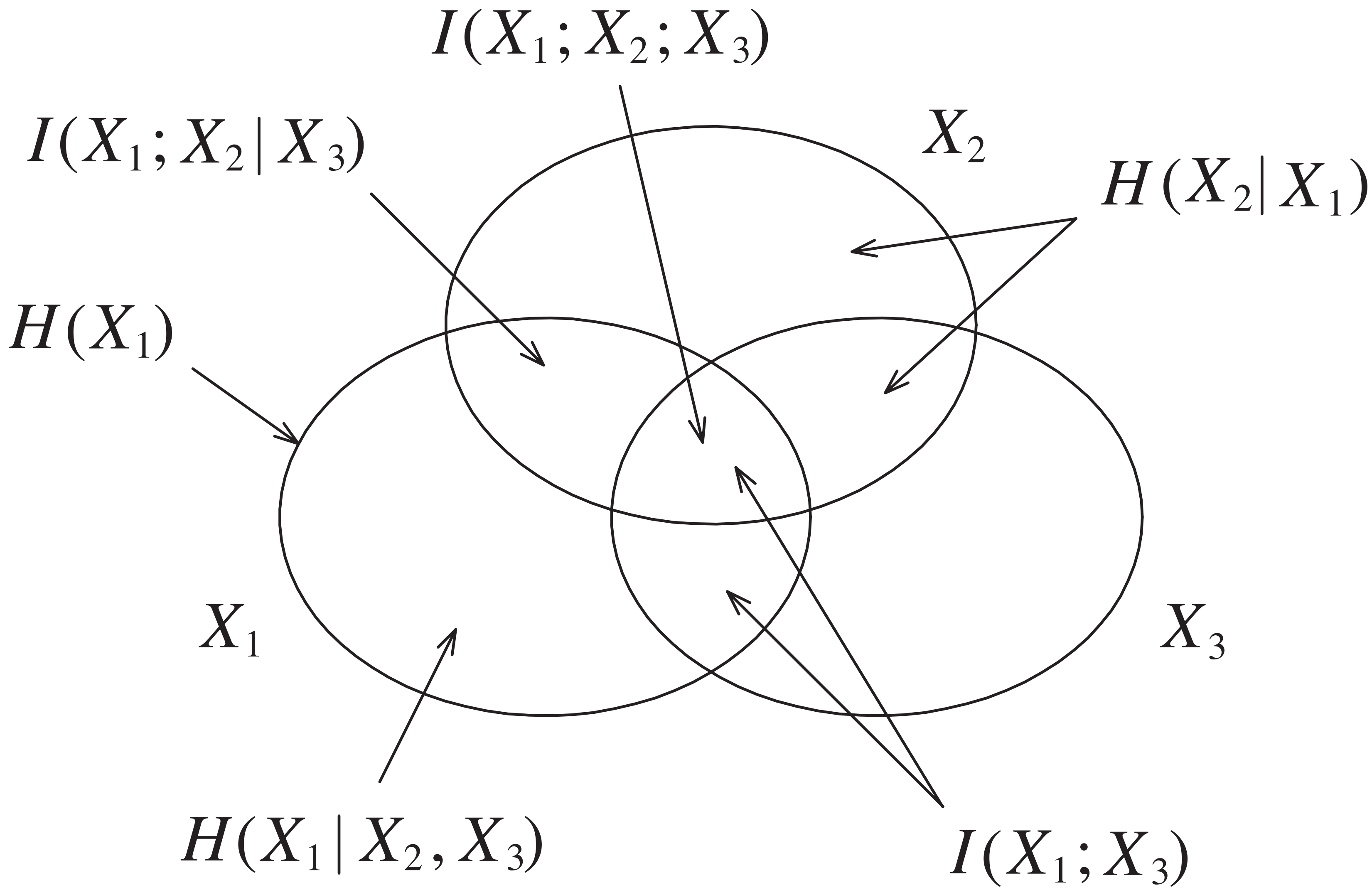
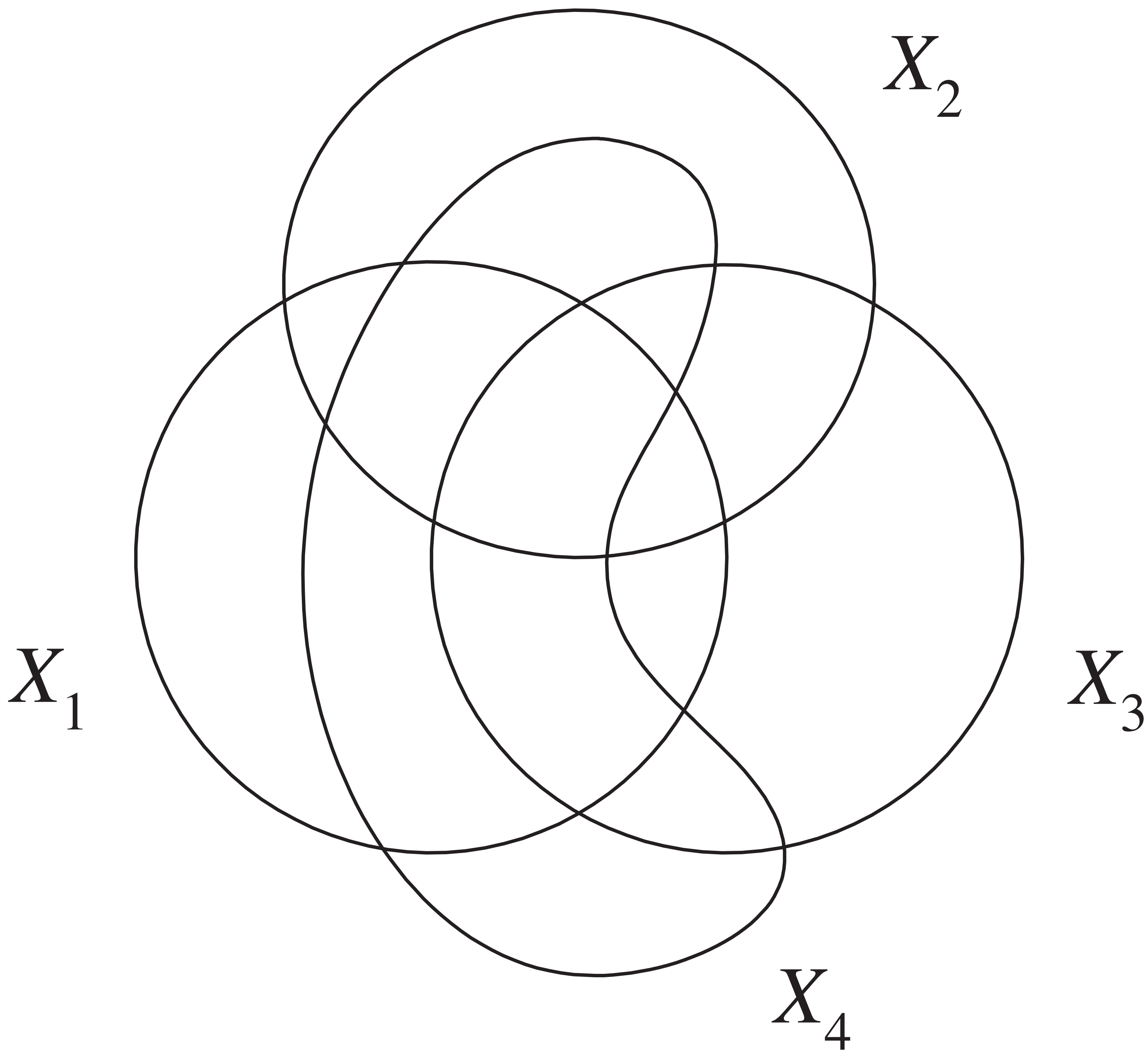
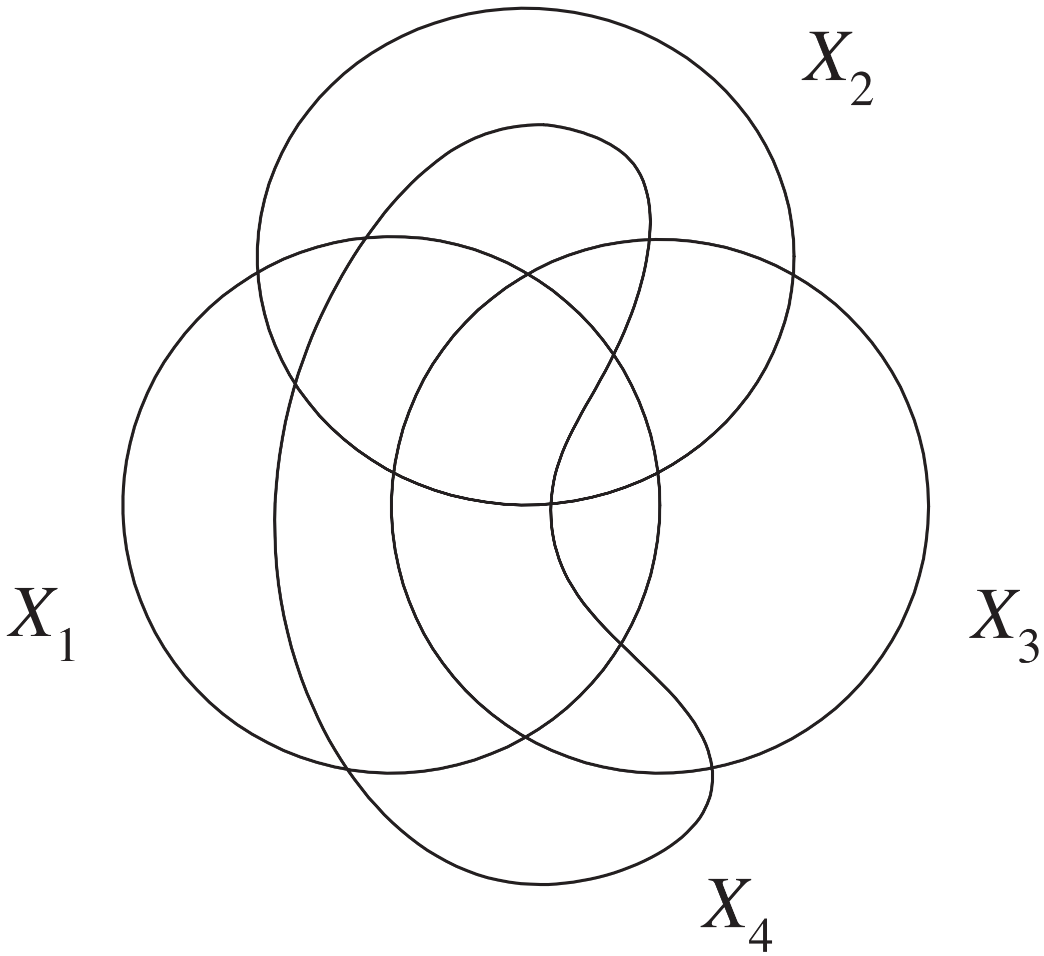
3.5 Information Diagrams


$I(X_1;X_2;X_3)$
$I(X_1;X_2|X_3)$
X_2
$H(X_2|X_1)$
$H(X_1)$
X_1
X_3
$H(X_1|X_2,X_3)$
$I(X_1;X_3)$



X_2
X_1
X_3
X_4



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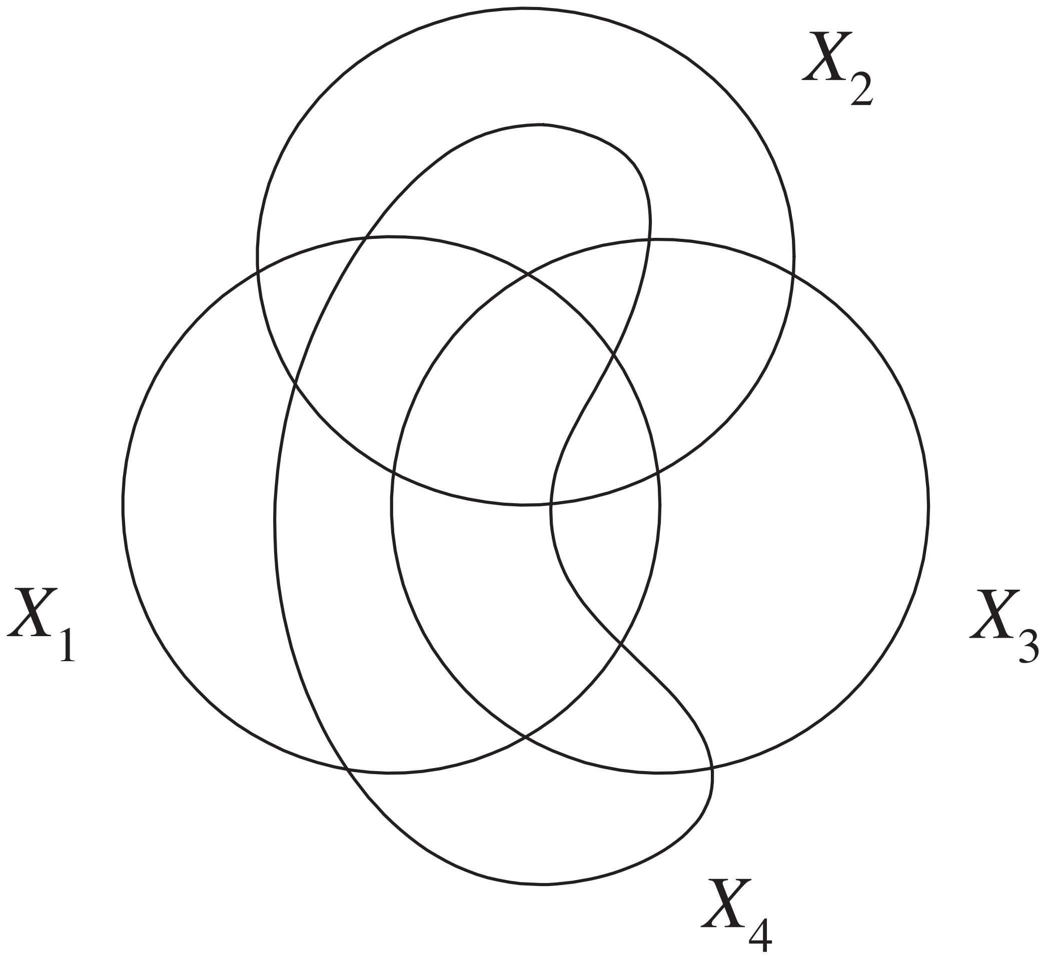
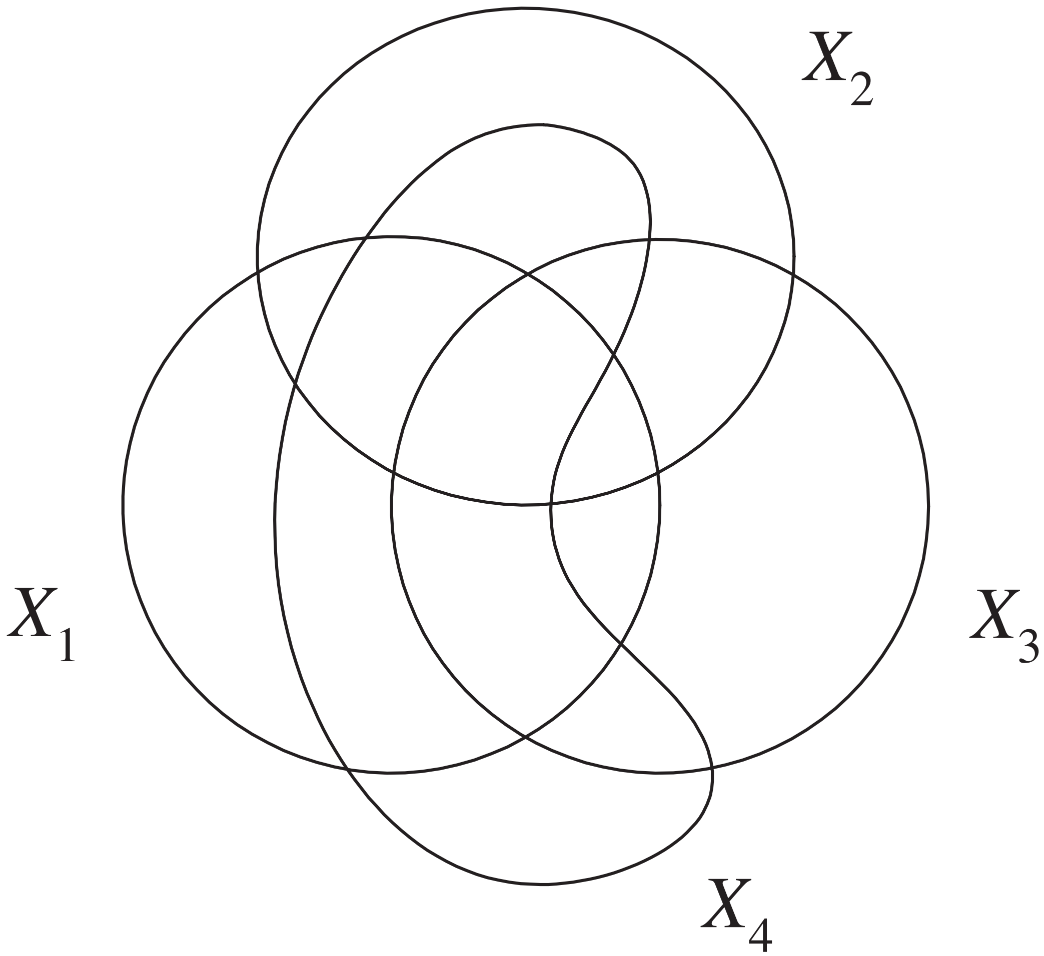
X_1
X_3
X_4



- This is a 2-dimensional representation of an information diagram for 4 r.v.'s.



