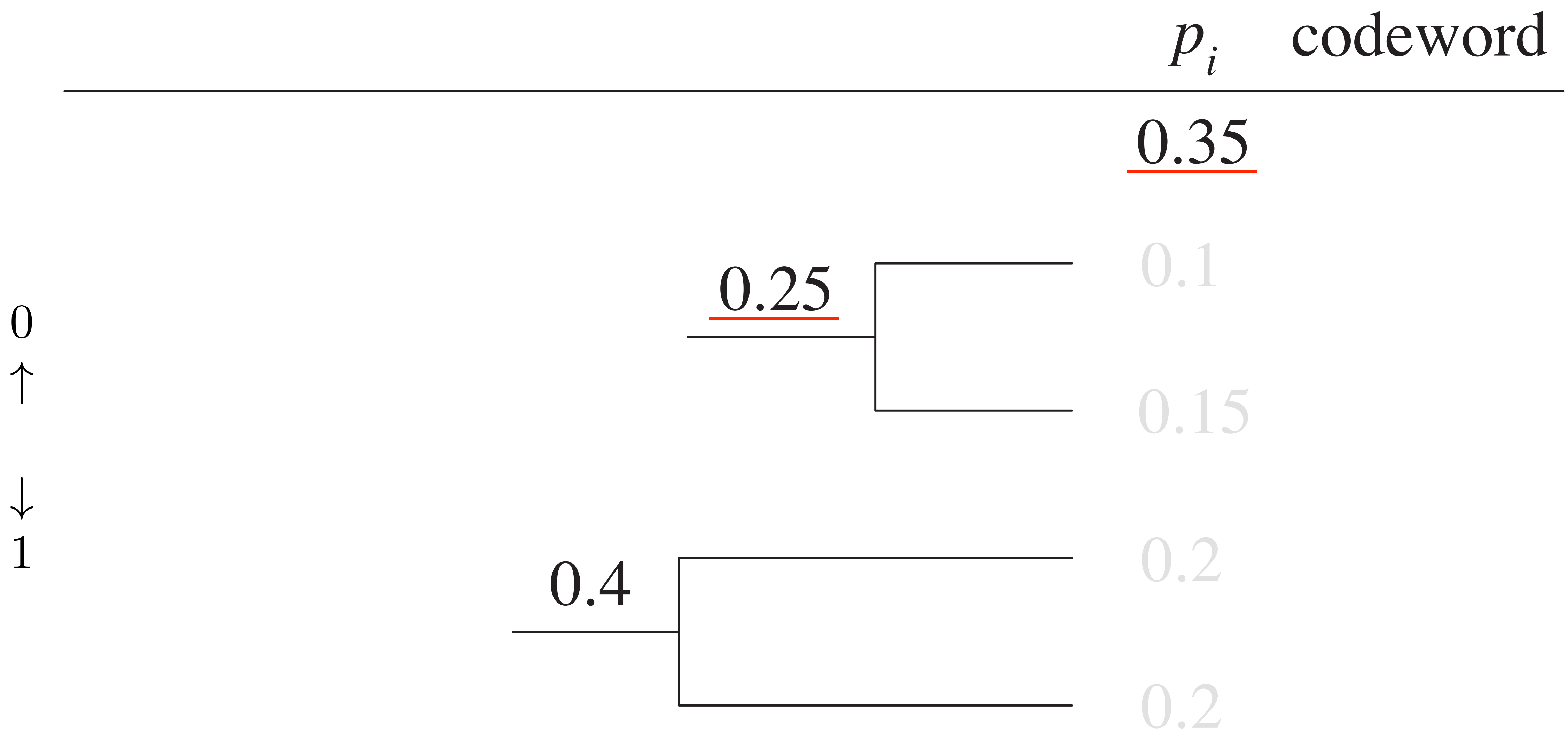

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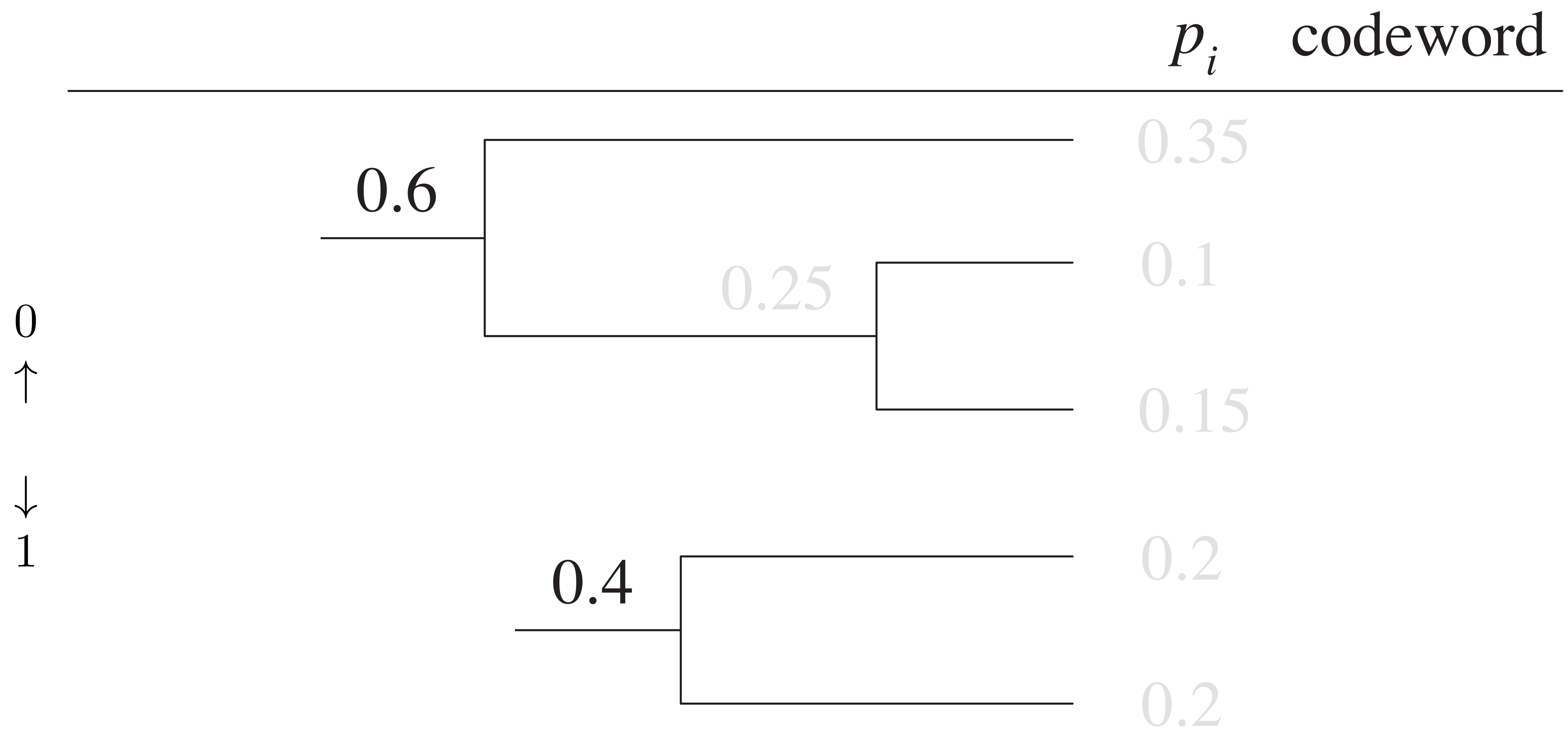

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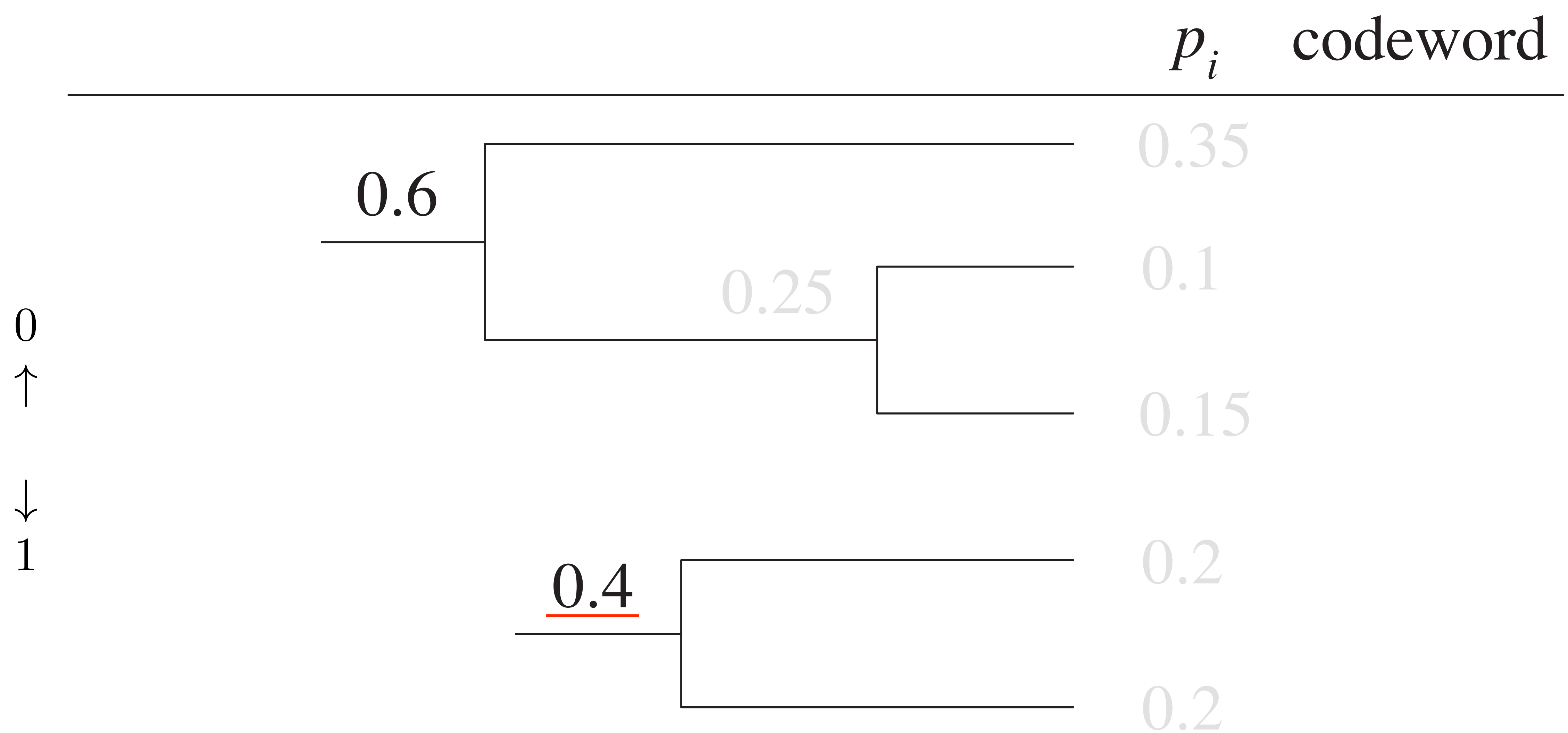

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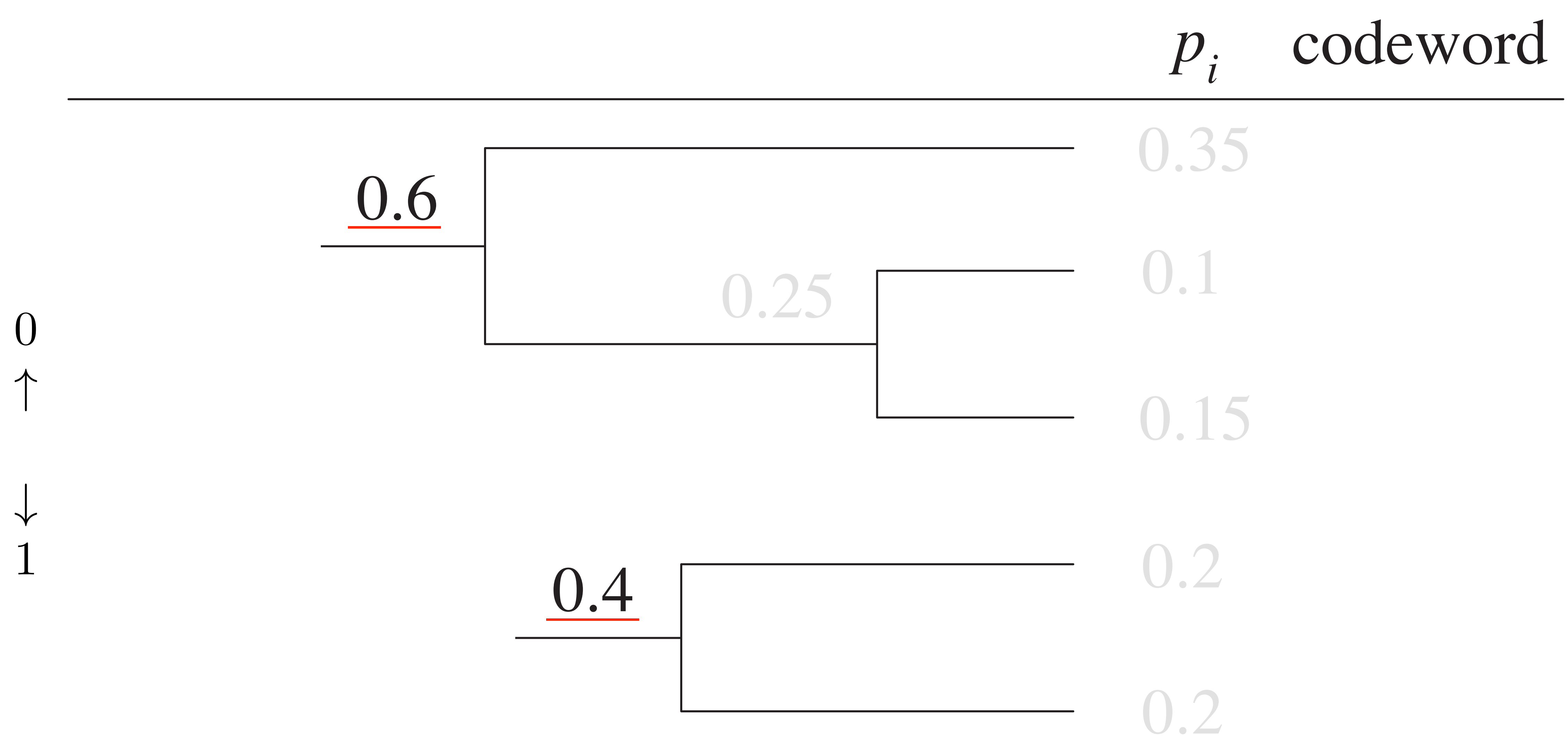