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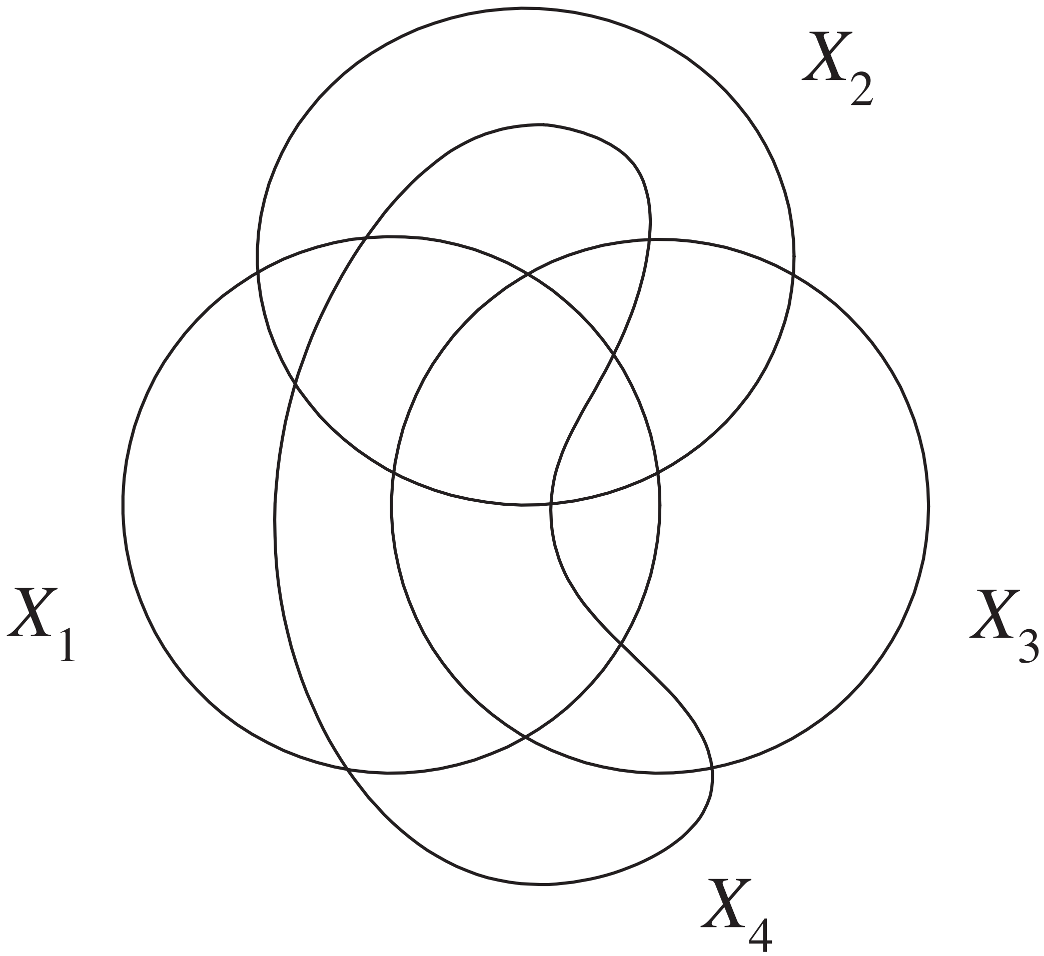
- First check that there are $2^4 - 1 = 15$ nonempty atoms.



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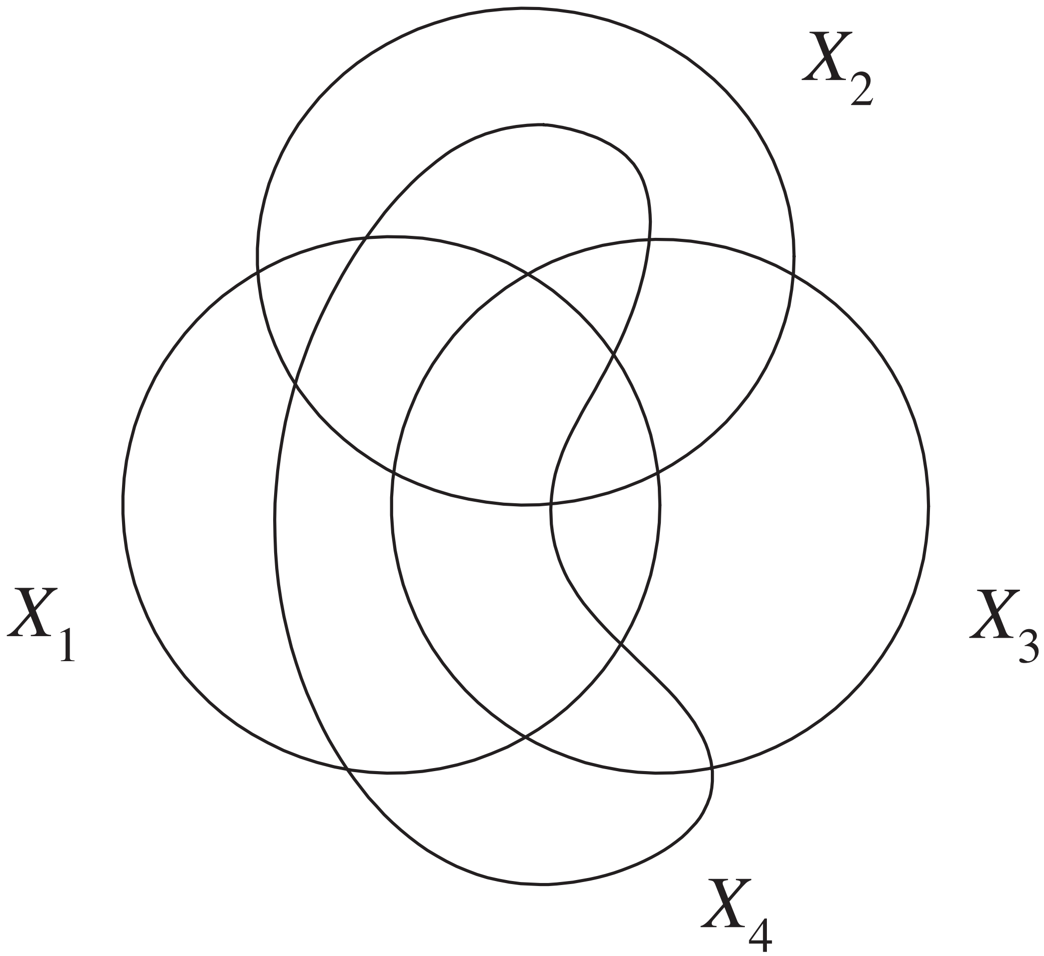
- This is a correct representation because the "kidney shape" representing X_4 splits each nonempty atom in $\mathcal{F}_3$ into 2 nonempty atoms in $\mathcal{F}_4$.



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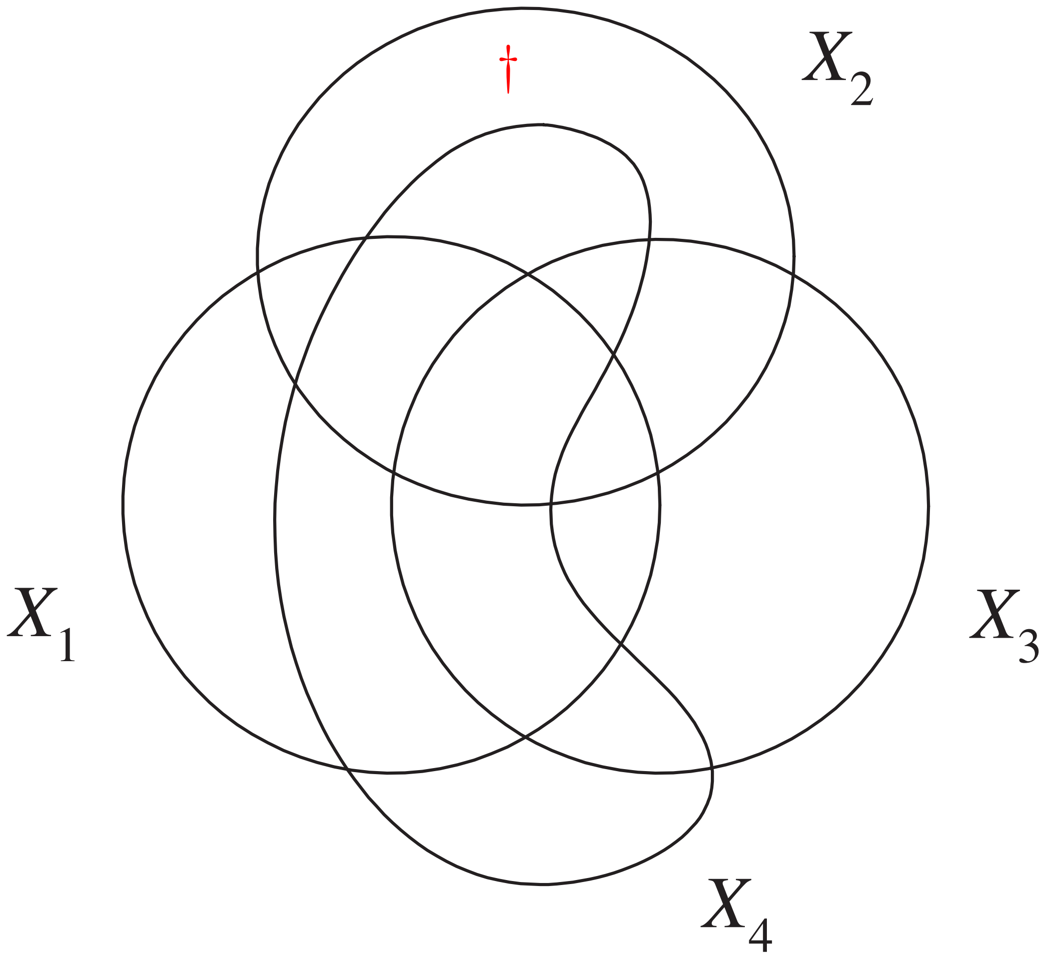

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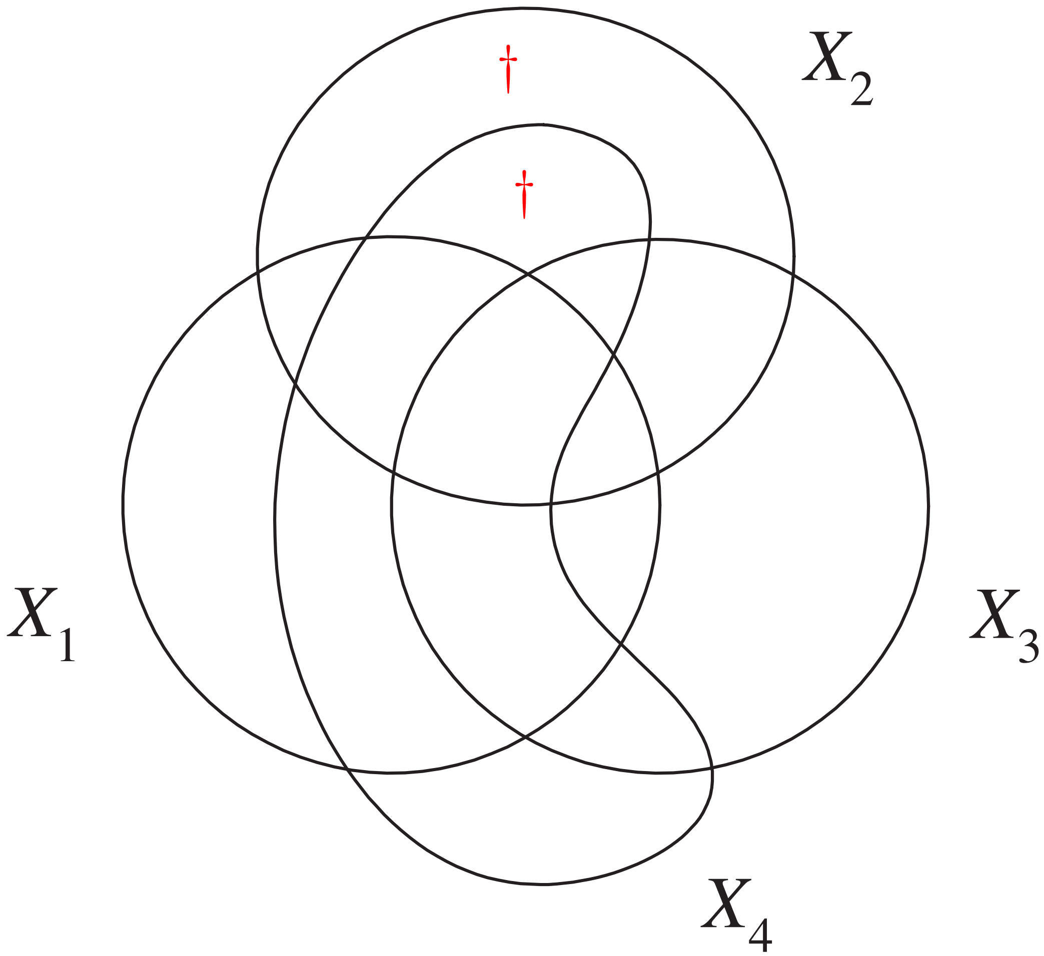

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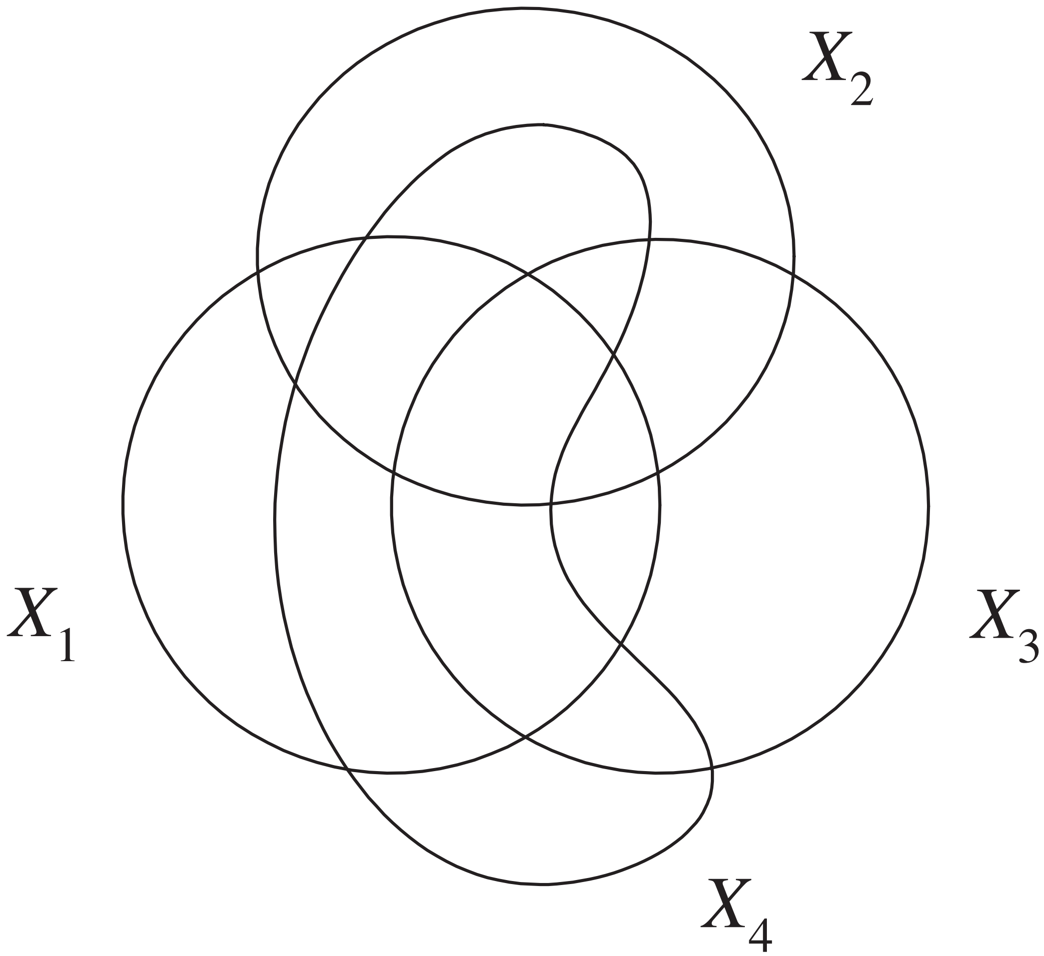

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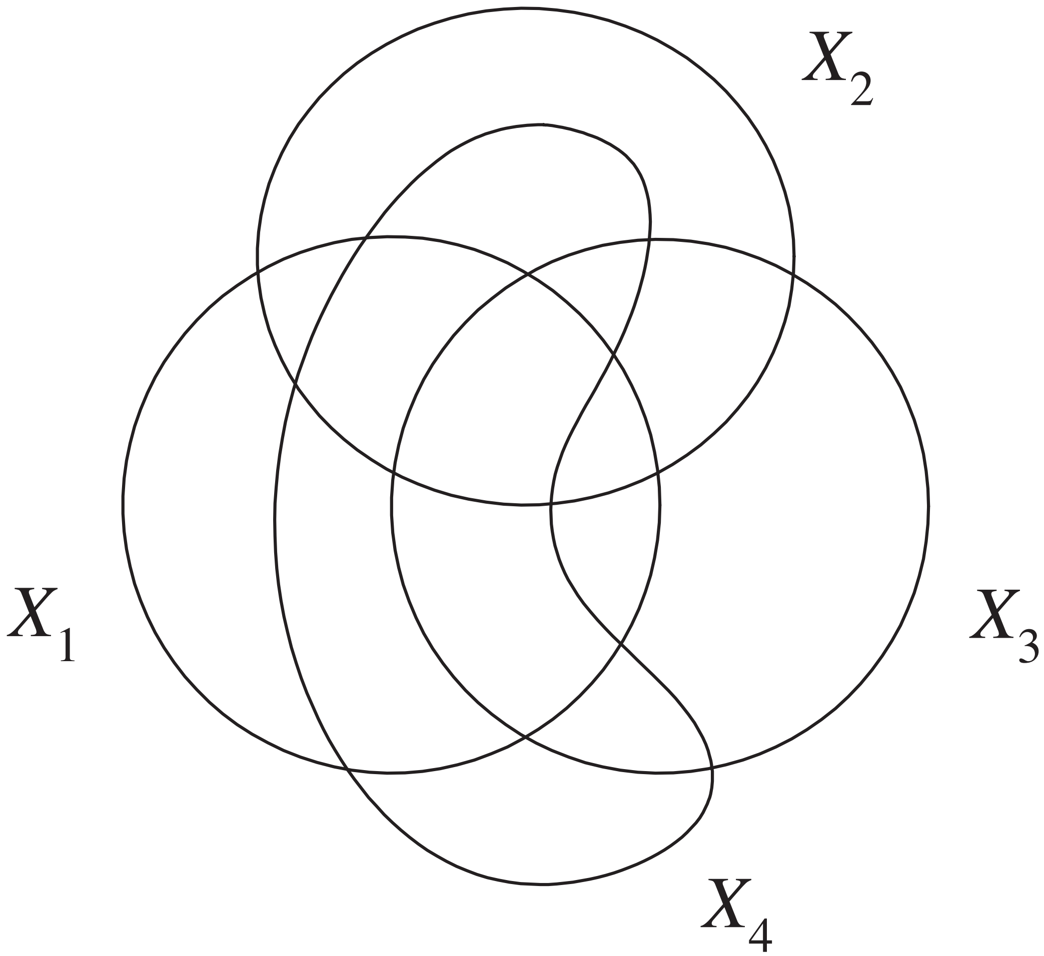
- Exercise: Check that all the other 6 atoms of $\mathcal{F}_3$ are split into two atoms in $\mathcal{F}_4$.



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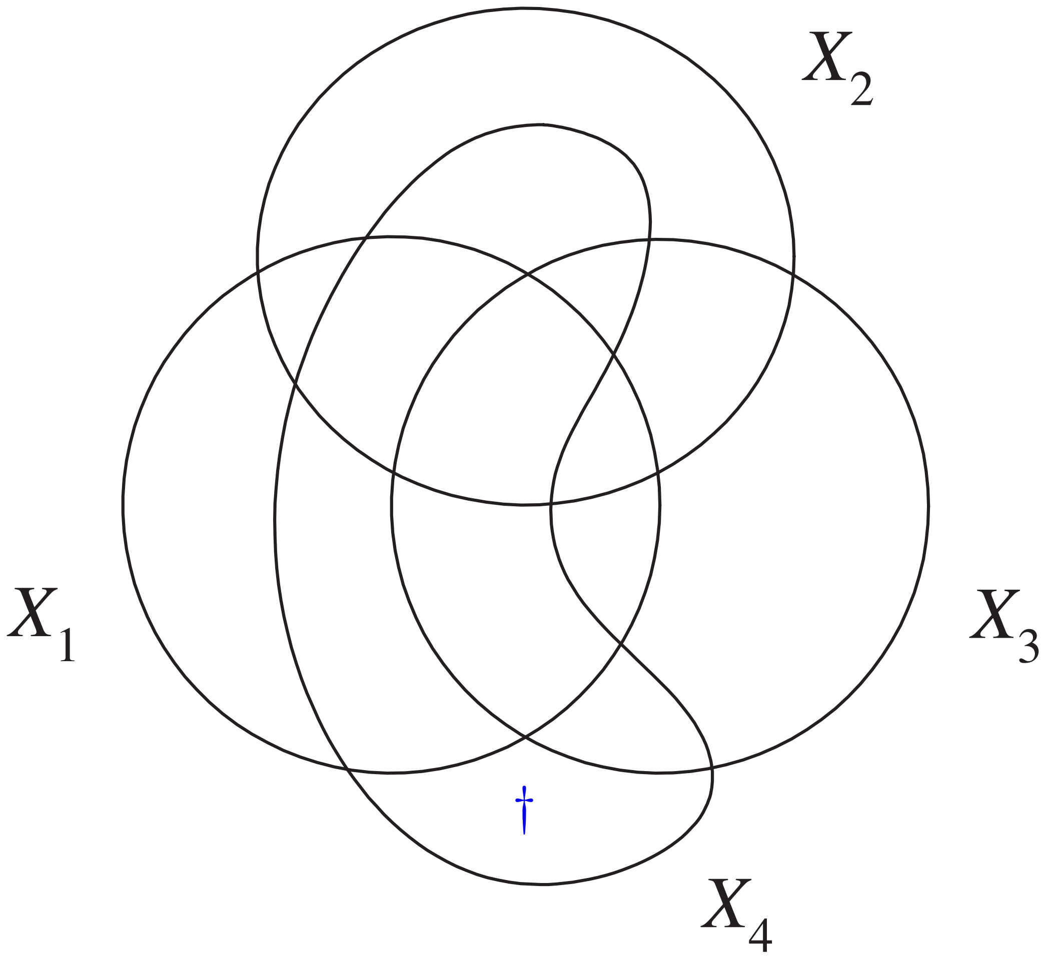

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Theorem 3.11 If there is no constraint on $X_1, X_2, \cdots, X_n$, then μ^* can take any set of nonnegative values on the nonempty atoms of $\mathcal{F}_n$.

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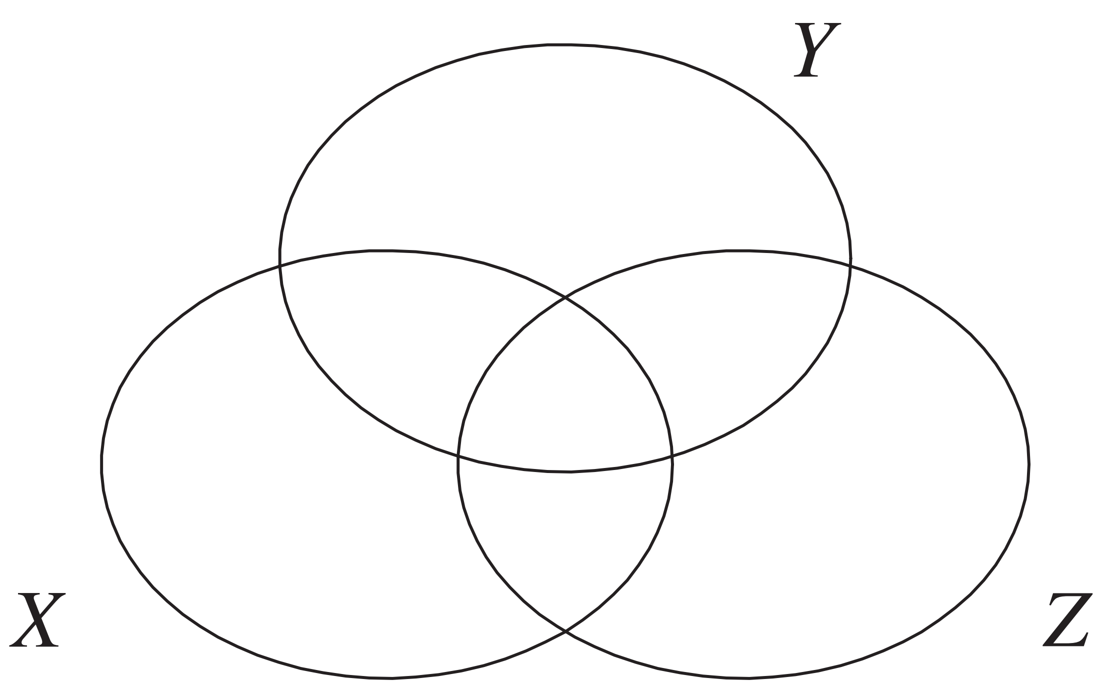

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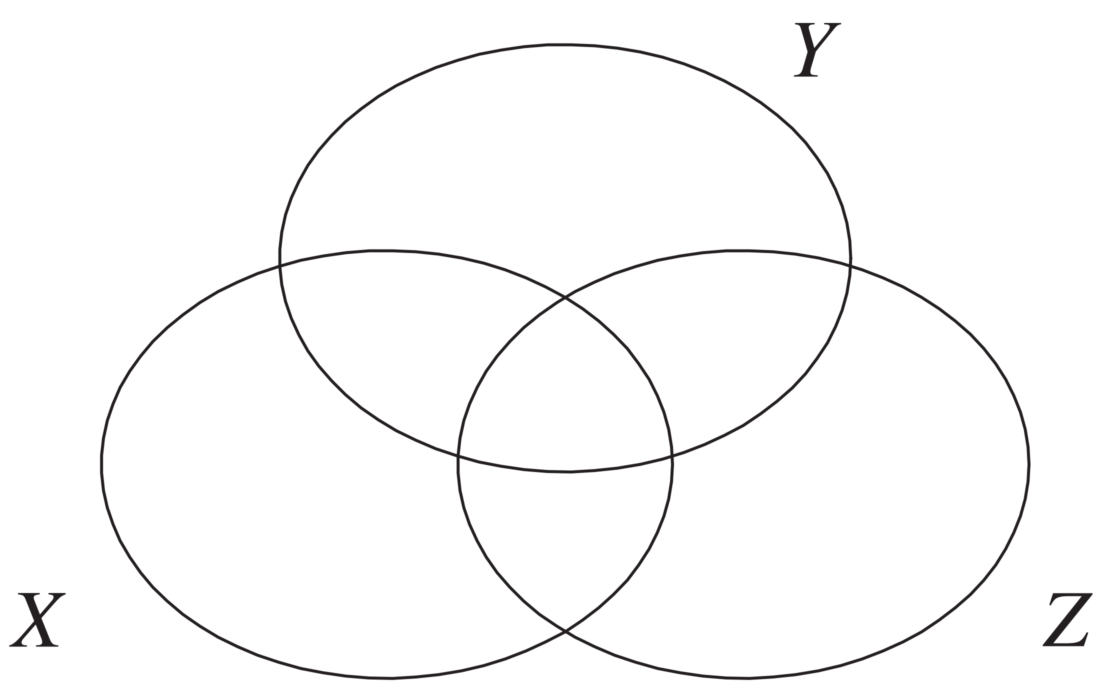