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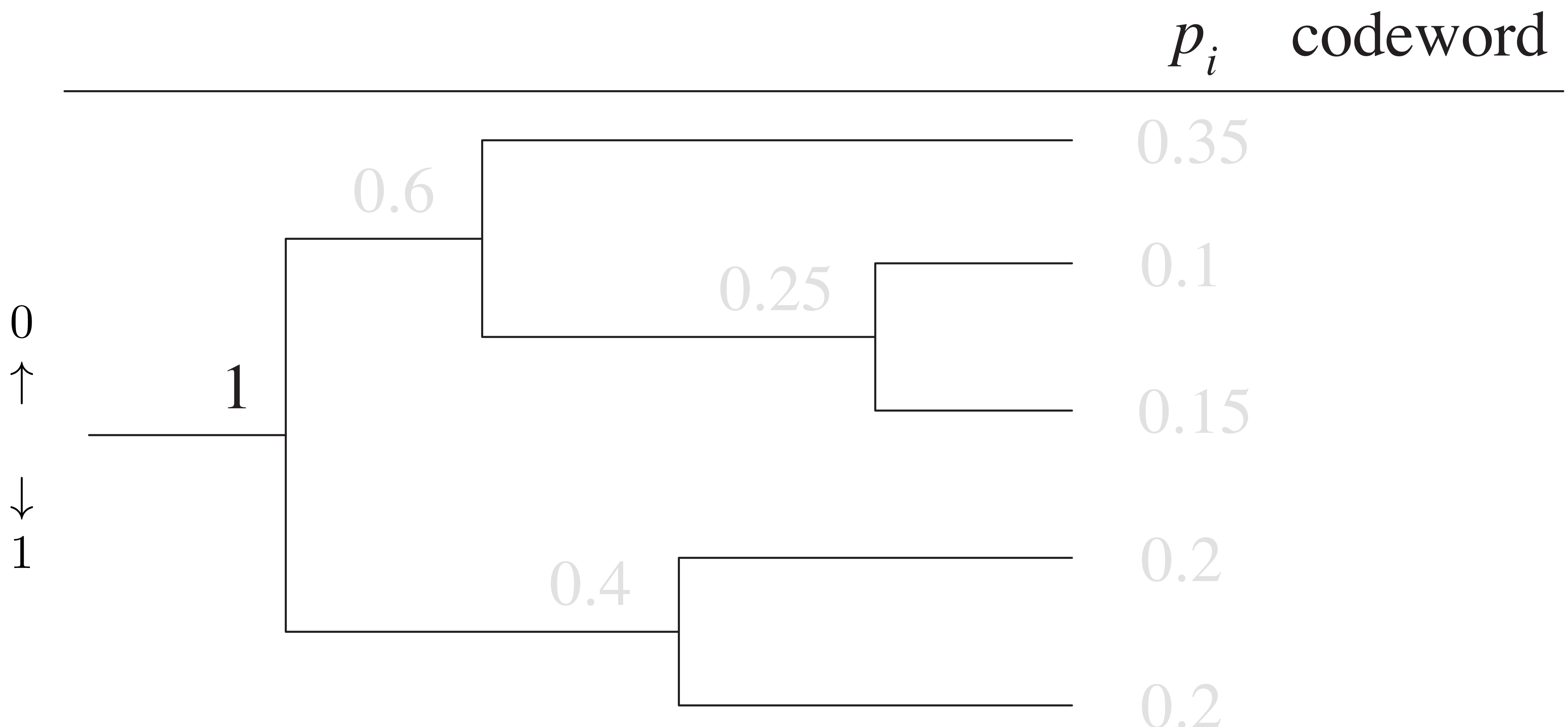

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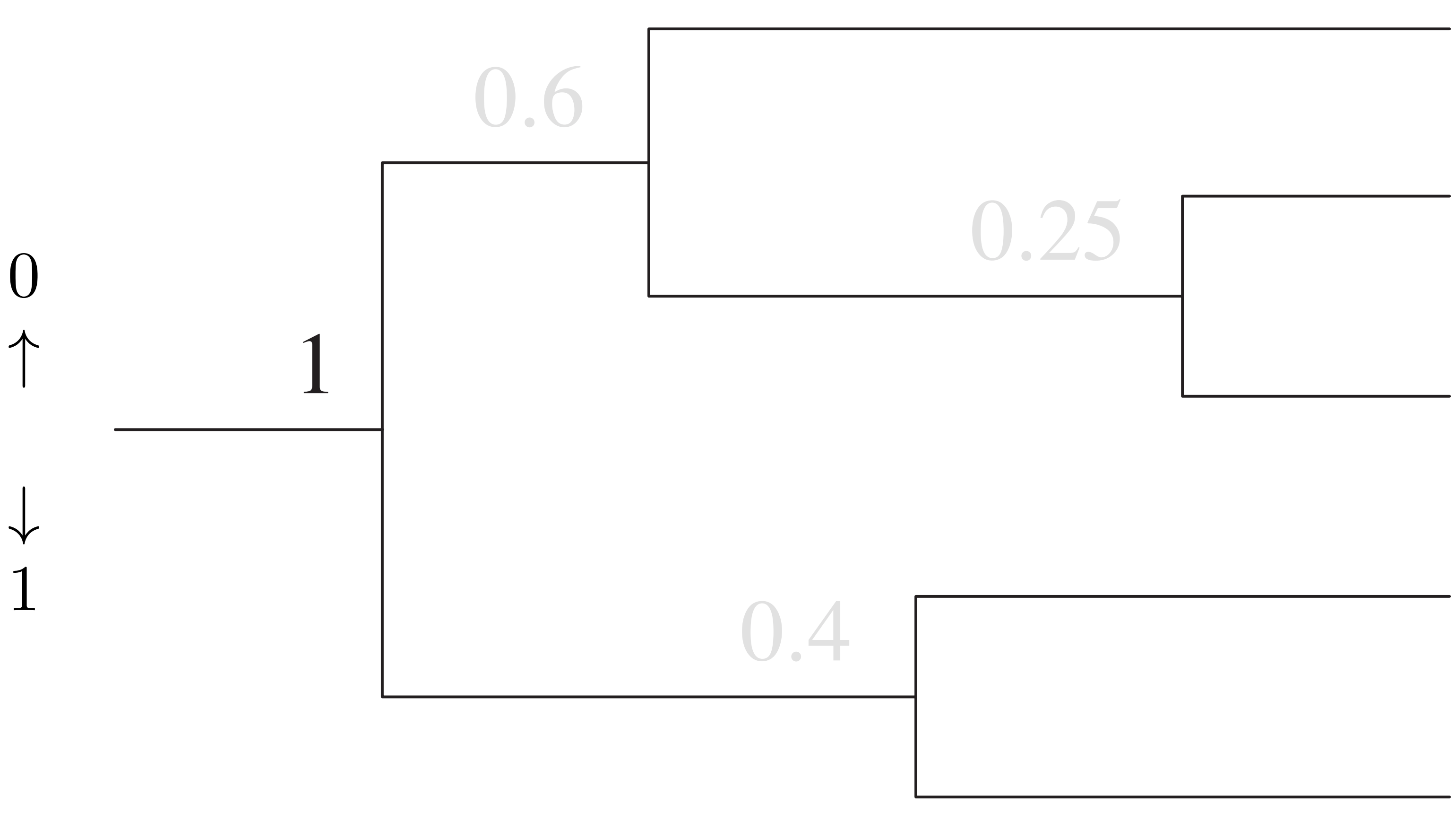

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0.35	00
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0.2	10
0.2	11

Optimality of Huffman Codes

- Without loss of generality, assume $p_1 \geq p_2 \geq \cdots \geq p_m$.
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Lemma 4.16 There exists an optimal code in which the codewords assigned to the two smallest probabilities are siblings, i.e., the two codewords have the same length and they differ only in the last symbol.

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$$l_1 \leq l_2 \leq \cdots \leq l_m. \qquad (1)$$

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1. Consider a probability distribution

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such that $p_i > p_j$. Assume that in a particular code, the codewords c_i and c_j are such that $l_i > l_j$, i.e., a shorter codeword is assigned to a smaller probability.

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4. Comparing L' and L, we see that

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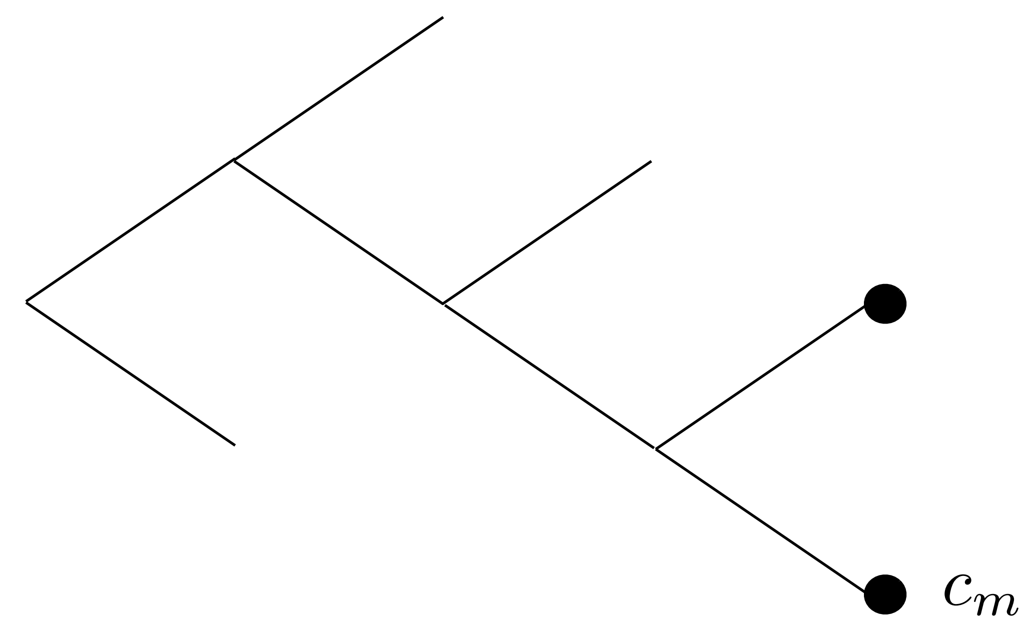

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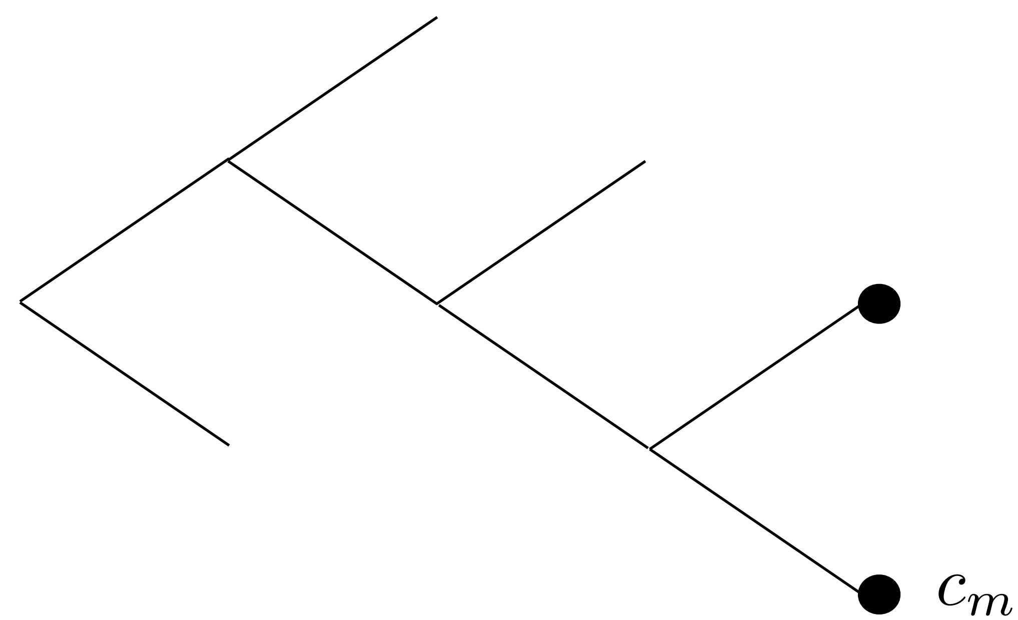

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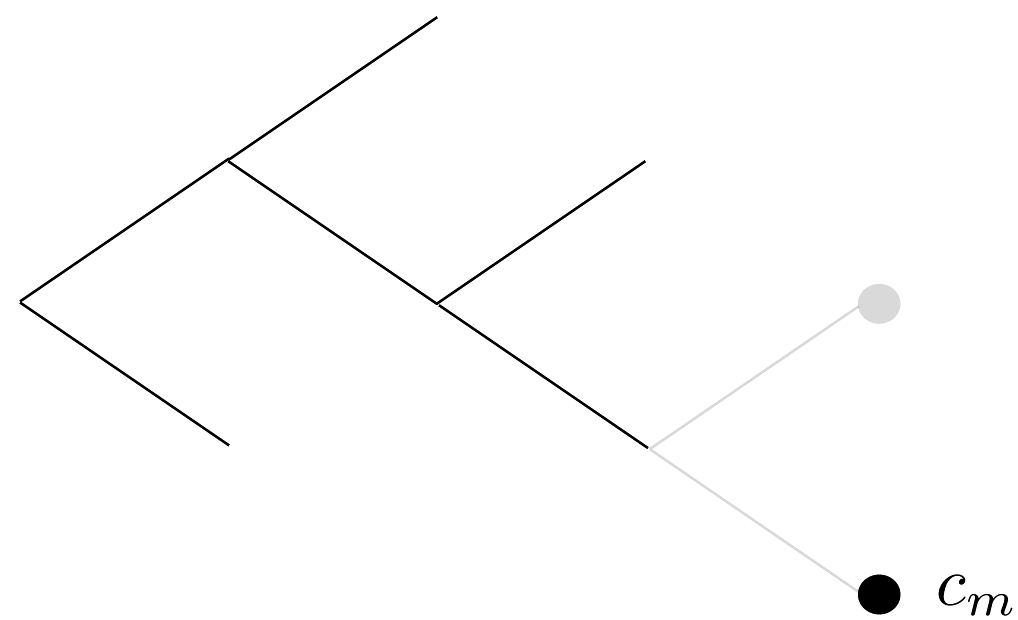