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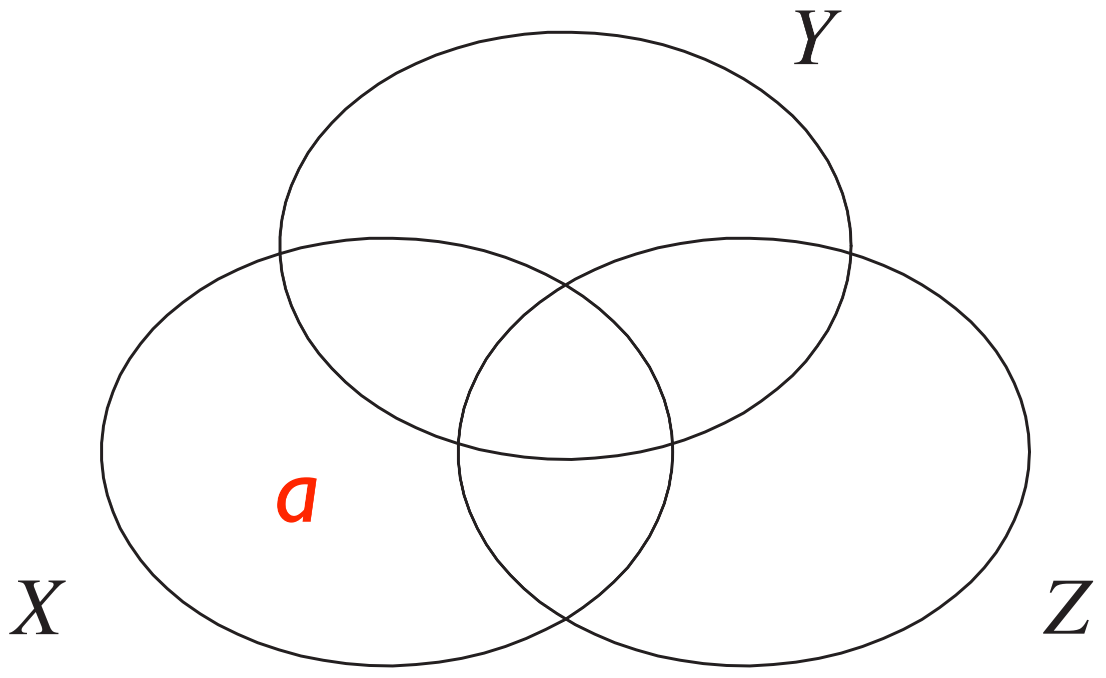

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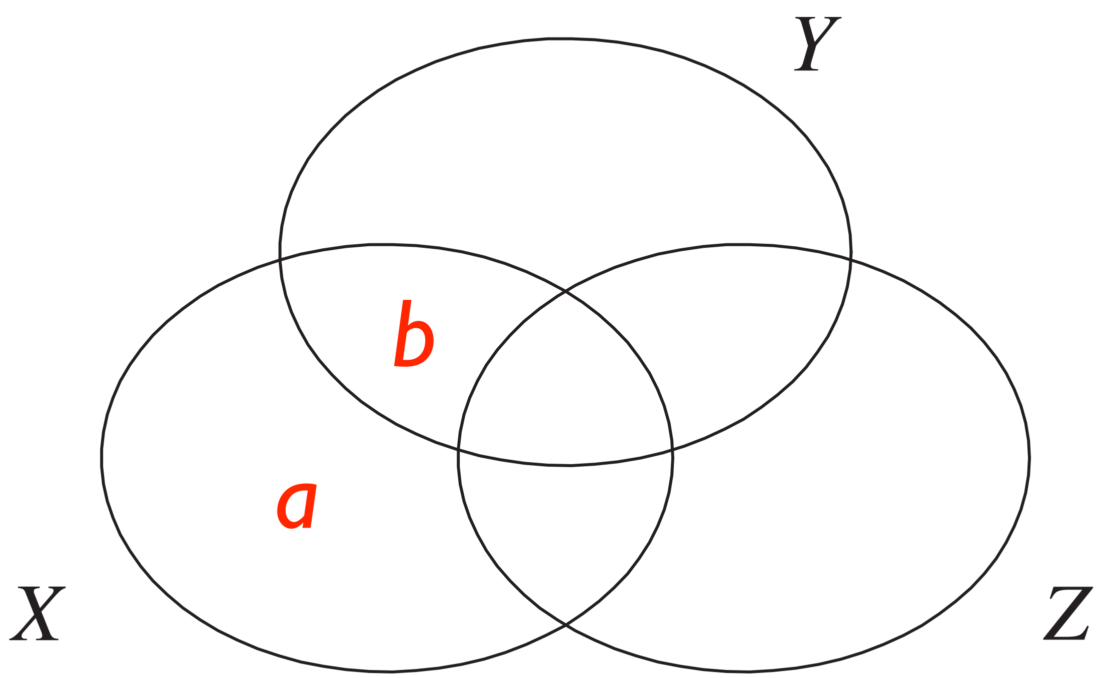

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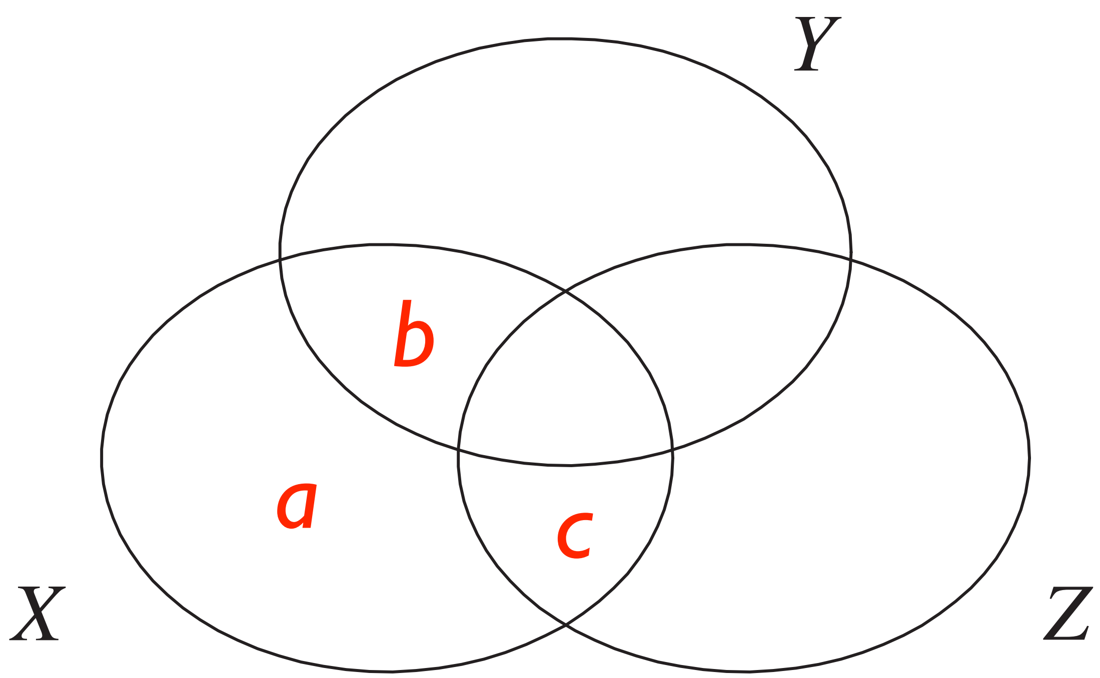

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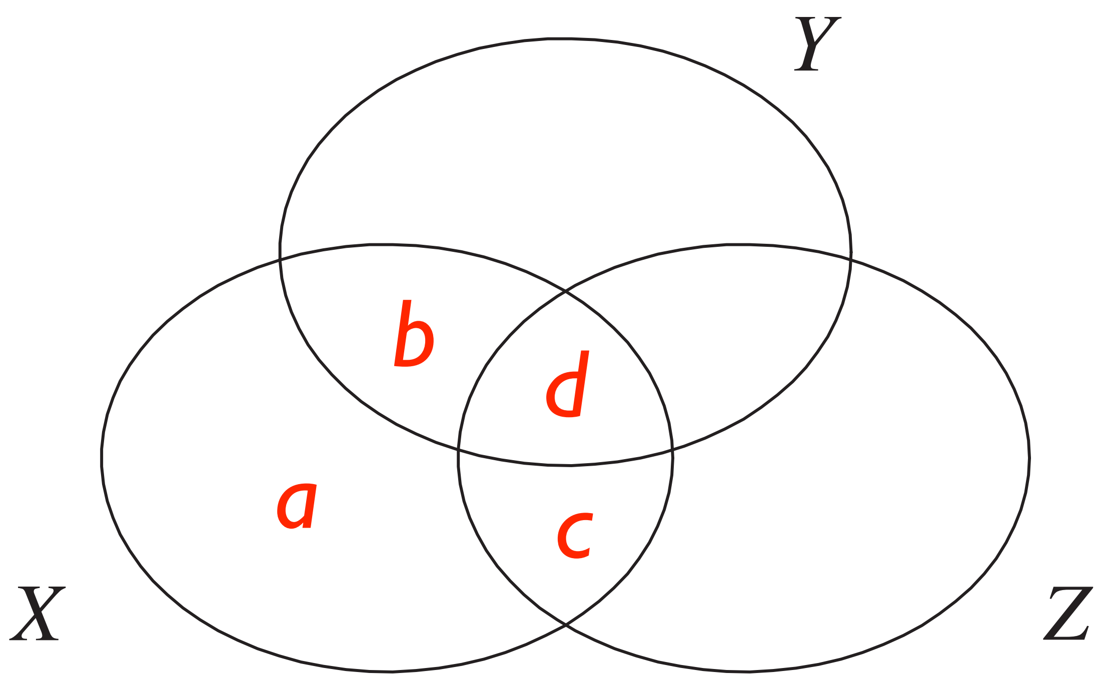

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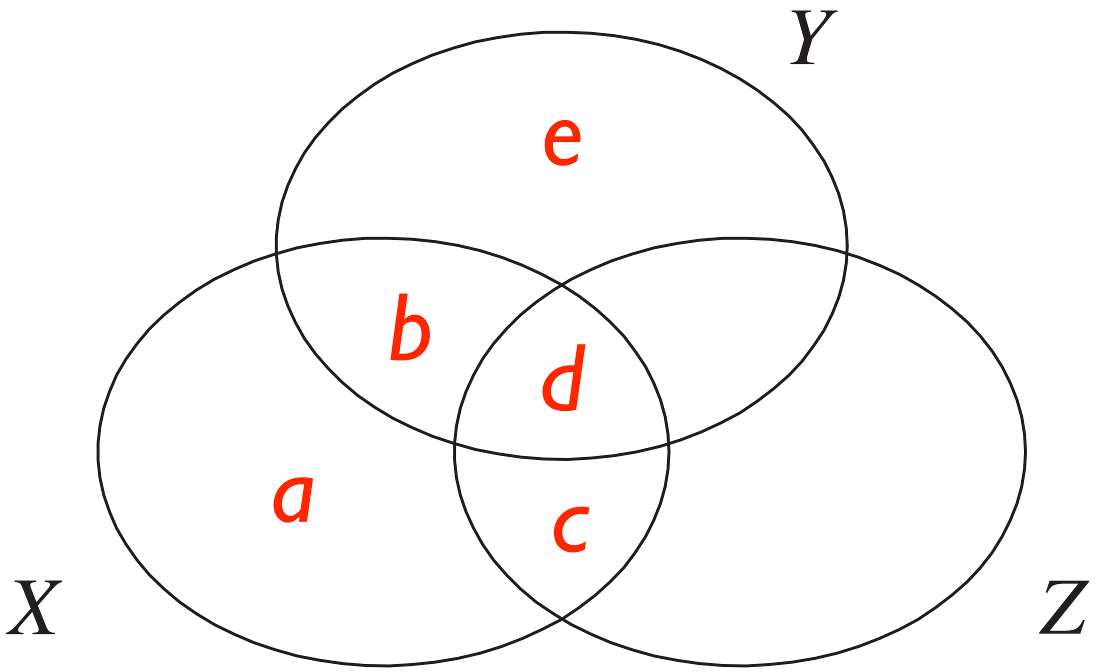

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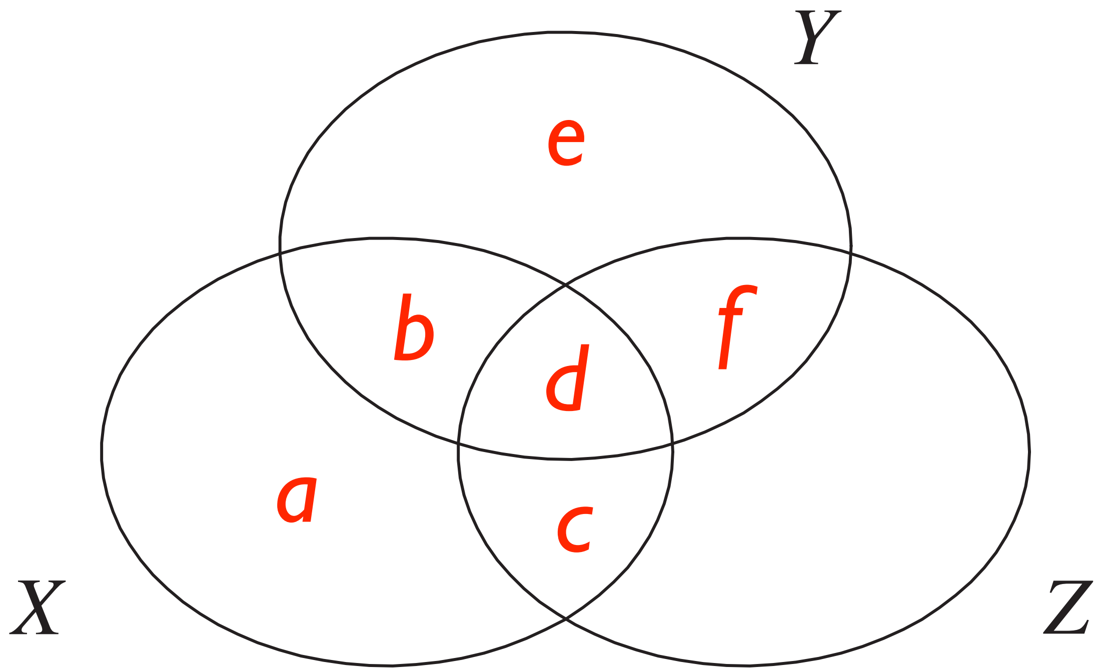

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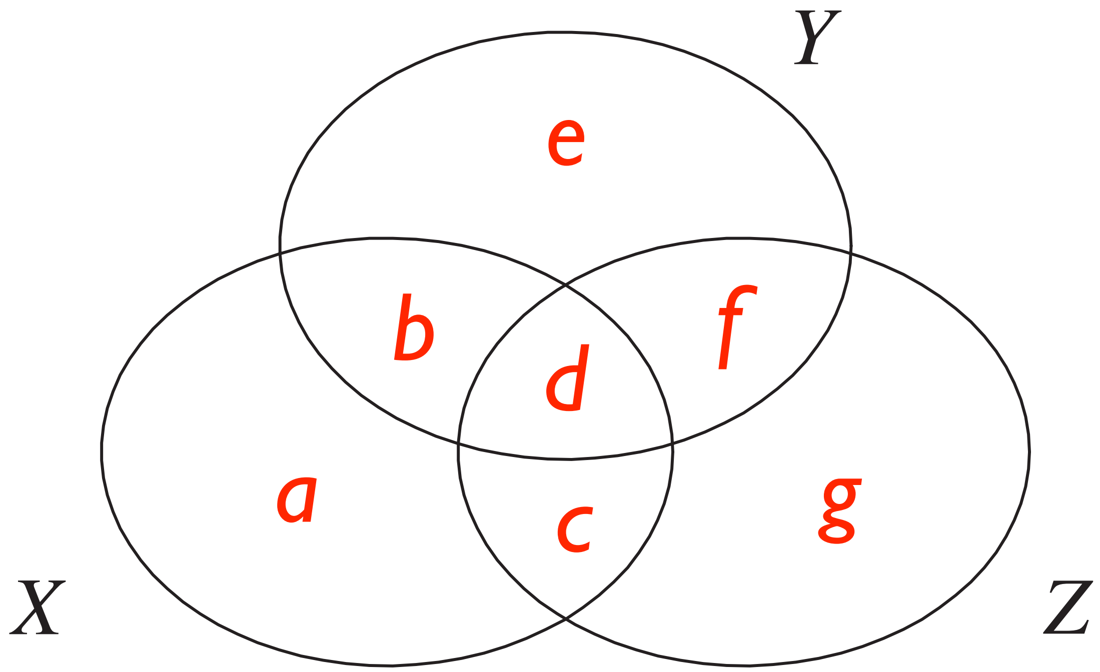

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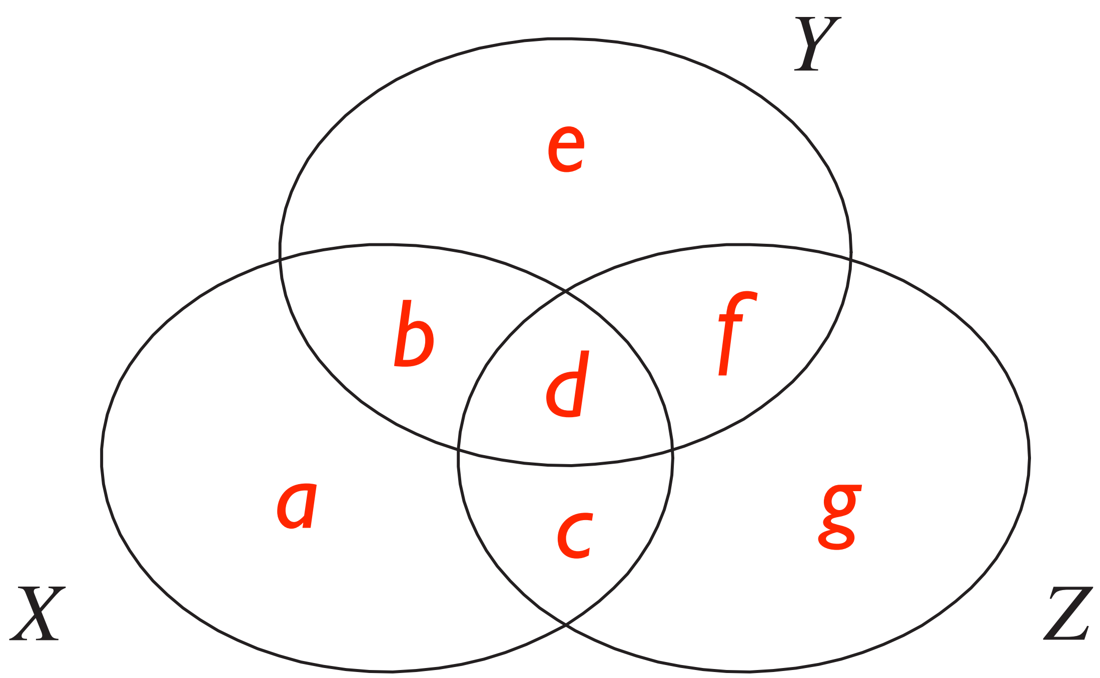

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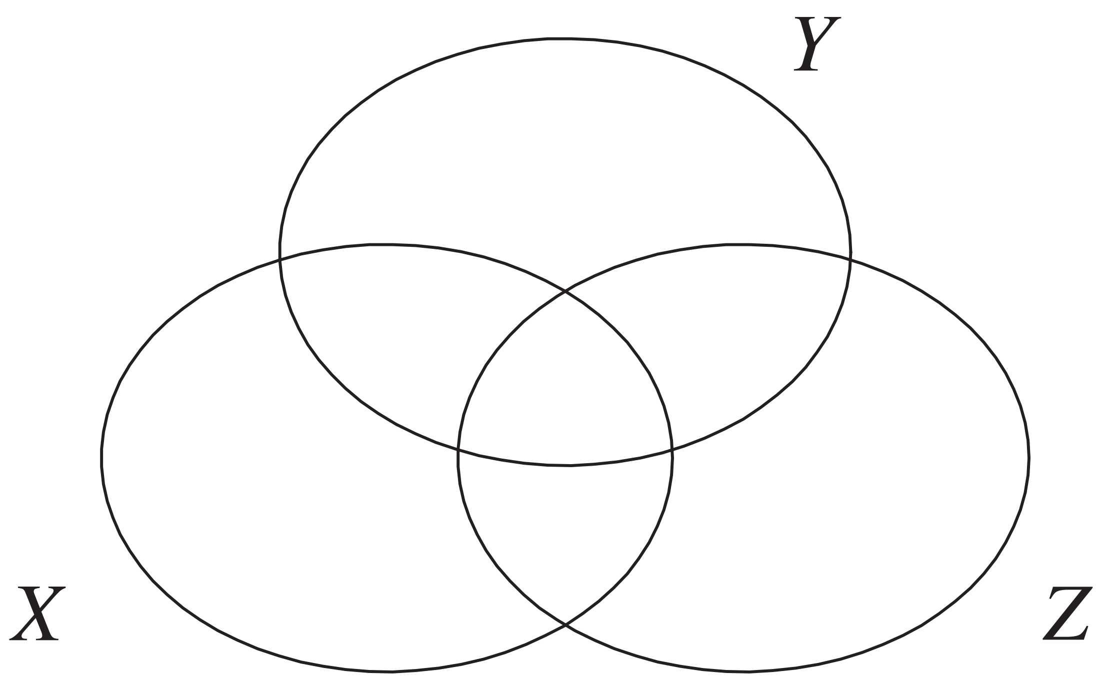

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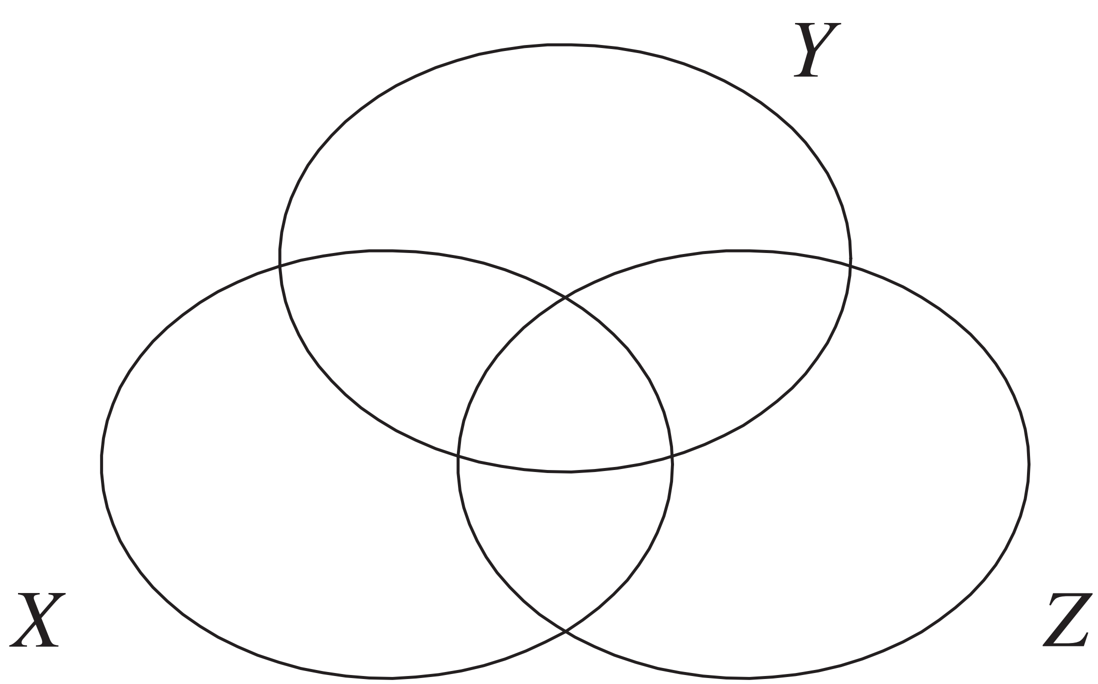

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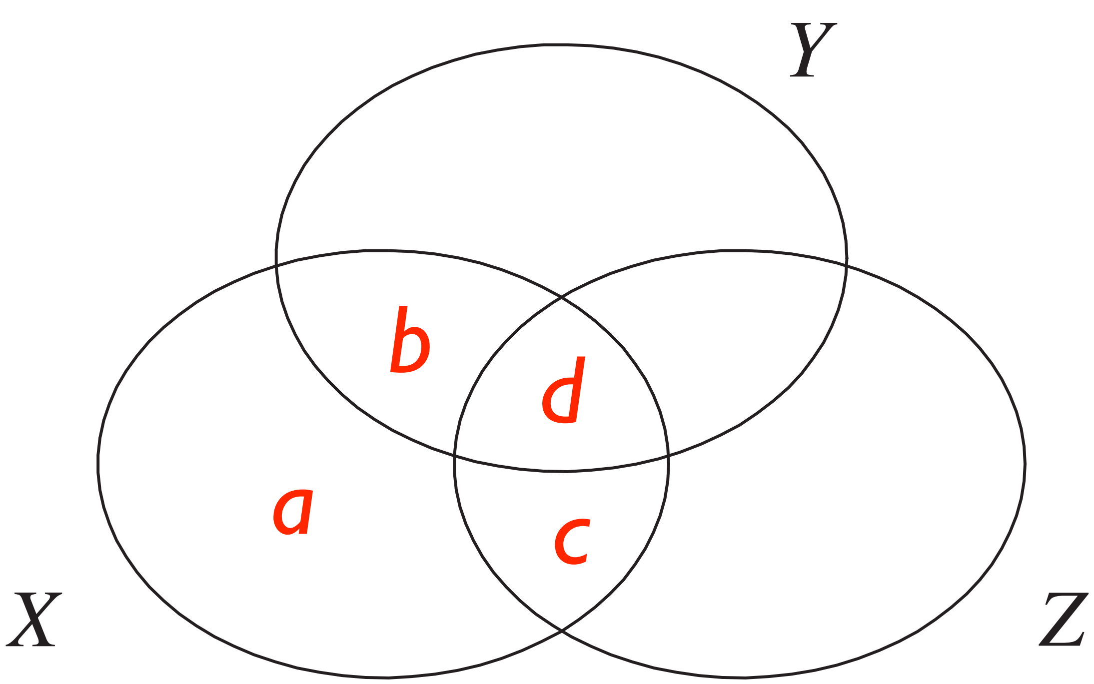

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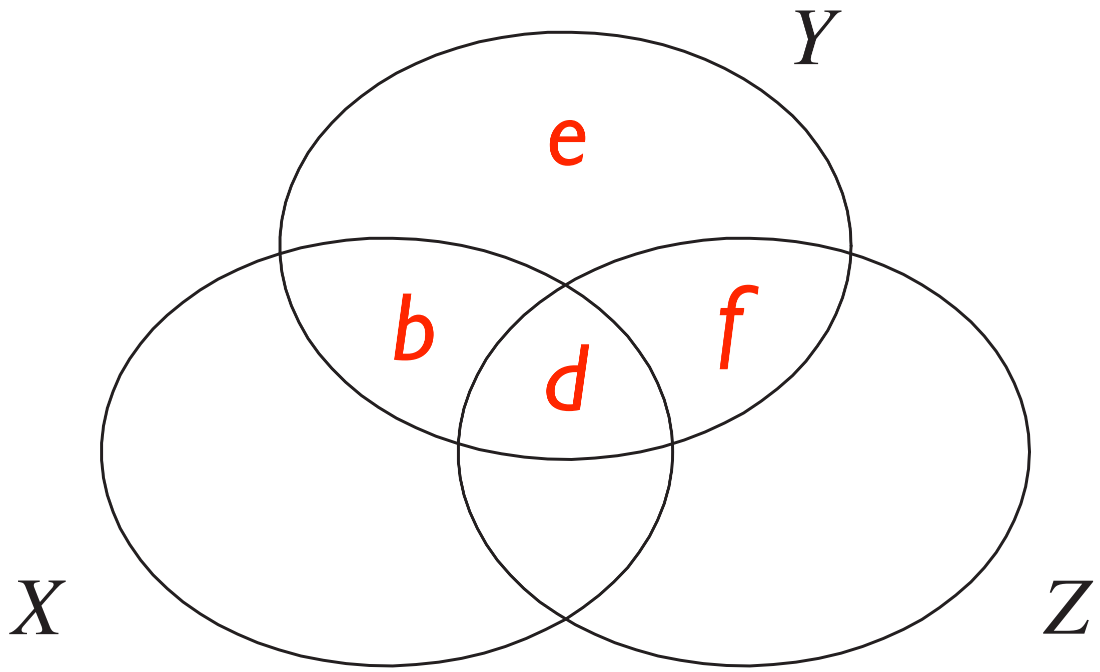

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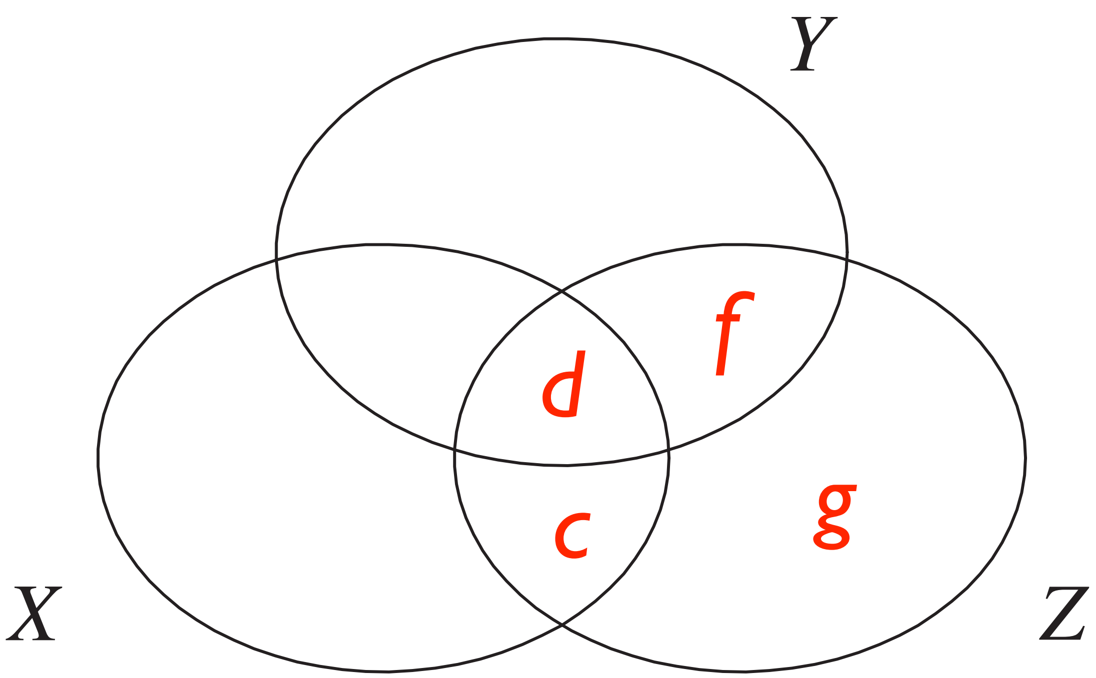

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$$
\begin{aligned}
X &= (A, B, C, D) \\
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Z &= (C, D, F, G).
\end{aligned}
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Proof

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Proof

1. Let $Y_A, A \in \mathcal{A}$ be mutually independent random variables. Note that these r.v.'s are labeled by the nonempty atoms of $\mathcal{F}_n$.

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$$\begin{aligned} H(X_G) &= H(X_i, i \in G) \\ &= H((Y_A : A \in \mathcal{A} \text{ and } A \subset \tilde{X}_i), i \in G) \end{aligned}$$

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for all $A \in \mathcal{A}$. By the uniqueness of μ^* (Theorem 3.9), this is also the only possibility for μ^*.

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Theorem 3.9 μ^* is the unique signed measure on $\mathcal{F}_n$ which is consistent with all Shannon's information measures.