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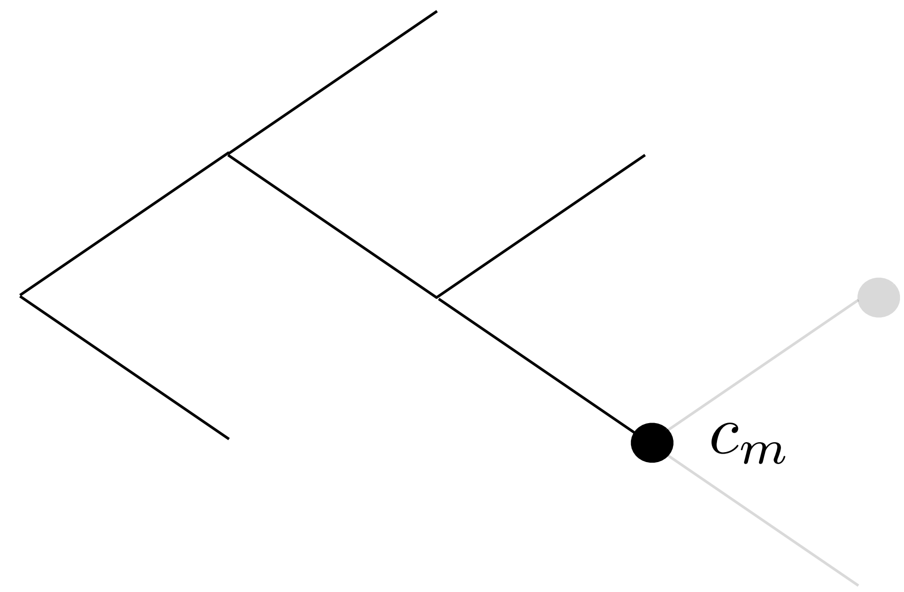

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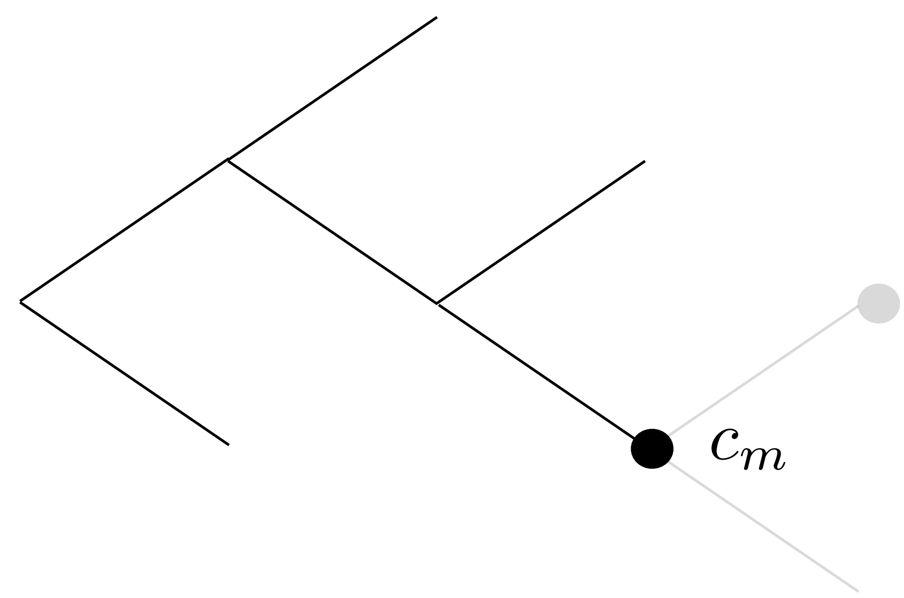

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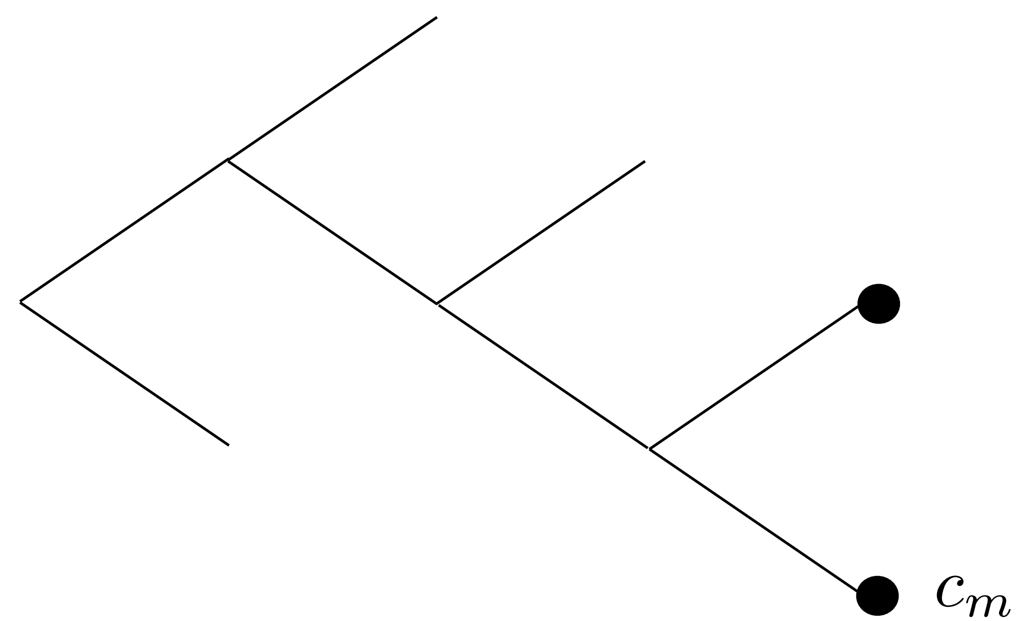

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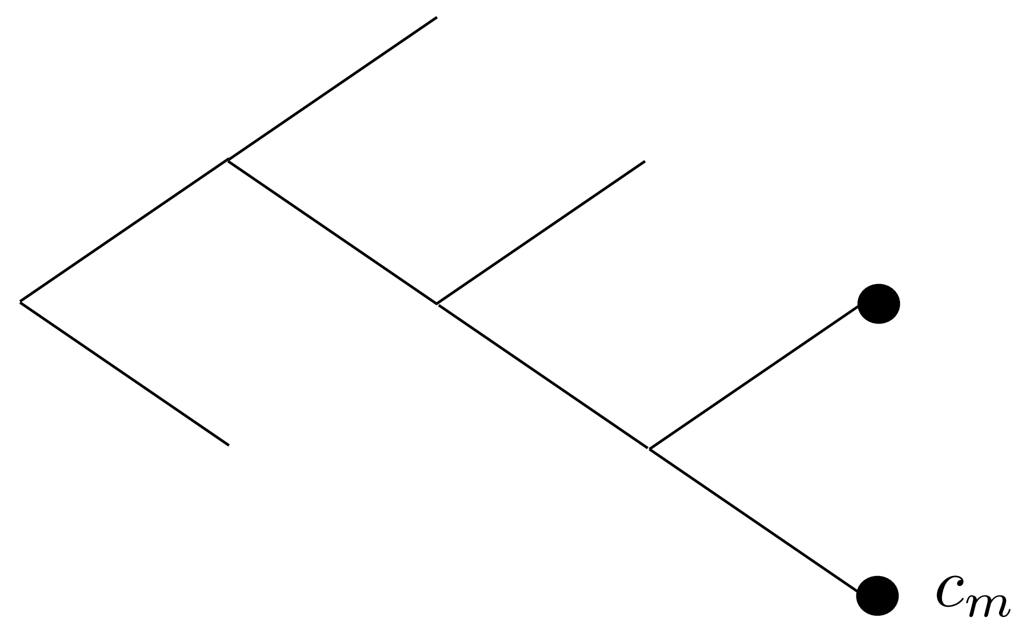

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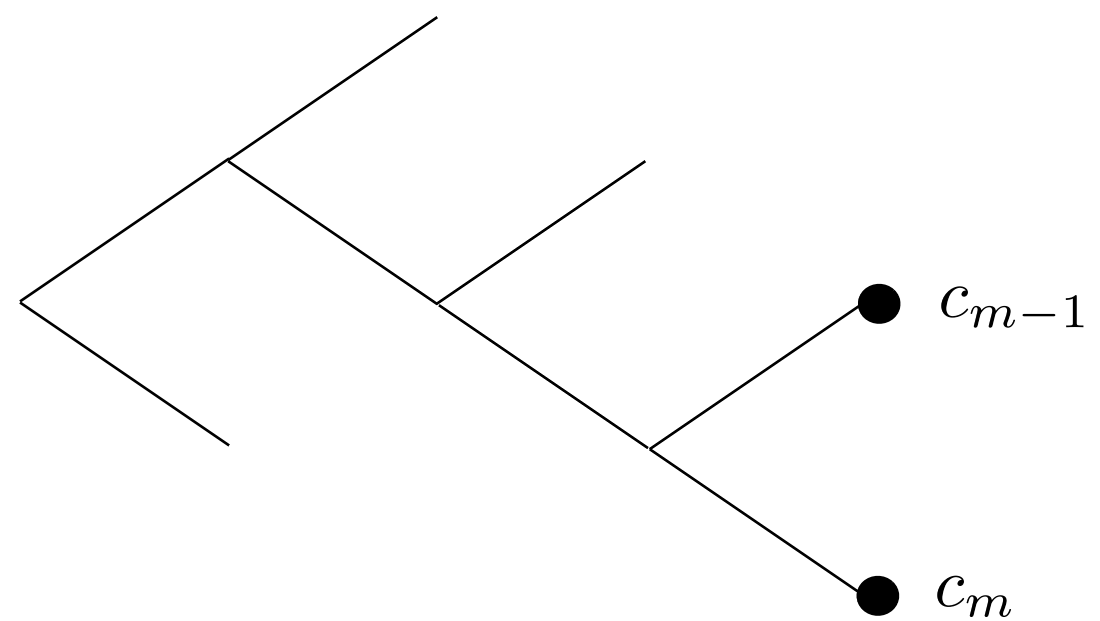

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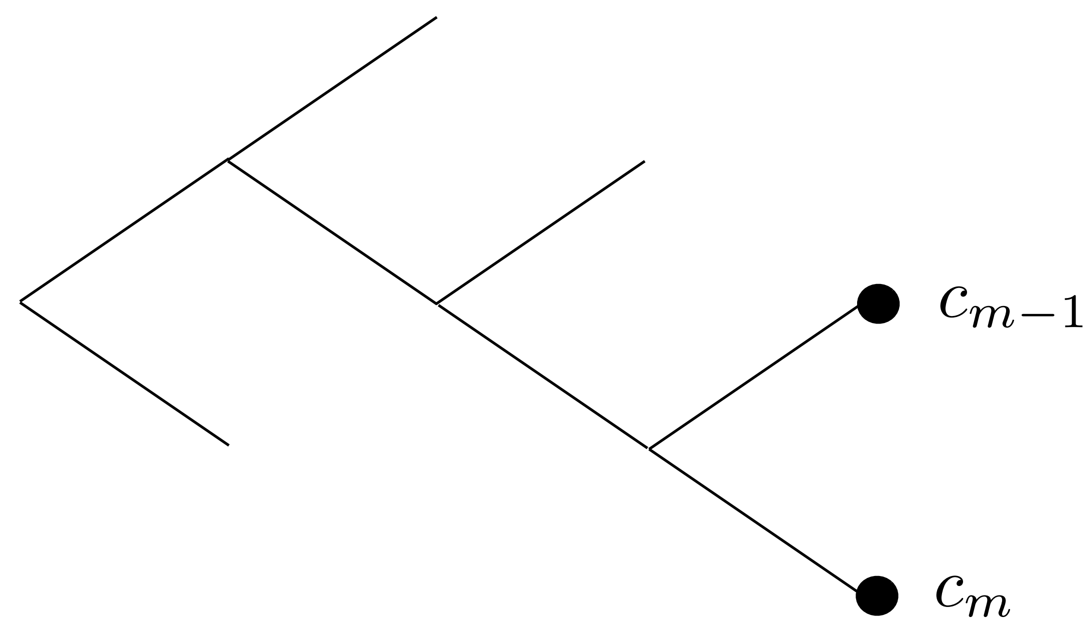

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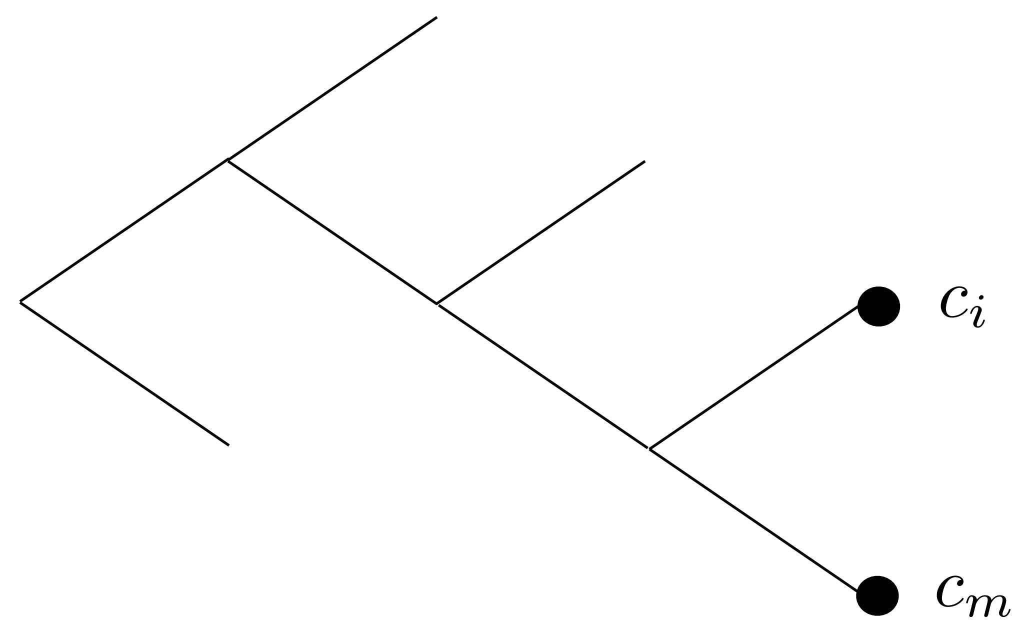

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i.e., c_i, c_{m-1}, and c_m all have the same order.