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3. Since Y_A are mutually independent, for any nonempty subsets G of $\mathcal{N}_n$, we have

$$\begin{aligned} H(X_G) &= H(X_i, i \in G) \\ &= H((Y_A : A \in \mathcal{A} \text{ and } A \subset \tilde{X}_i), i \in G) \\ &= H(Y_A : A \in \mathcal{A} \text{ and } A \subset \tilde{X}_G) \\ &= \sum_{A \in \mathcal{A} : A \subset \tilde{X}_G} H(Y_A). \end{aligned} \qquad (1)$$

4. On the other hand, by set-additivity, we have

$$H(X_G) = \mu^*(\tilde{X}_G) = \sum_{A \in \mathcal{A} : A \subset \tilde{X}_G} \mu^*(A). \qquad (2)$$

Equating the right hand sides of (1) and (2), we have

$$\sum_{A \in \mathcal{A} : A \subset \tilde{X}_G} H(Y_A) = \sum_{A \in \mathcal{A} : A \subset \tilde{X}_G} \mu^*(A).$$

5. Here $\mu^*(A)$ are the unknowns. Evidently, we can make the above equality hold for all nonempty subsets G of $\mathcal{N}_n$ by taking

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for all $A \in \mathcal{A}$. By the uniqueness of μ^* (Theorem 3.9), this is also the only possibility for μ^*.

6. Since $H(Y_A)$ can take any nonnegative value by Corollary 2.44, μ^* can take any set of nonnegative values on the nonempty atoms of $\mathcal{F}_n$. The theorem is proved.

Theorem 3.9 μ^* is the unique signed measure on $\mathcal{F}_n$ which is consistent with all Shannon's information measures.

Theorem 3.11 If there is no constraint on $X_1, X_2, \cdots, X_n$, then μ^* can take any set of nonnegative values on the nonempty atoms of $\mathcal{F}_n$.

Proof

1. Let $Y_A, A \in \mathcal{A}$ be mutually independent random variables. Note that these r.v.'s are labeled by the nonempty atoms of $\mathcal{F}_n$.

2. Define the random variables $X_i, i = 1, 2, \cdots, n$ by

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Theorem 3.9 μ^* is the unique signed measure on $\mathcal{F}_n$ which is consistent with all Shannon's information measures.

Corollary 2.44 The entropy of a random variable may take any nonnegative real value.

Information Diagrams for Markov Chains

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- For $n = 3$, $X_1 \to X_2 \to X_3$ iff $I(X_1; X_3|X_2) = 0$, or $\mu^*(\tilde{X}_1 \cap \tilde{X}_3 - \tilde{X}_2) = 0$.

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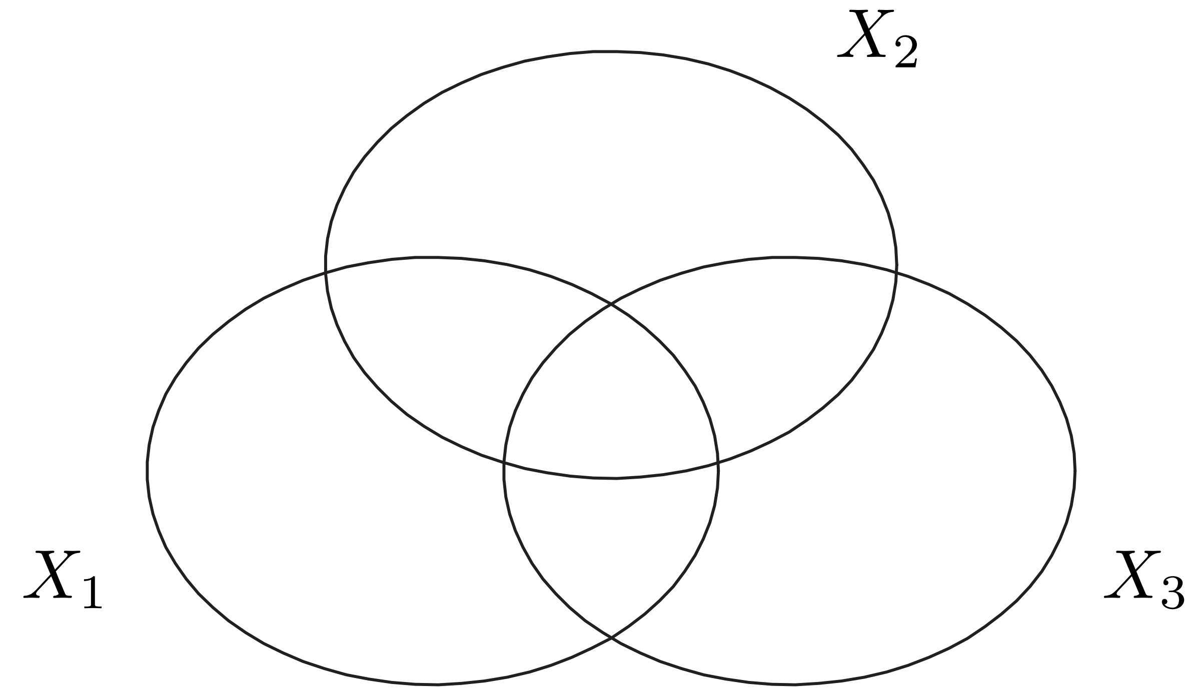
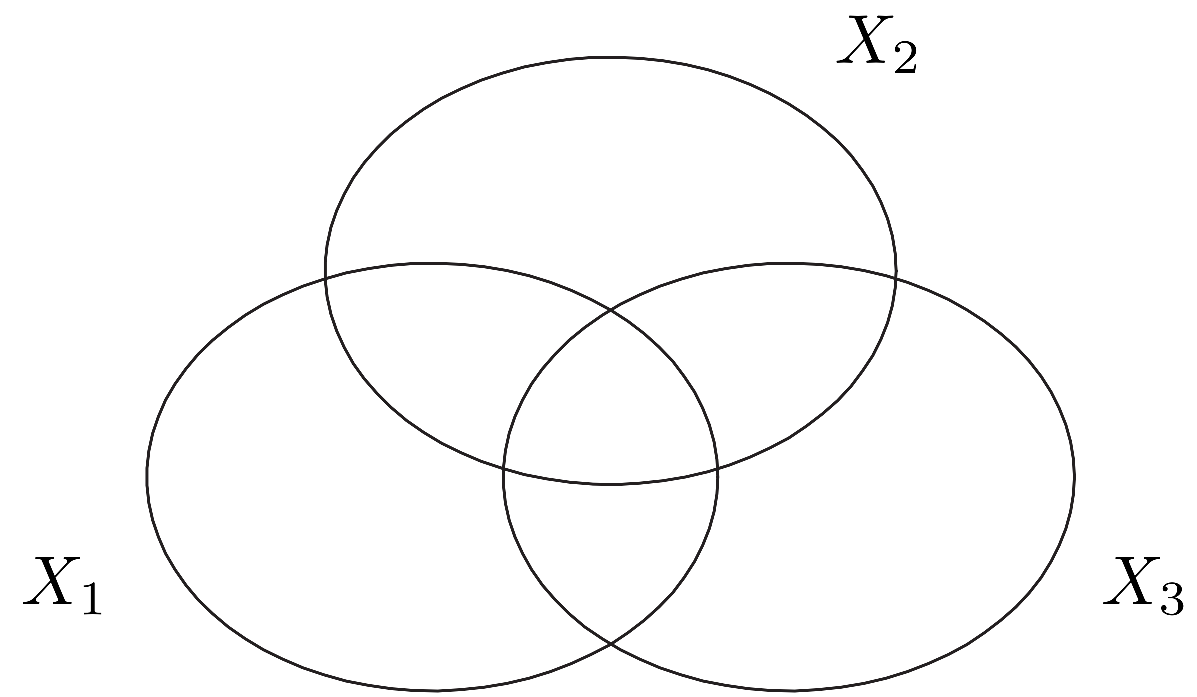
- So the atom $\tilde{X}_1 \cap \tilde{X}_3 - \tilde{X}_2$ can be suppressed in the information diagram.


X_2
X_1
X_3

- Suppose $I(X_1; X_3|X_2) = 0$.



- Suppose $I(X_1; X_3|X_2) = 0$.
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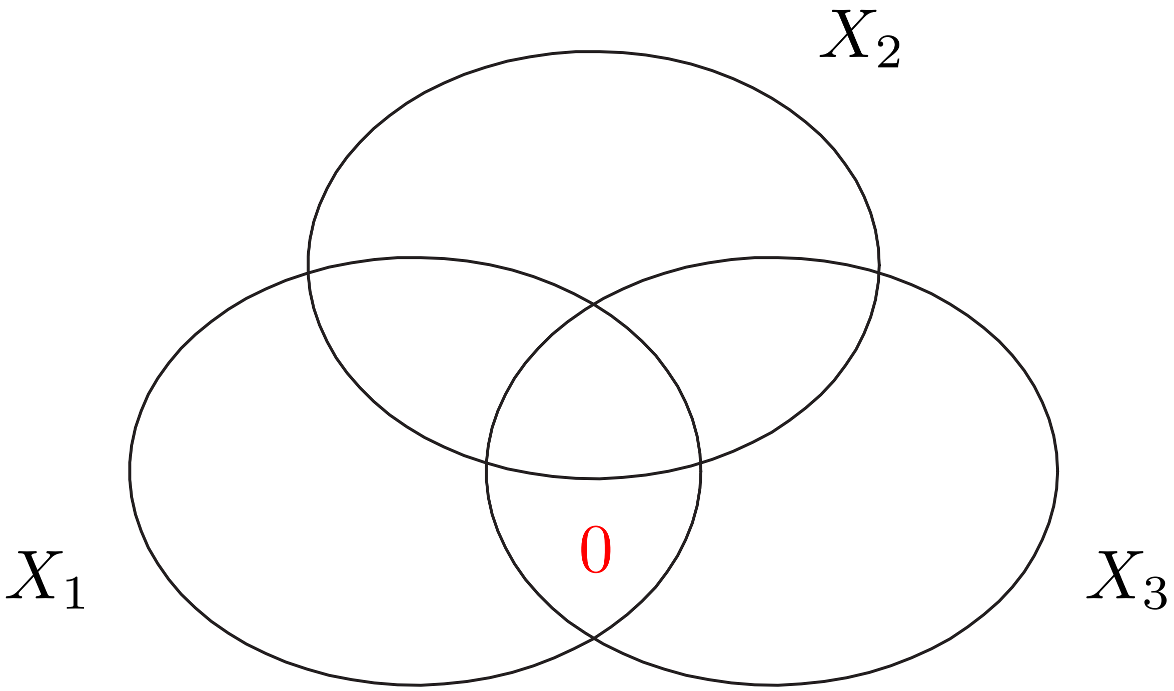

- Suppose $I(X_1; X_3|X_2) = 0$.
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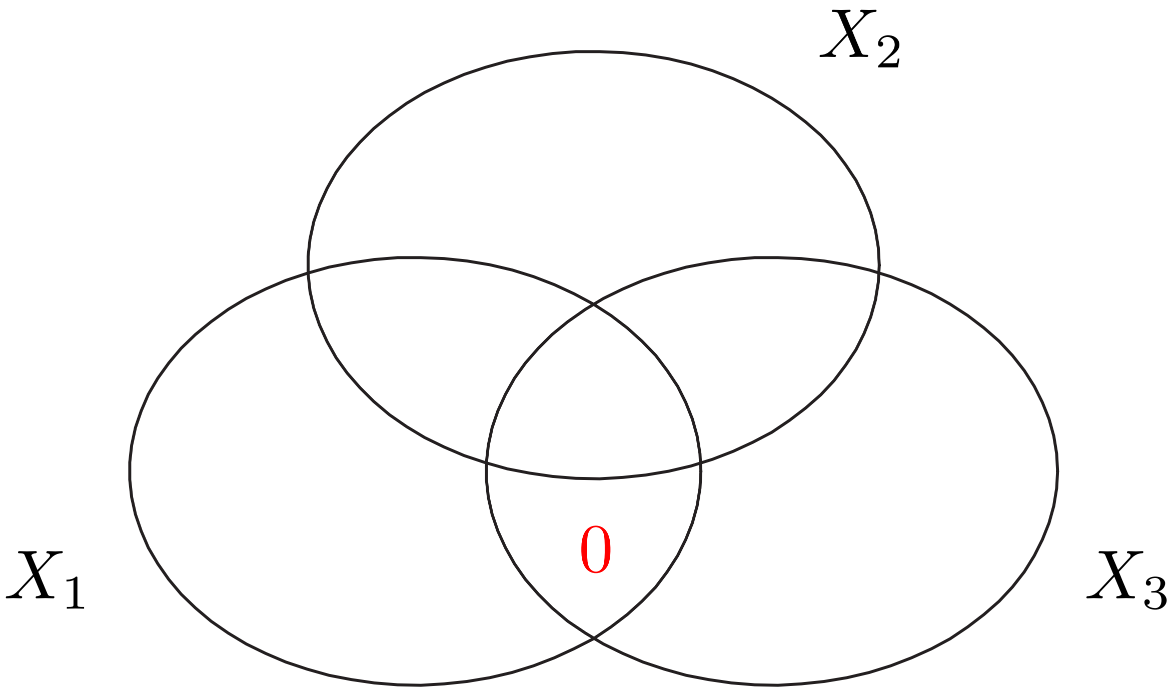

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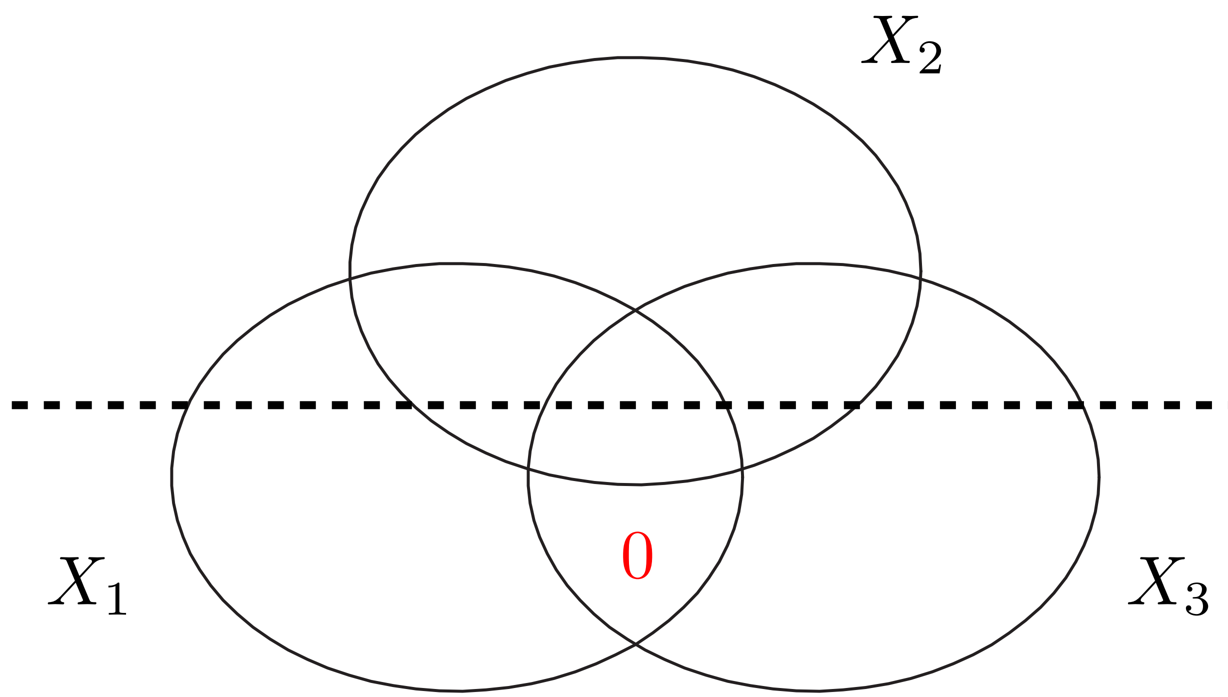