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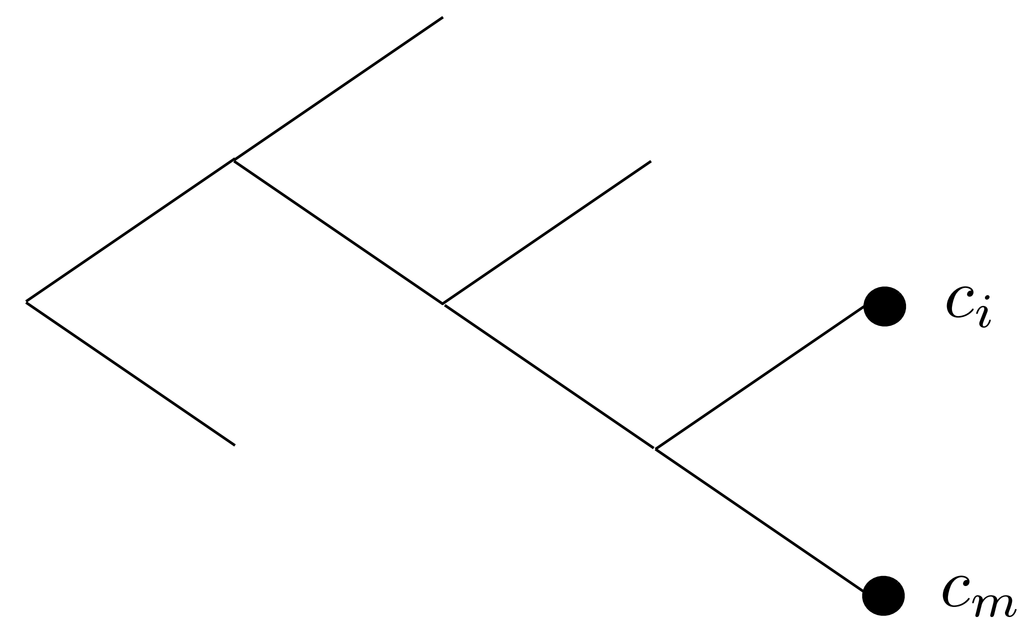

Lemma 4.16 There exists an optimal code in which the codewords assigned to the two smallest probabilities are siblings, i.e., the two codewords have the same length and they differ only in the last symbol.

Lemma 4.15 In an optimal code,

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3. This is a contradiction to the assumption that the code is optimal. Therefore, the sibling of c_m must be a codeword.

4. If the sibling of c_m is c_{m-1}, then the code already has the desired property, i.e., the codewords assigned to the two smallest probabilities are siblings.

5. If not, assume that the sibling of c_m is c_i, where $i < m - 1$. Since the code is optimal, by Lemma 4.15,

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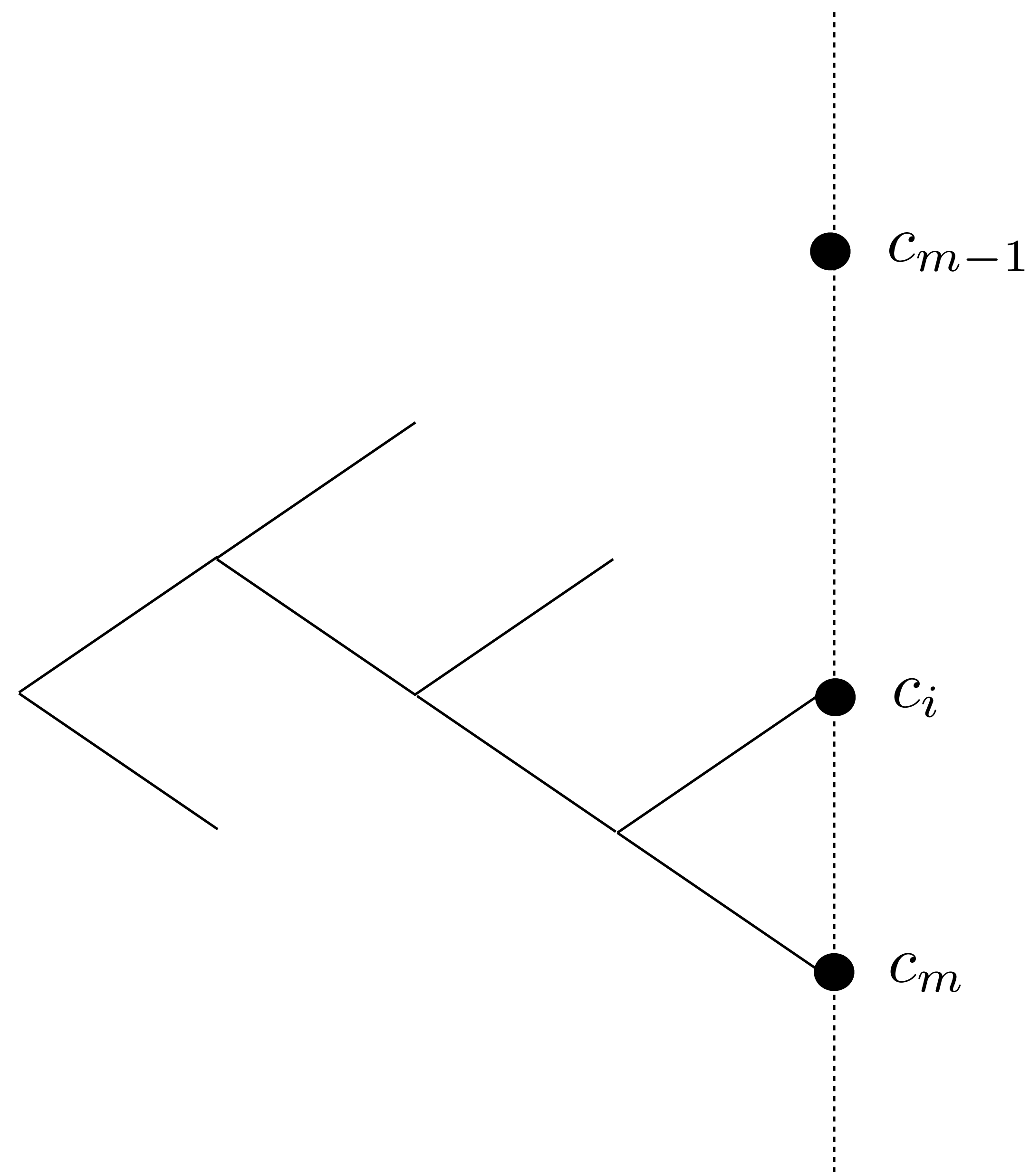

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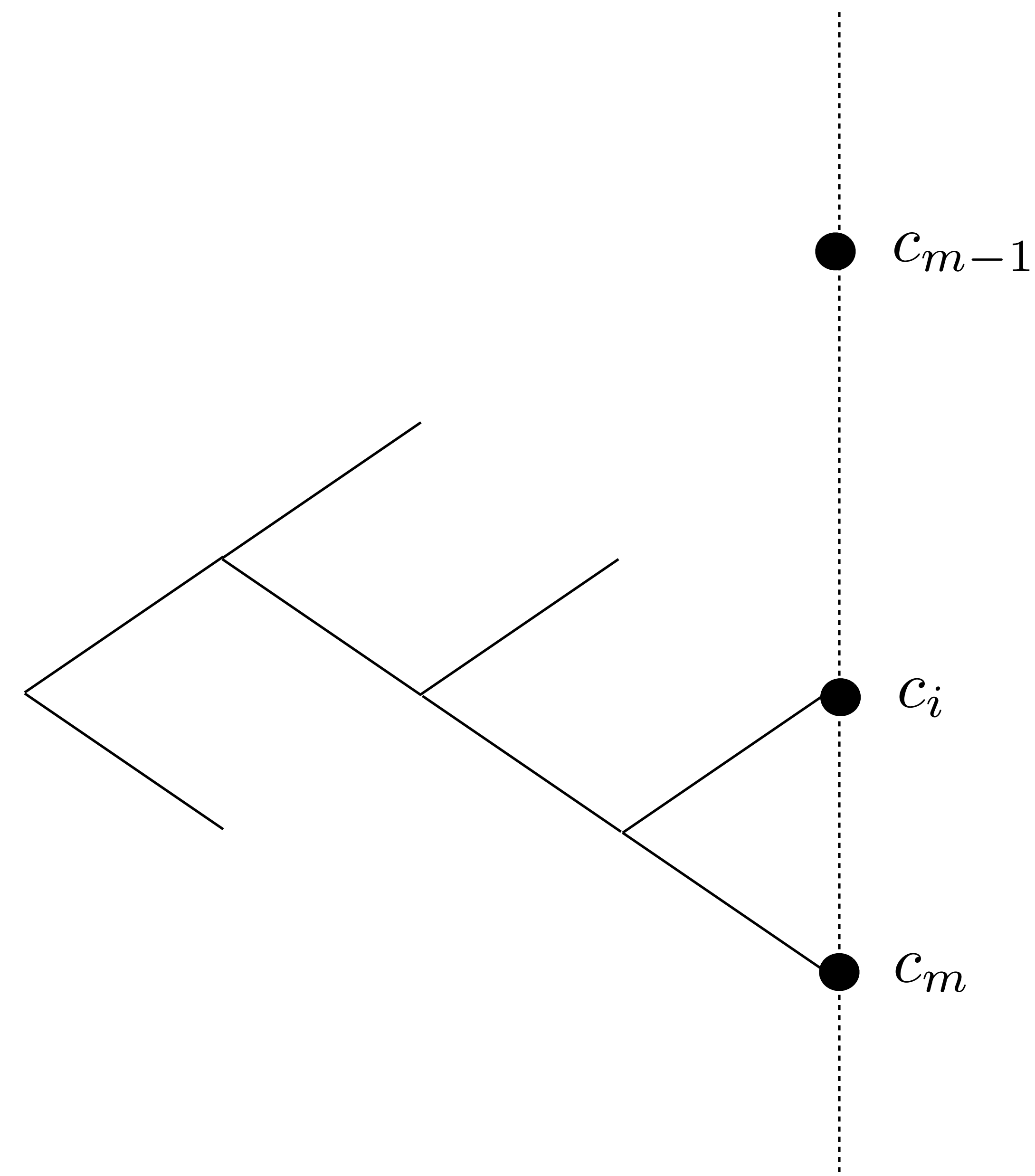

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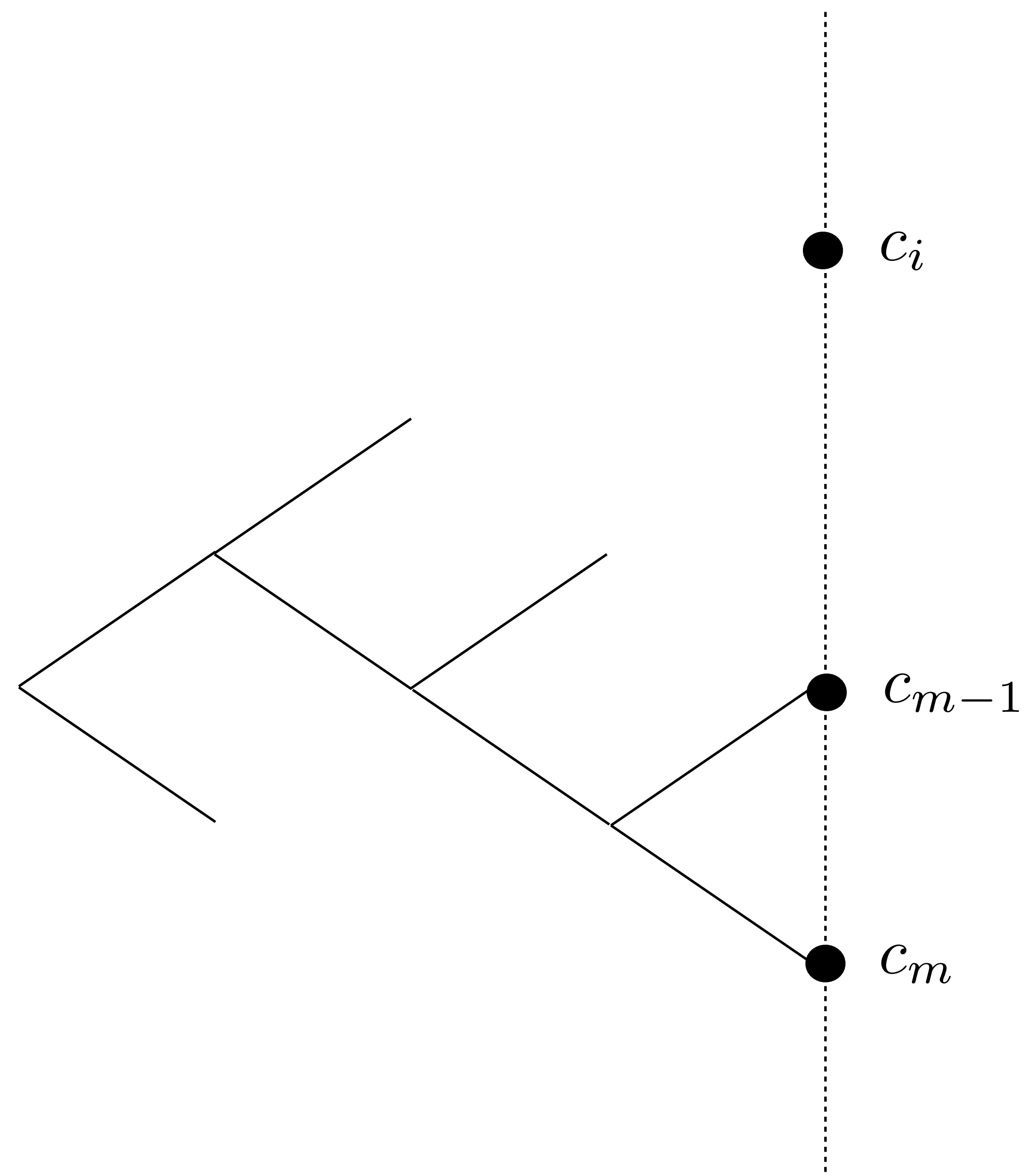

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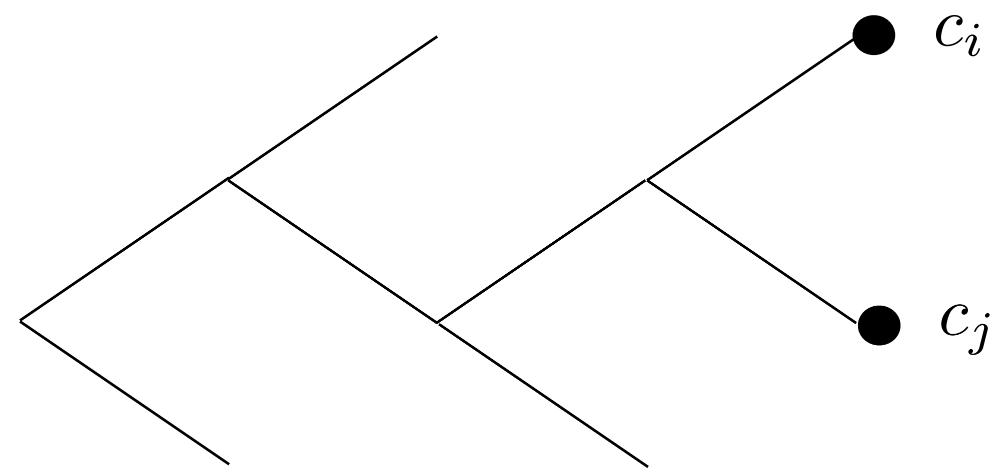

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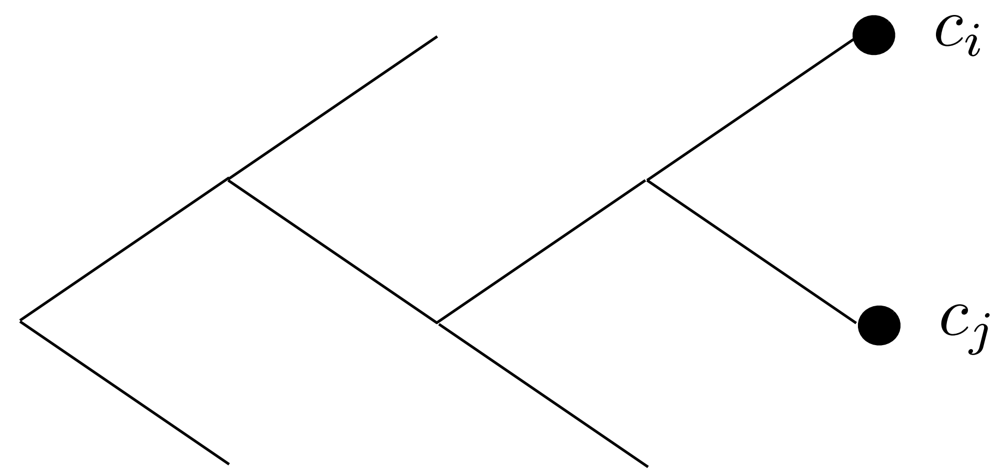

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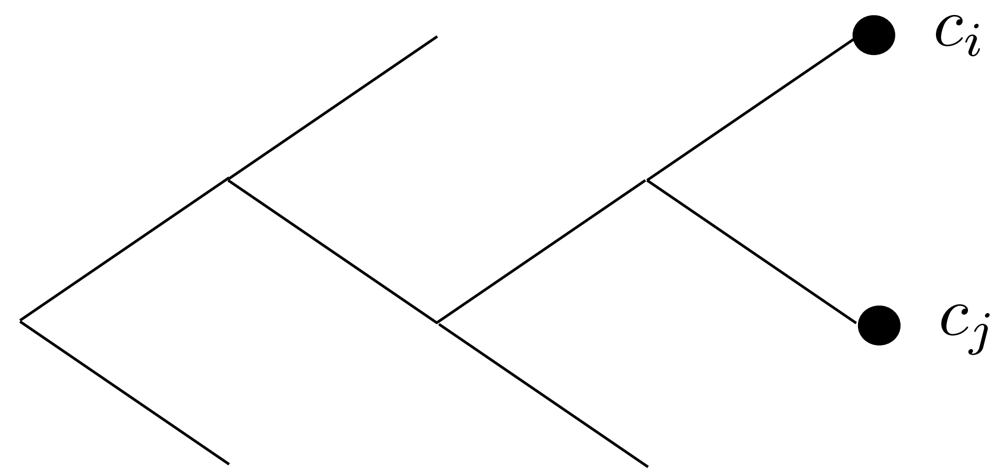

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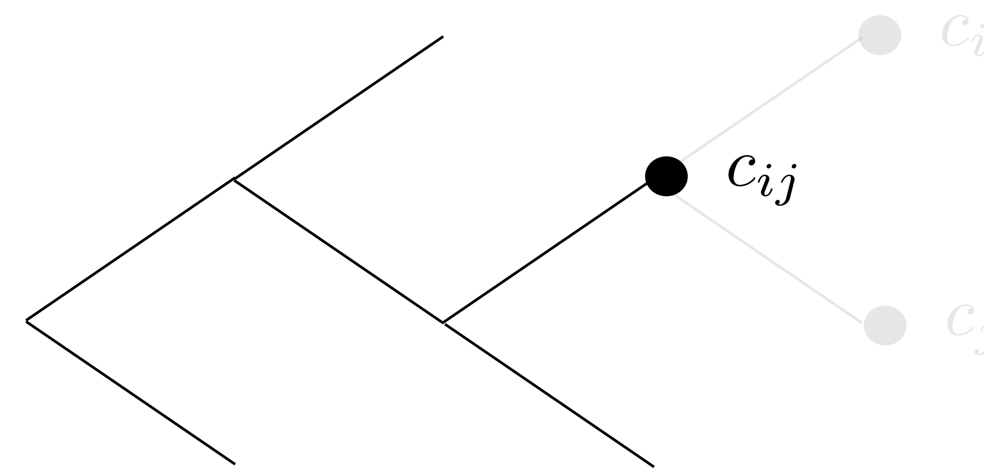

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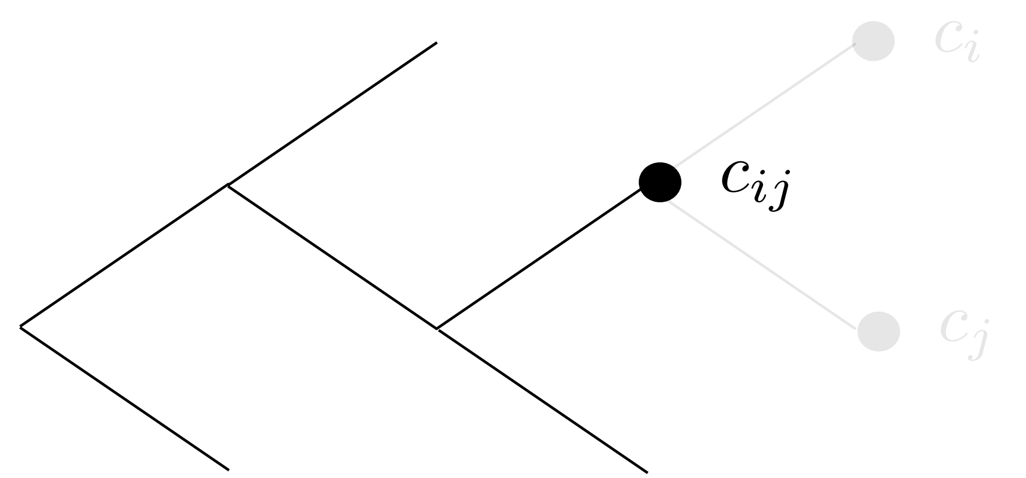

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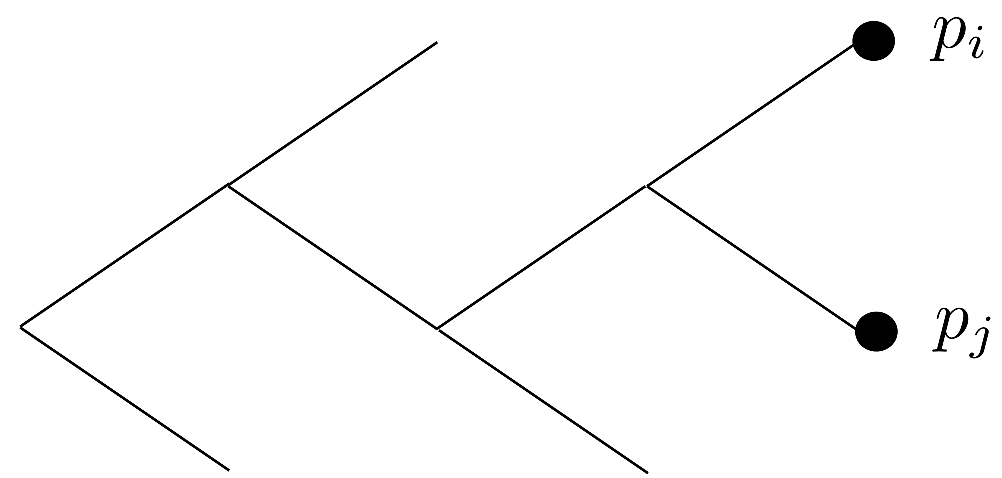

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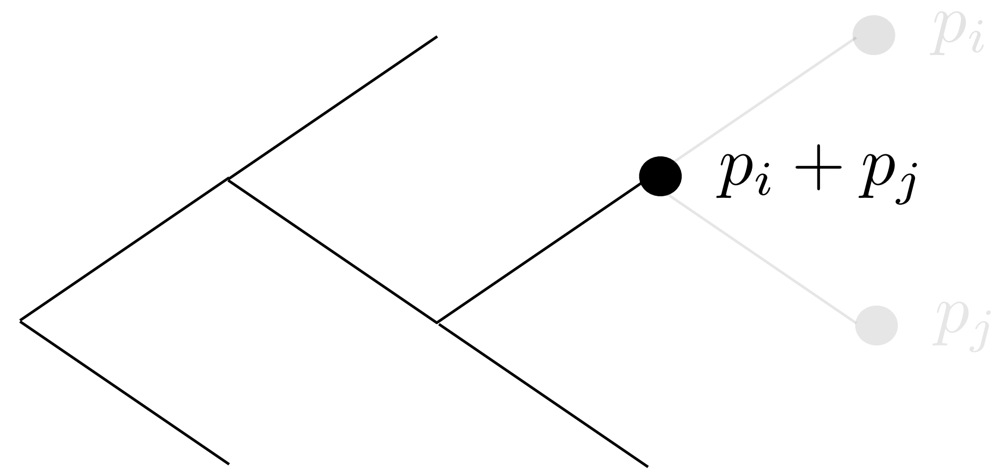

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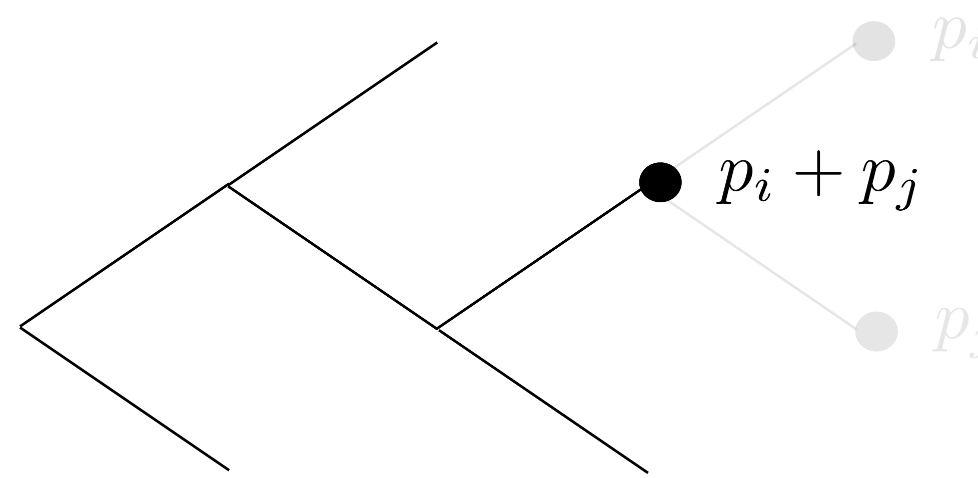

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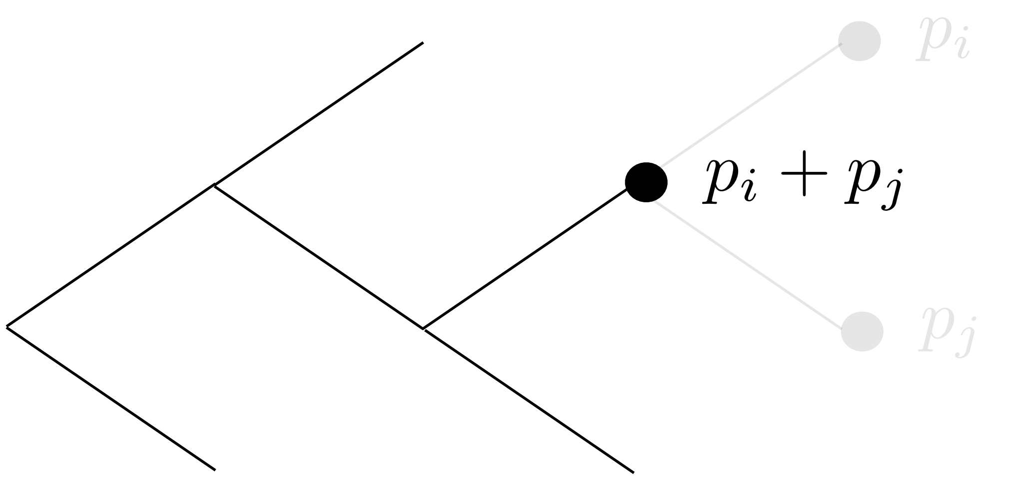