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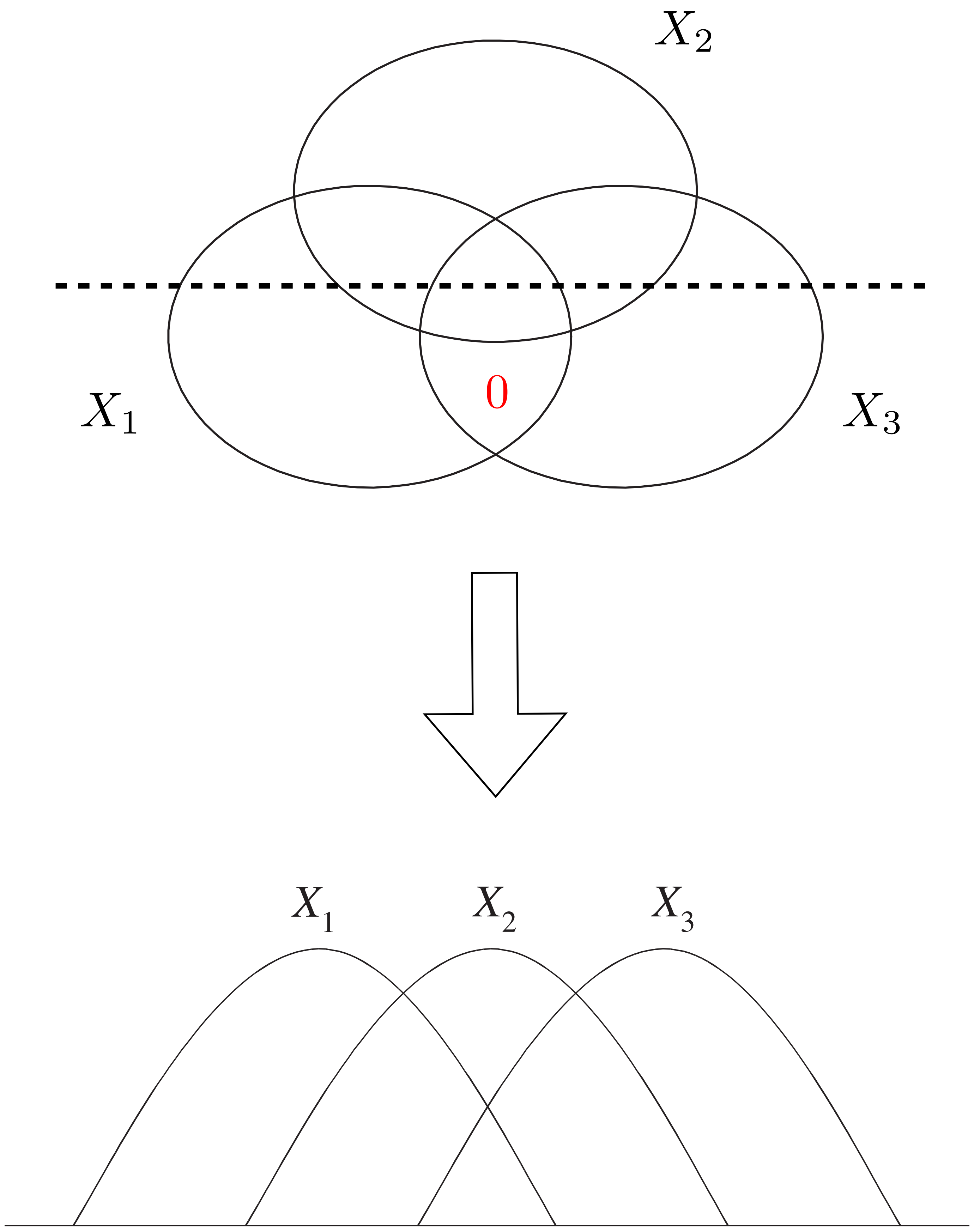
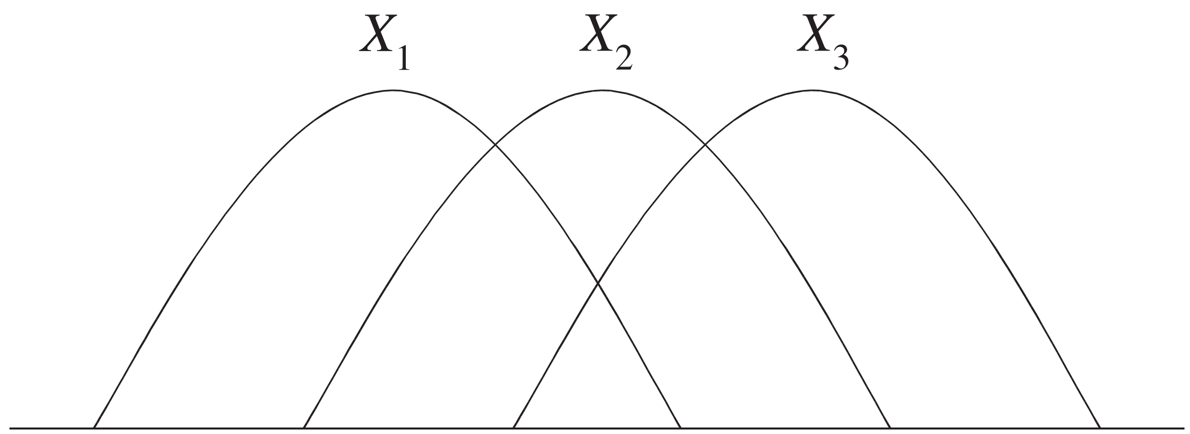


μ^* **for** $X_1 \rightarrow X_2 \rightarrow X_3$



μ^* for $X_1 \to X_2 \to X_3$

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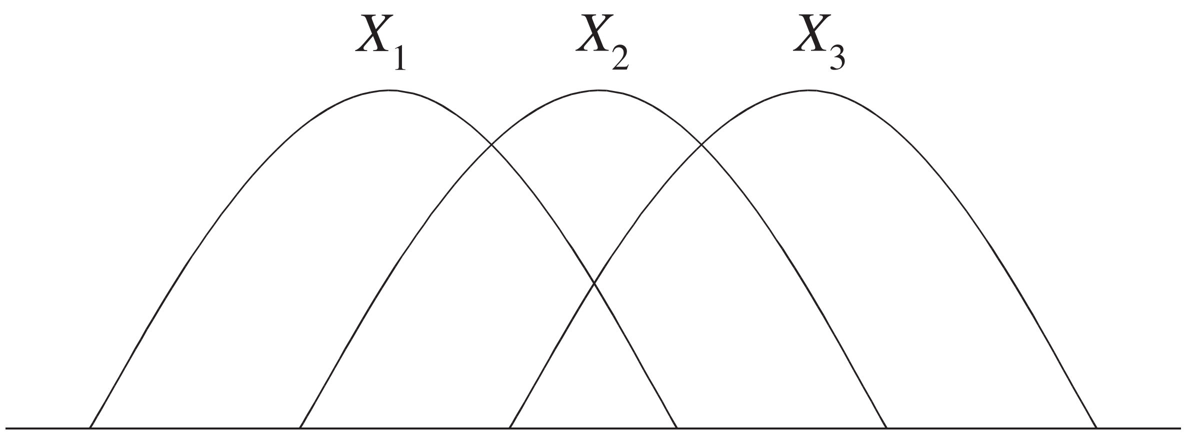

μ^* for $X_1 \to X_2 \to X_3$

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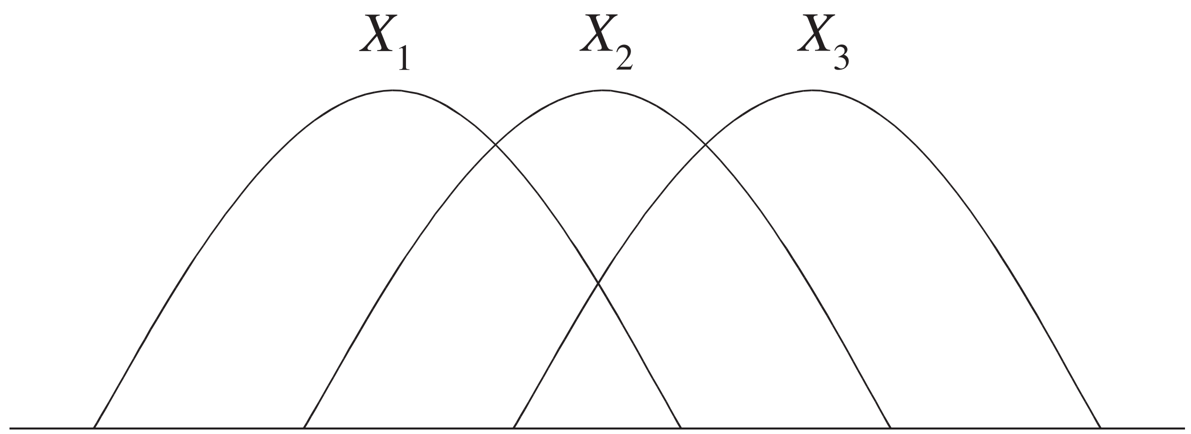
$$I(X_1;X_3|X_2) = \mu^*(\tilde{X}_1 \cap \tilde{X}_3 - \tilde{X}_2)$$



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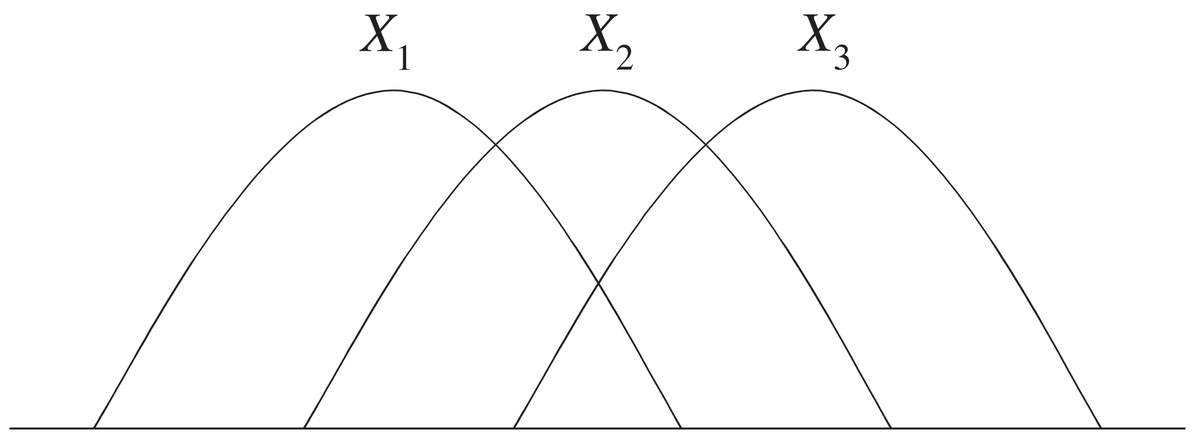

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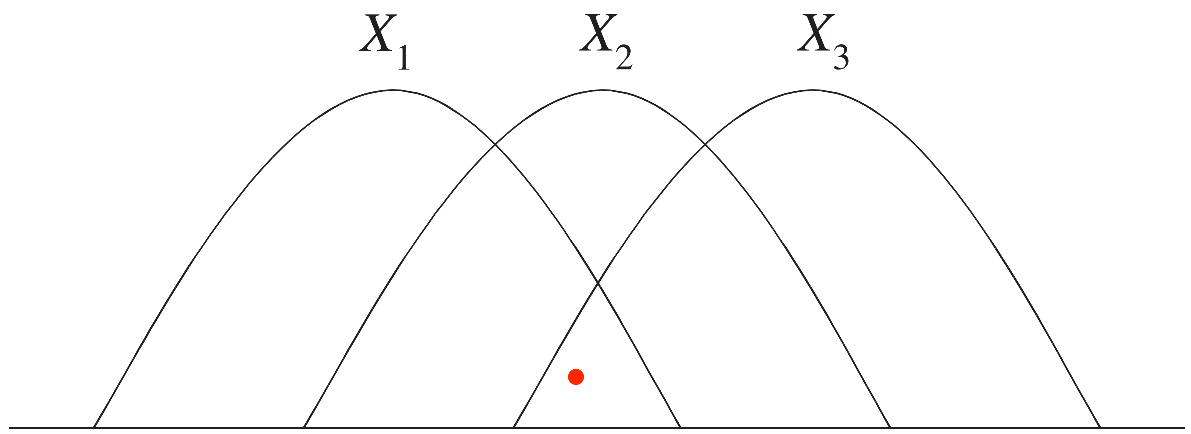

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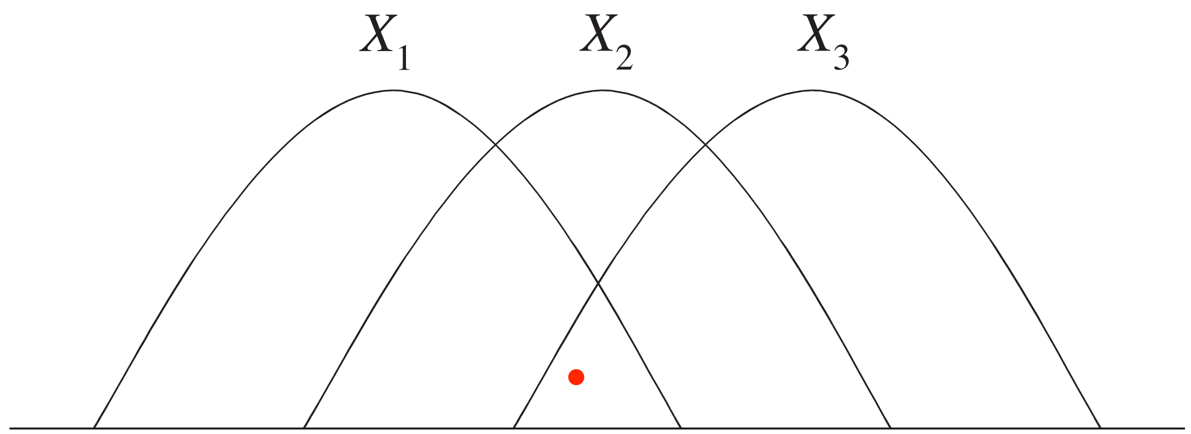