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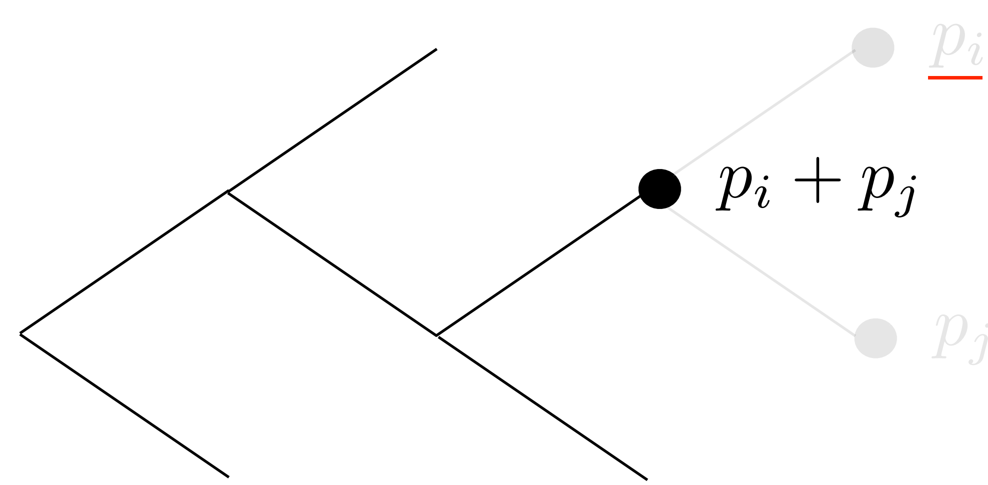

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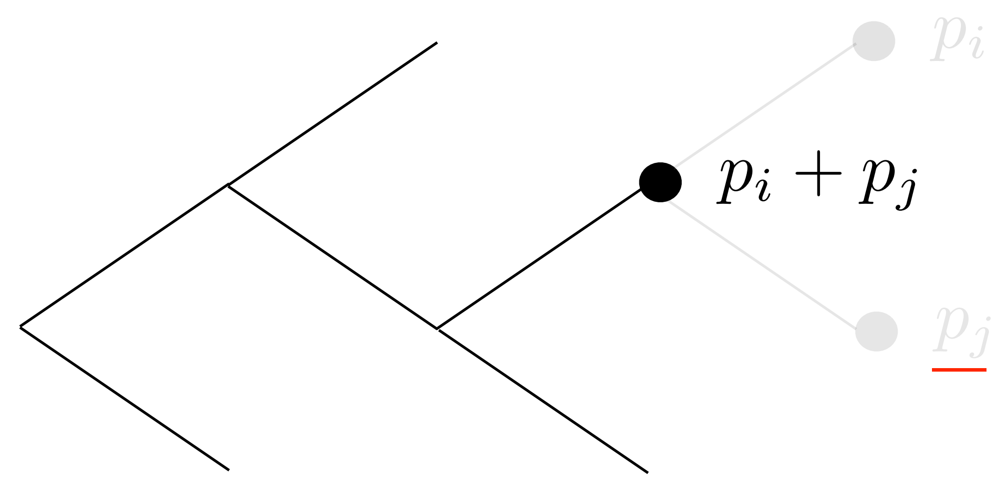

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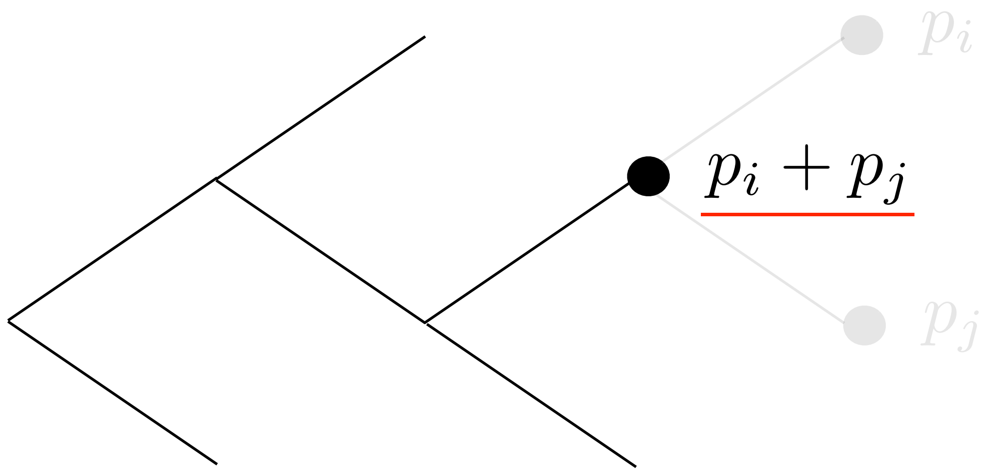

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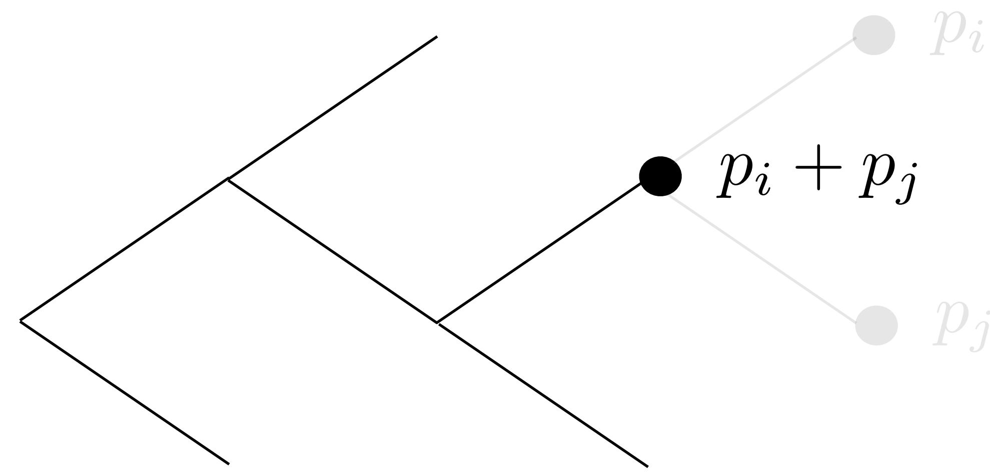

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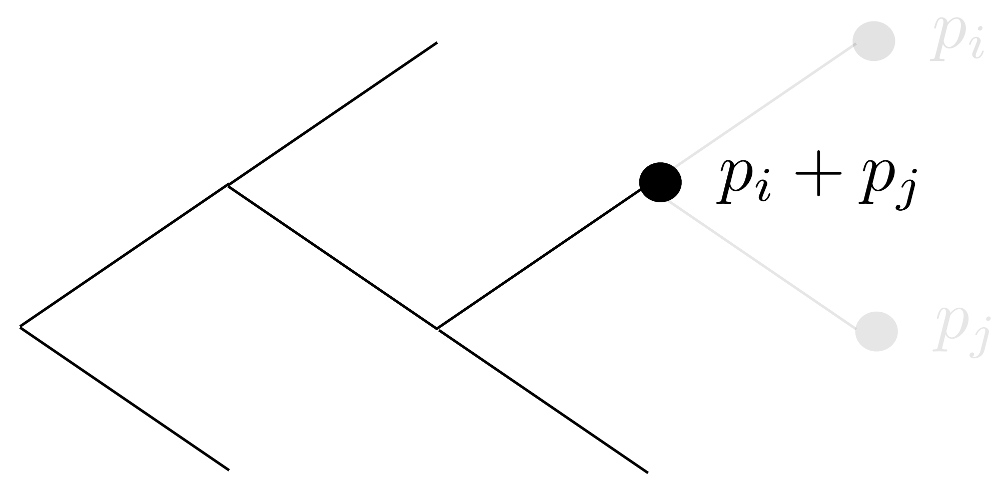

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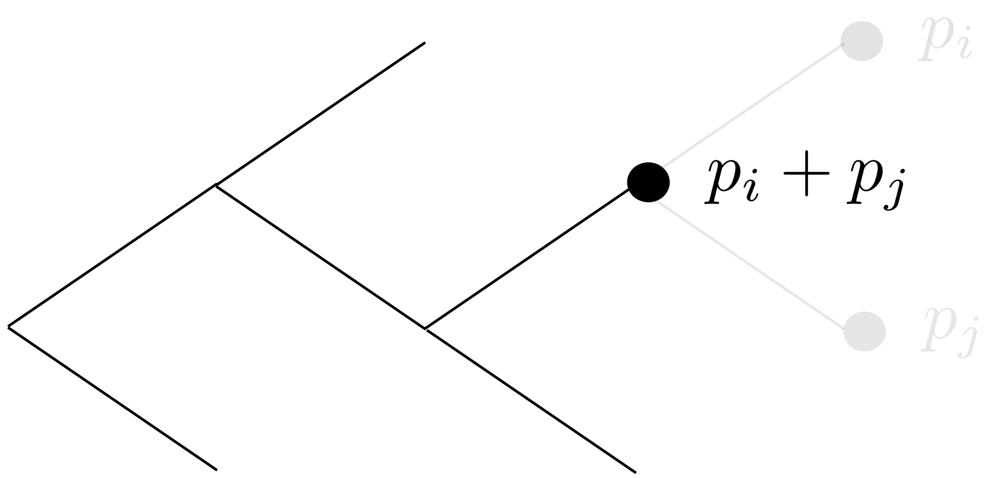

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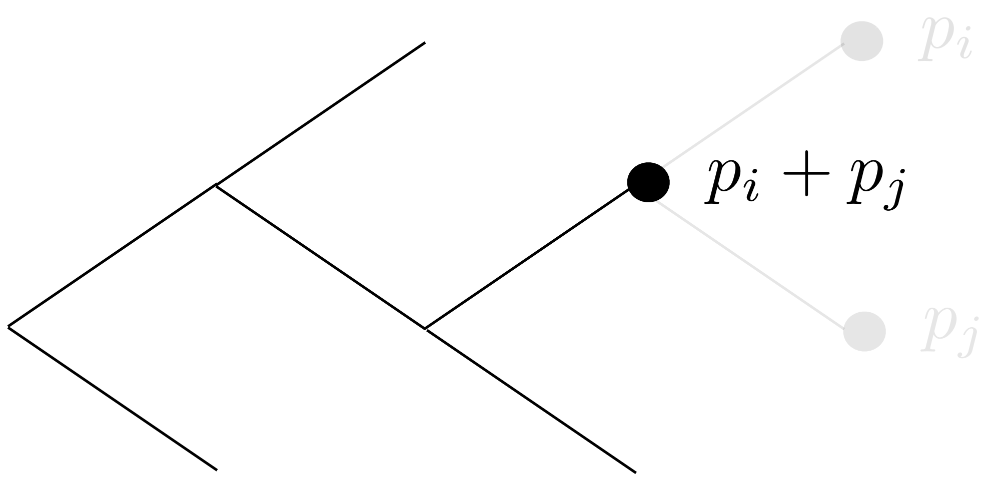

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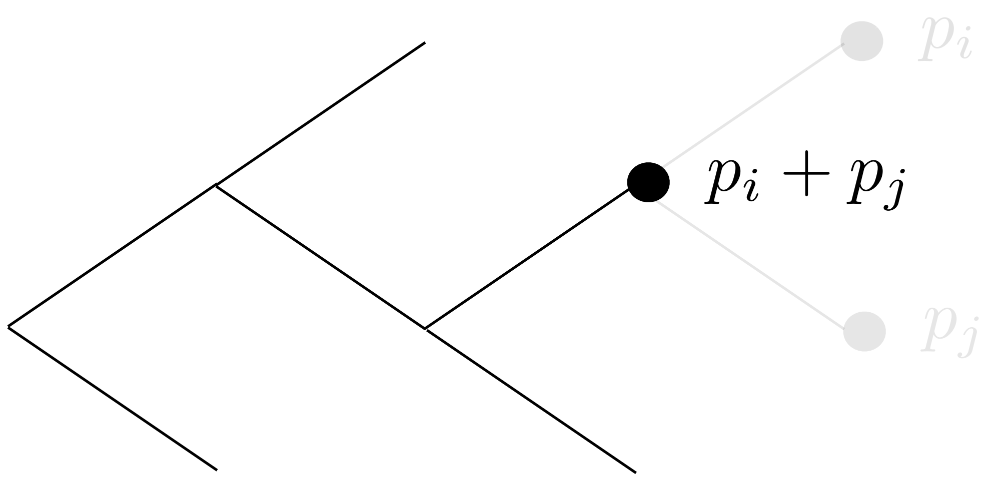

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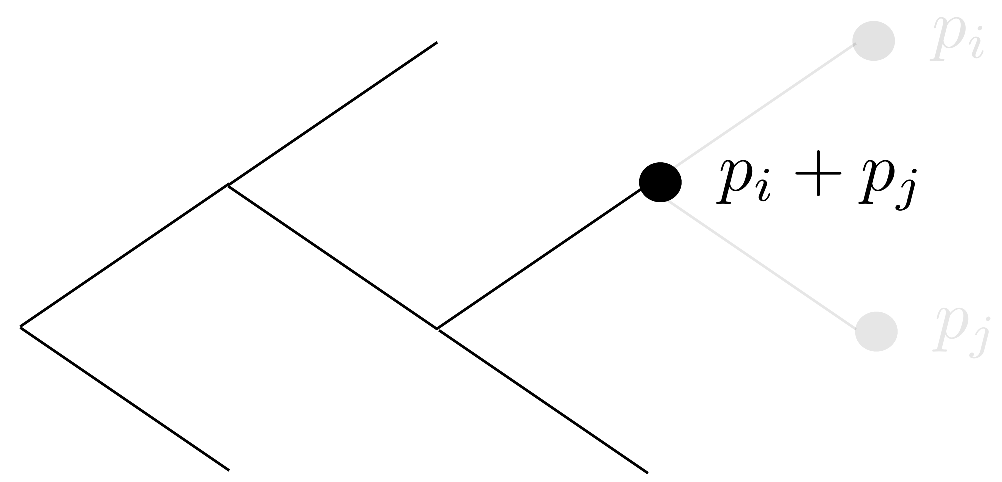

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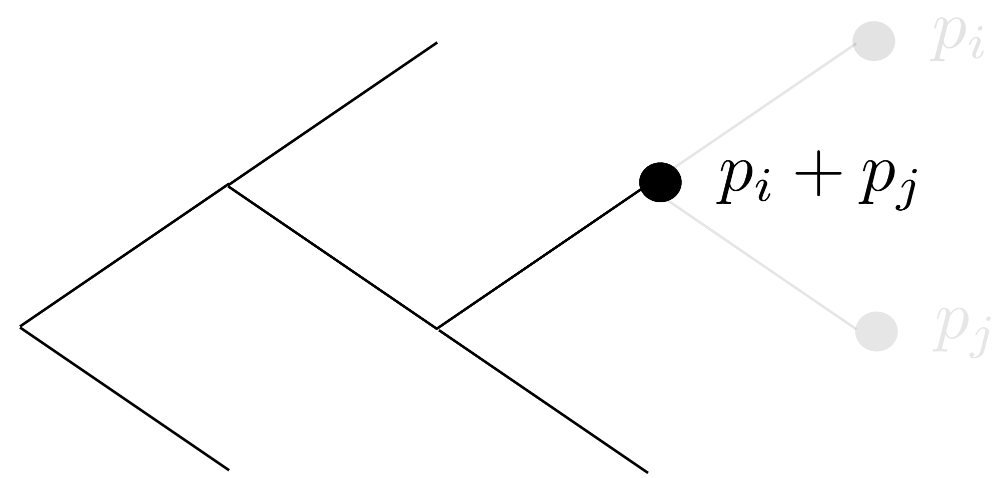

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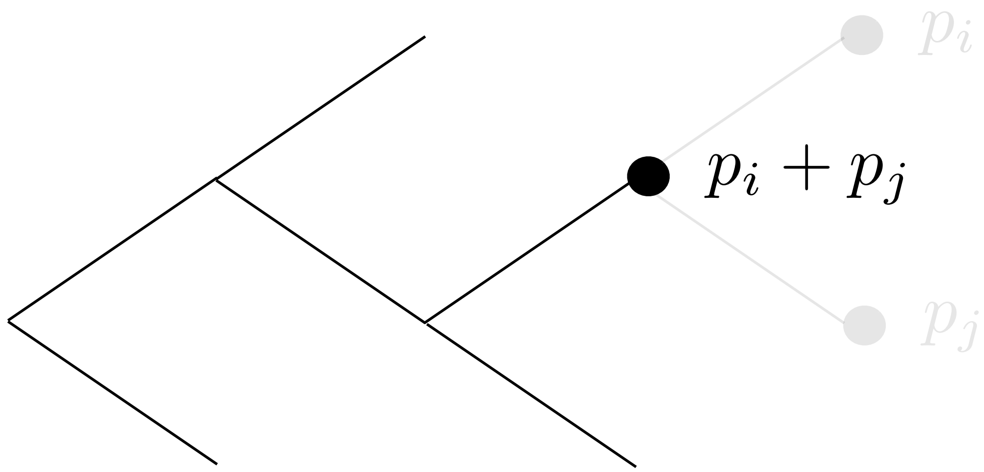

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4. Since

$$L = L' + (p_{m-1} + p_m),$$

we see that L is the expected length of an optimal code for $\{p_i\}$ if and only if L' is the expected length of an optimal code for $\{p'_i\}$.

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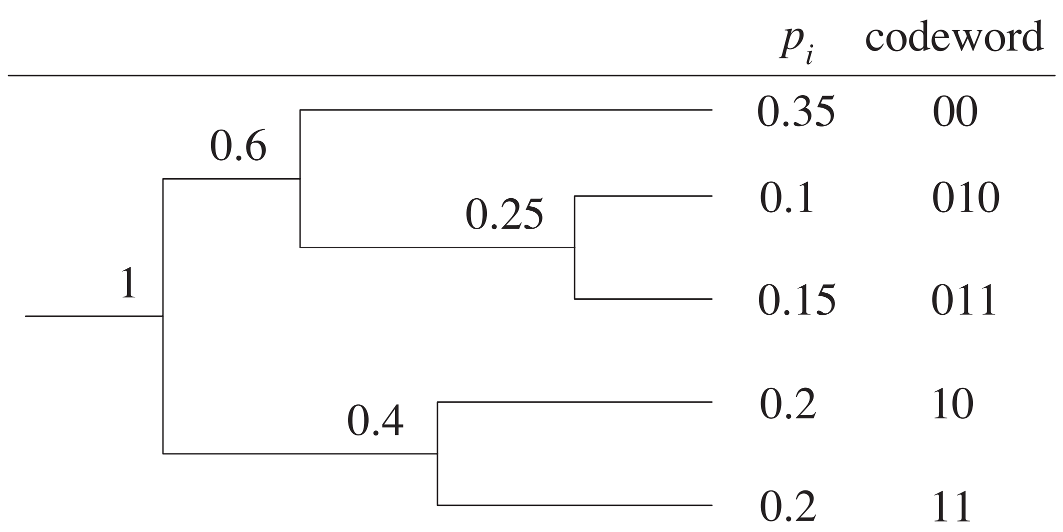

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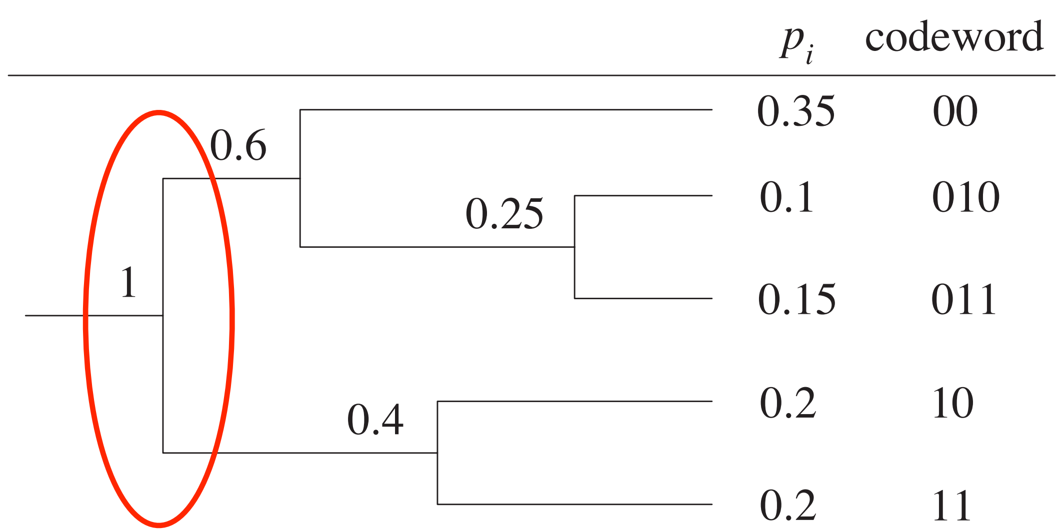

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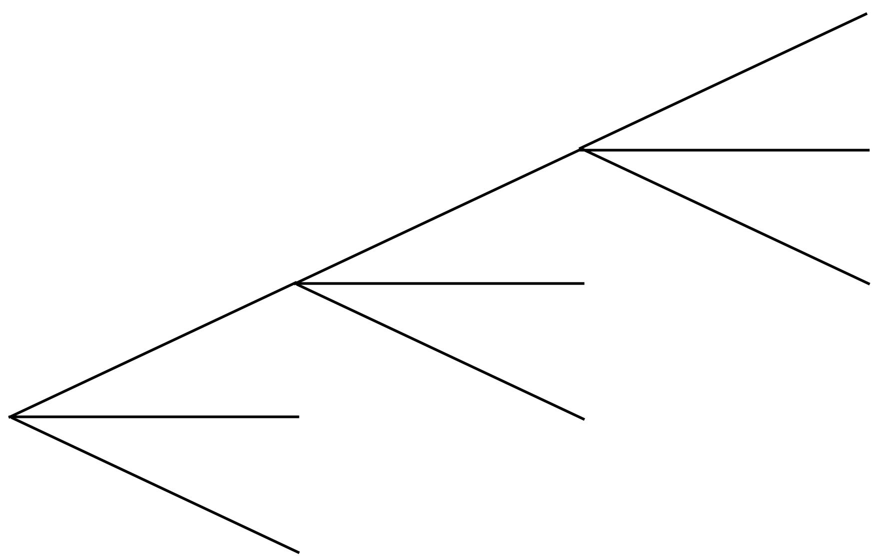

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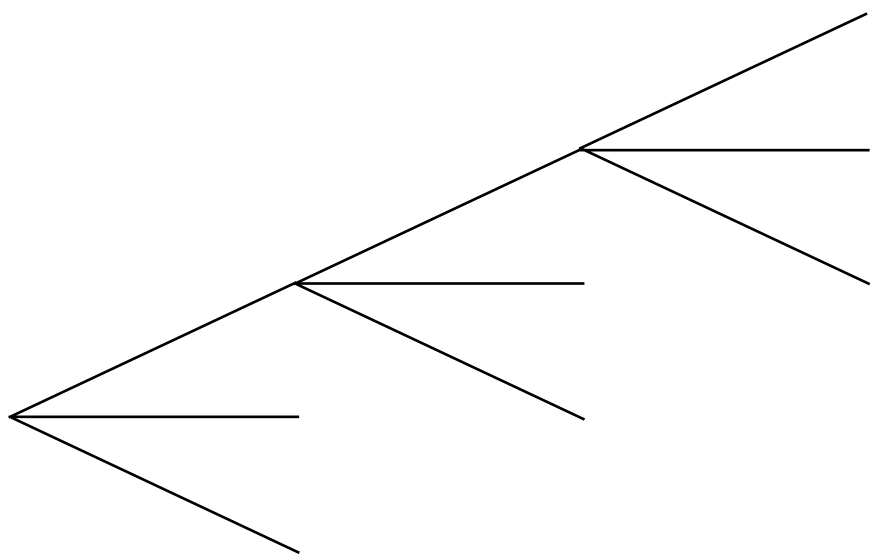

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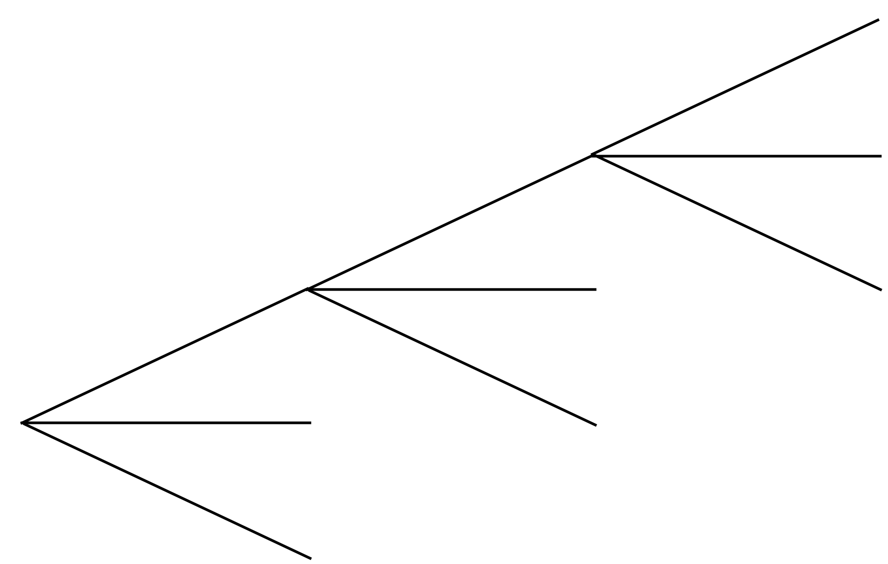

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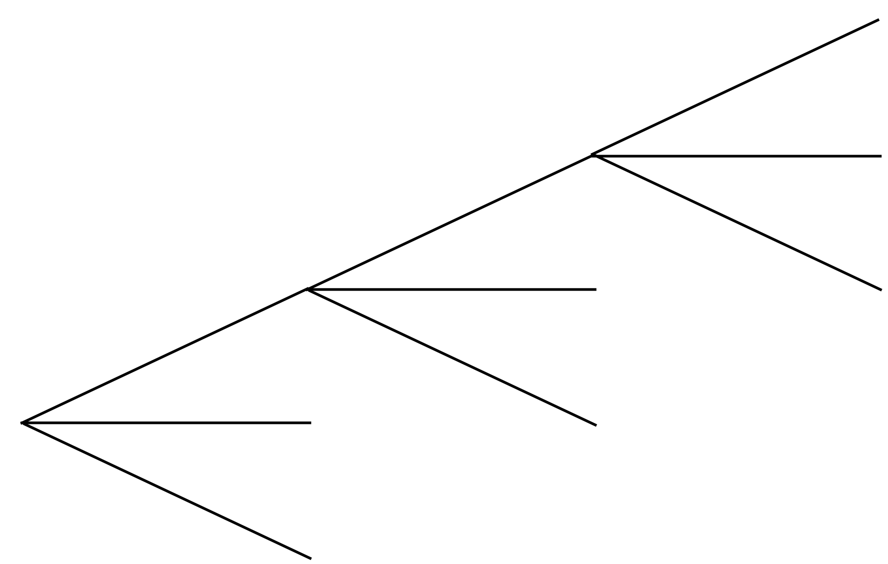

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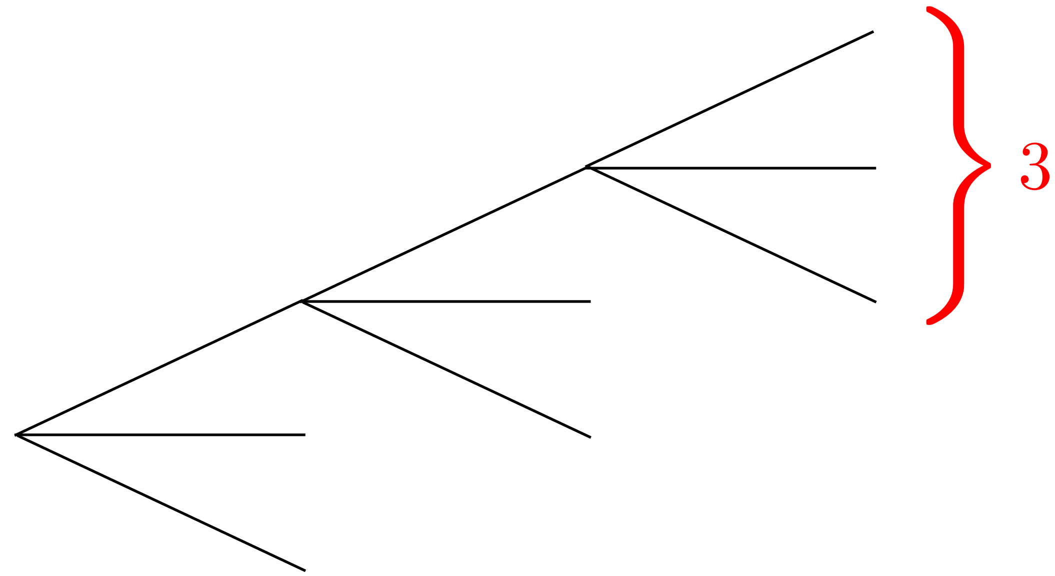

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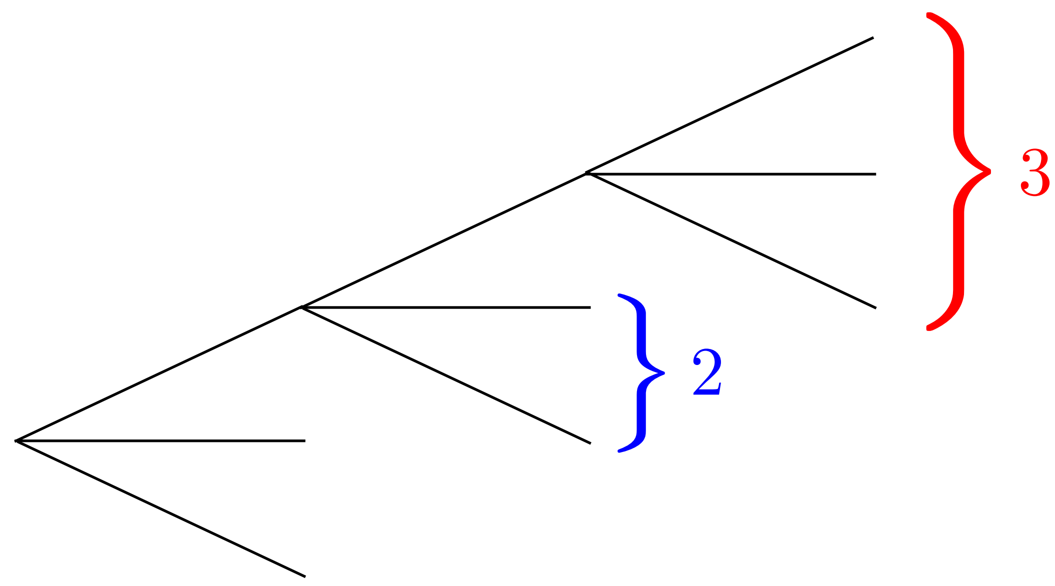

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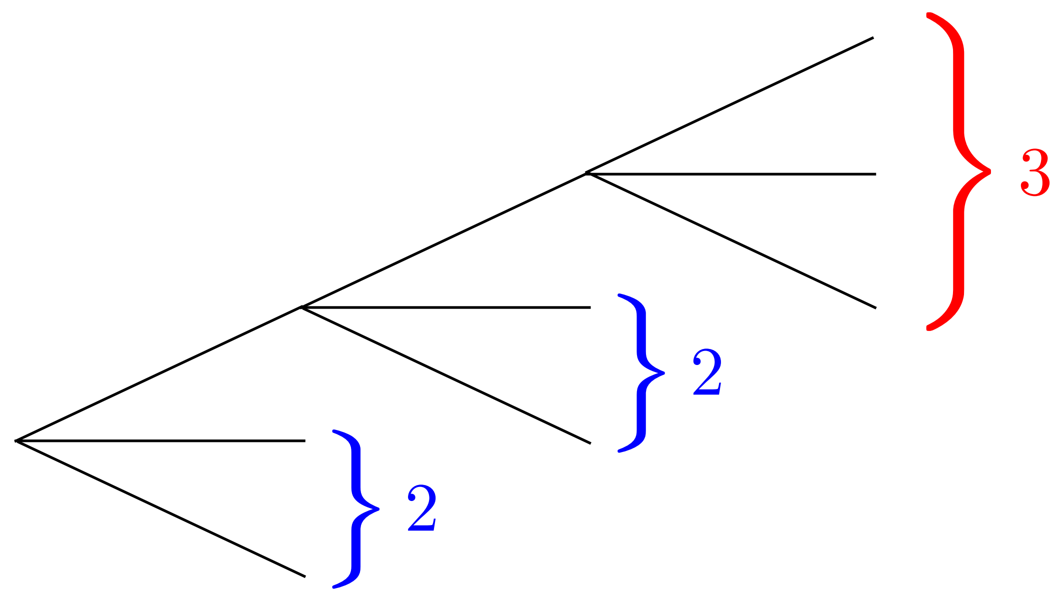

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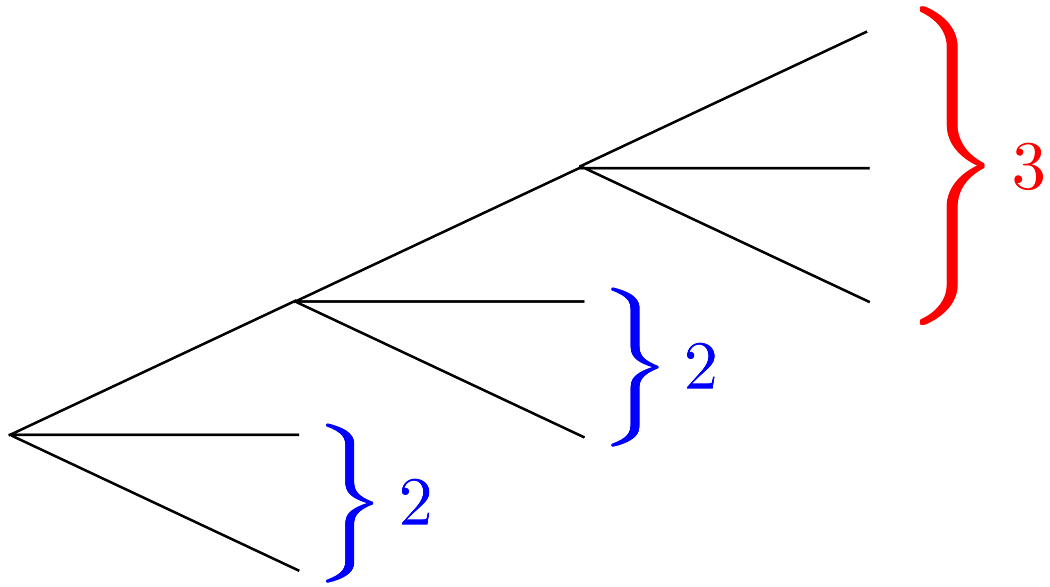

Upper Bound on L_{Huff}

Theorem 4.18 The expected length of a Huffman code, denoted by L_{Huff}, satisfies

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This bound is the tightest among all the upper bounds on L_{Huff} which depend only on the source entropy.

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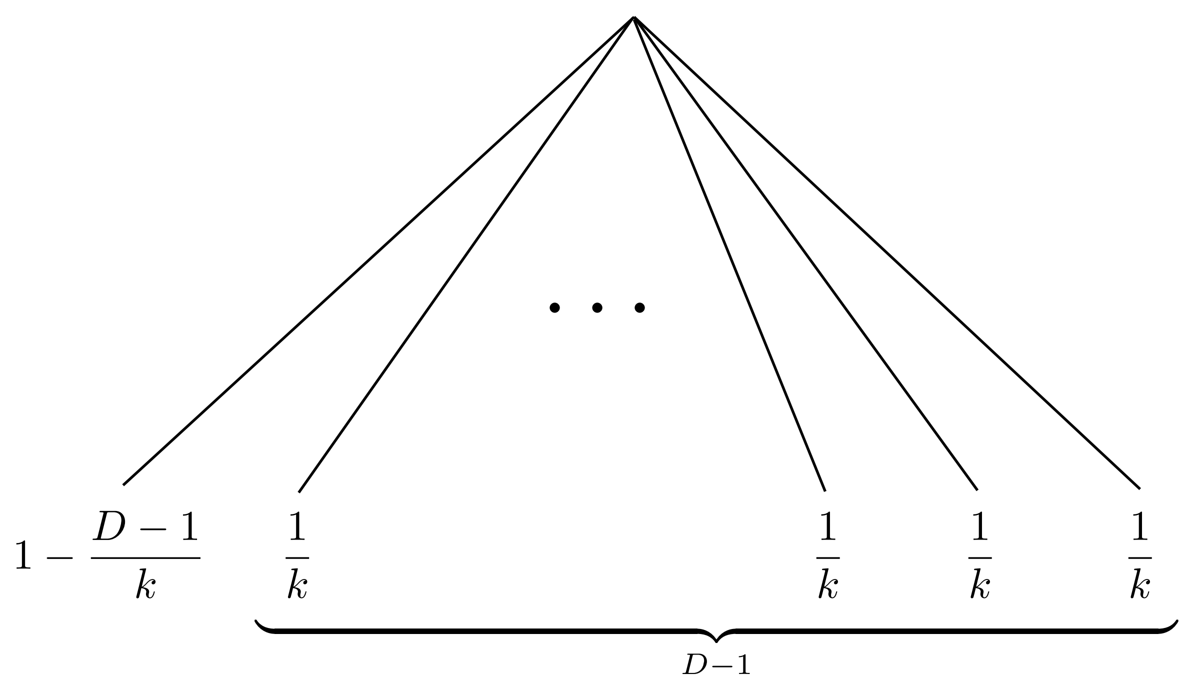

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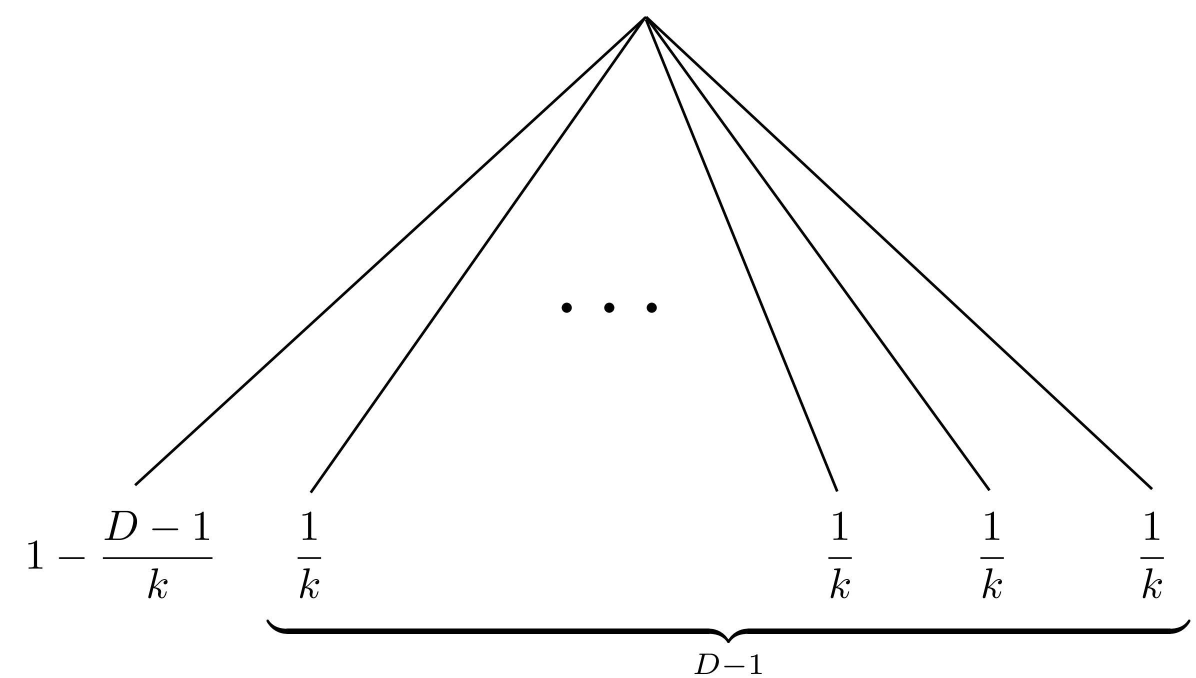

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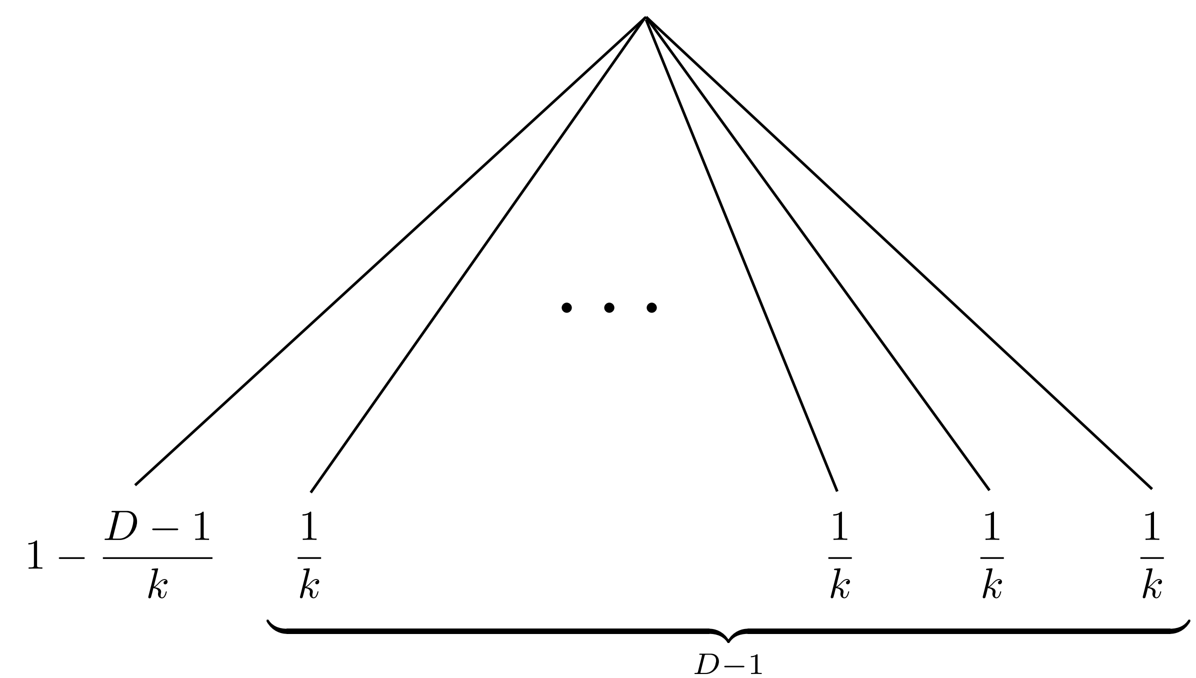

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