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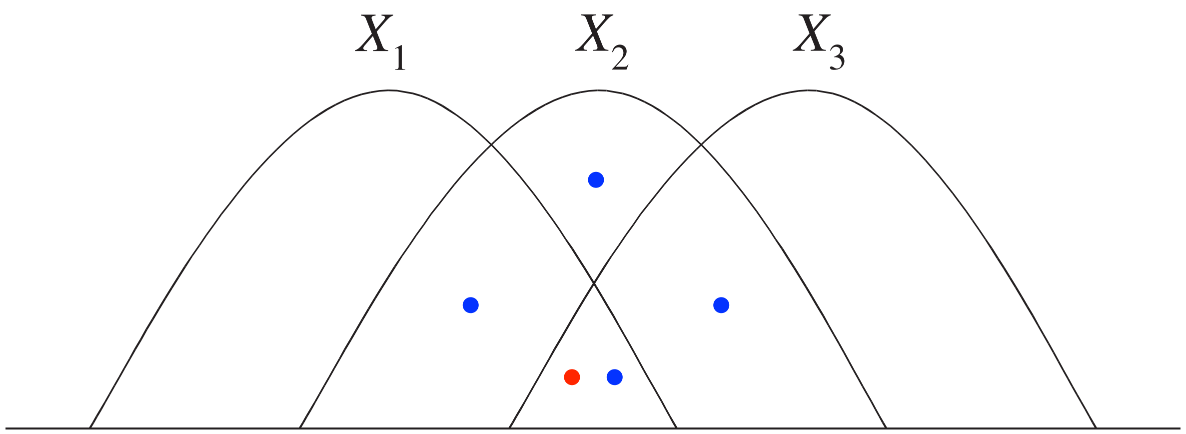
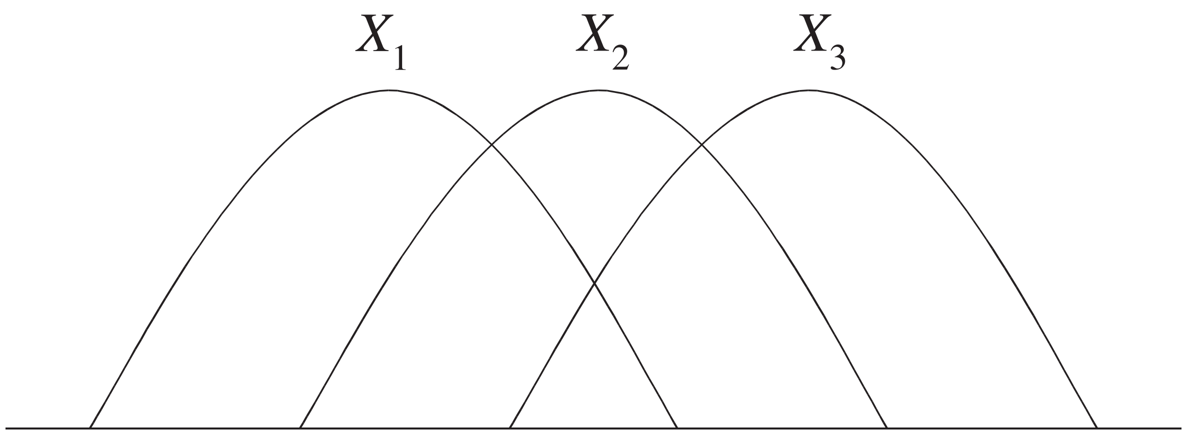


μ^* for $X_1 \to X_2 \to X_3$



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I(X_1; X_3 | X_2) &= \mu^*(\tilde{X}_1 \cap \tilde{X}_3 - \tilde{X}_2) \\
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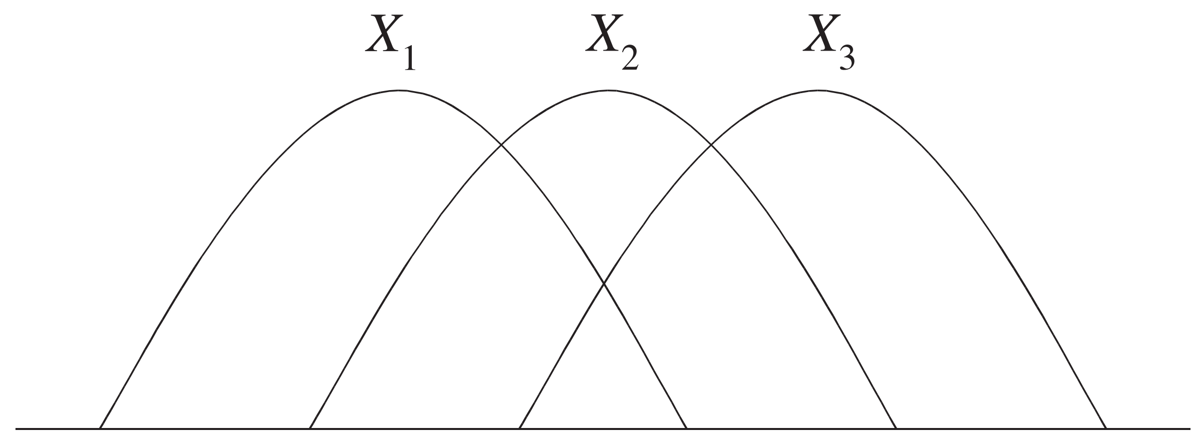

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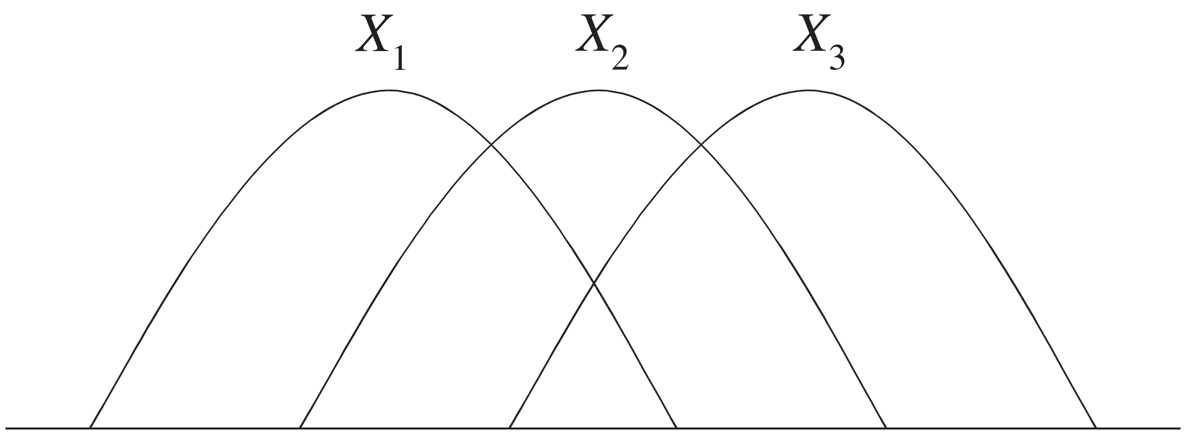

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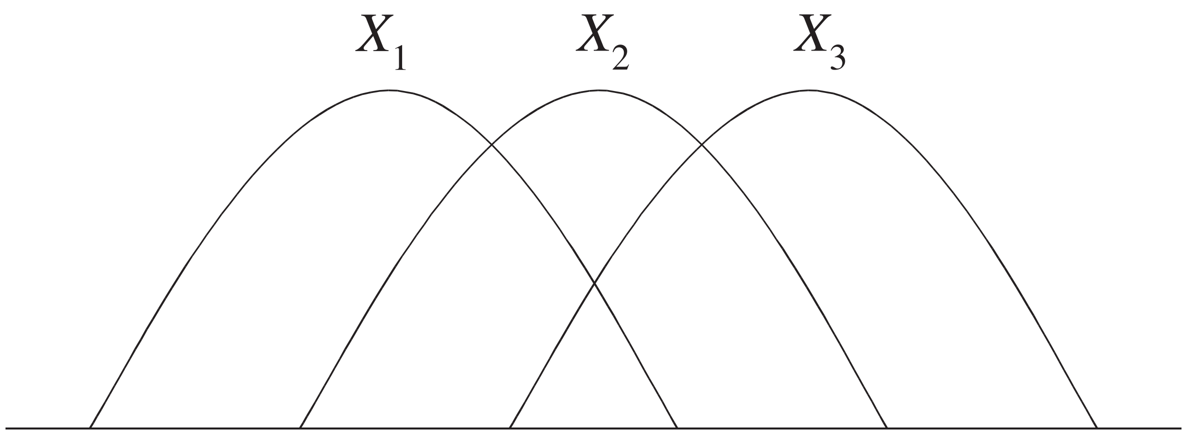

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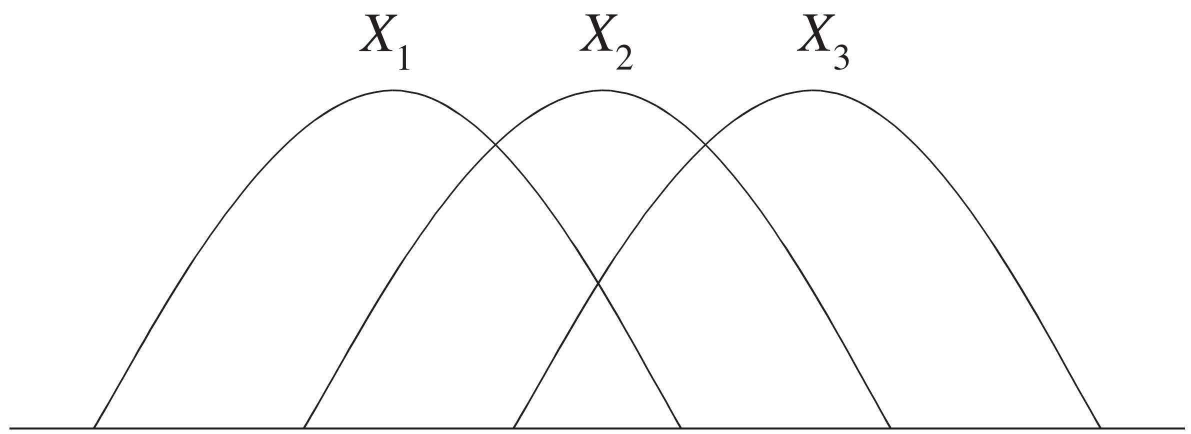

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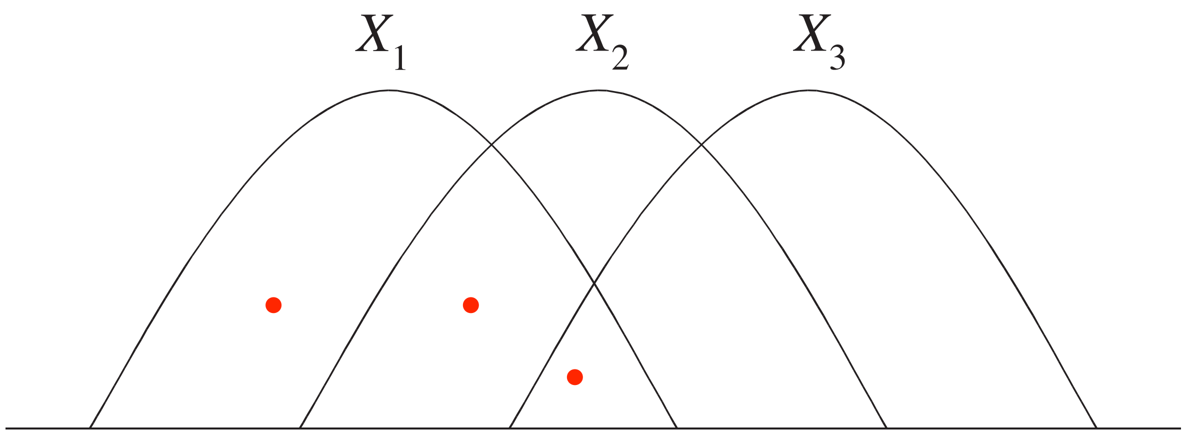

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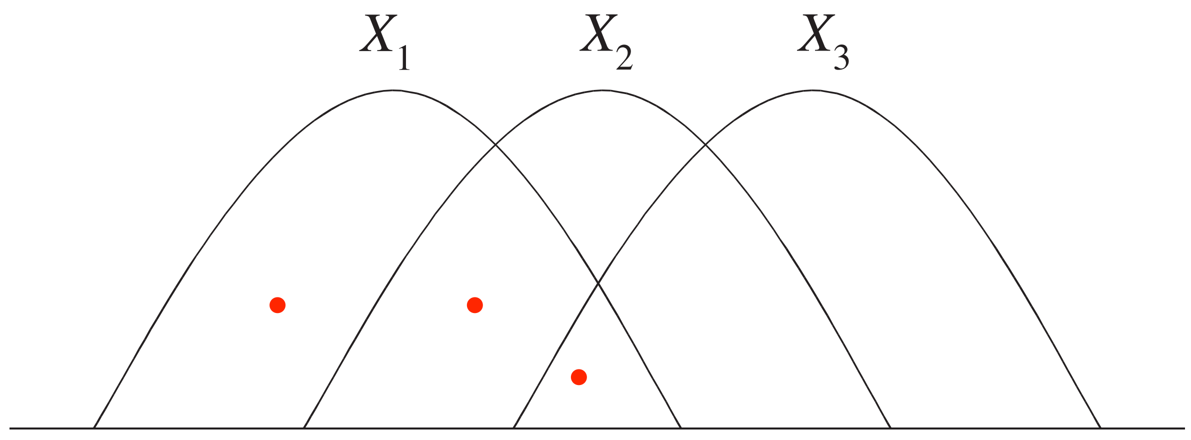

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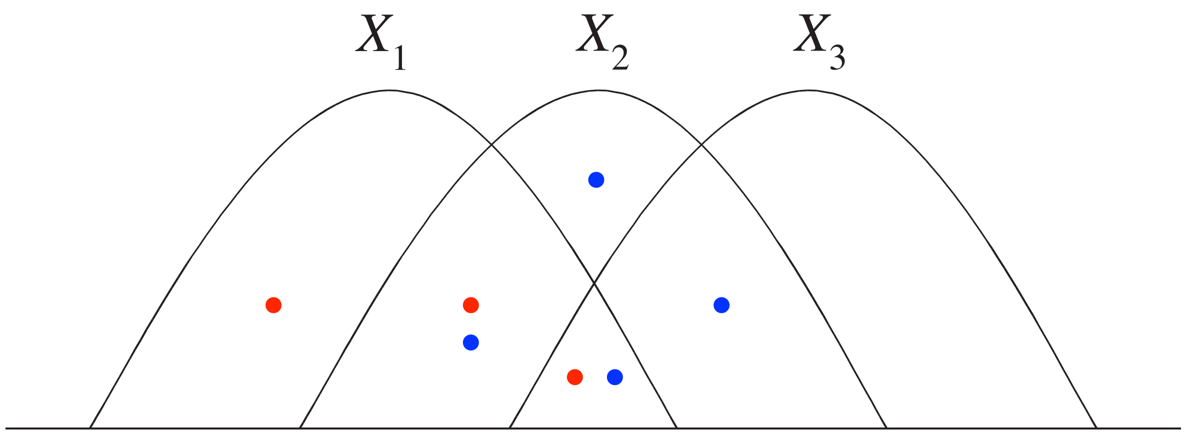

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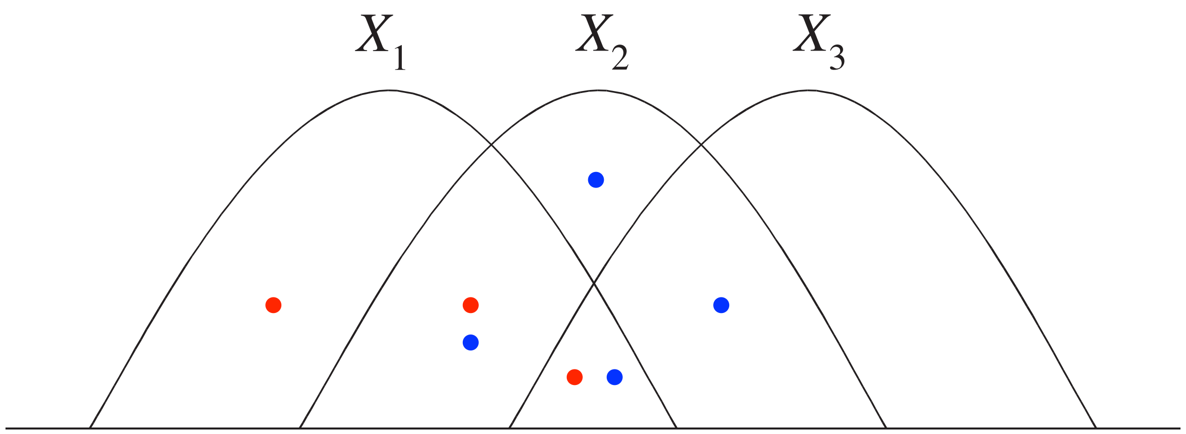

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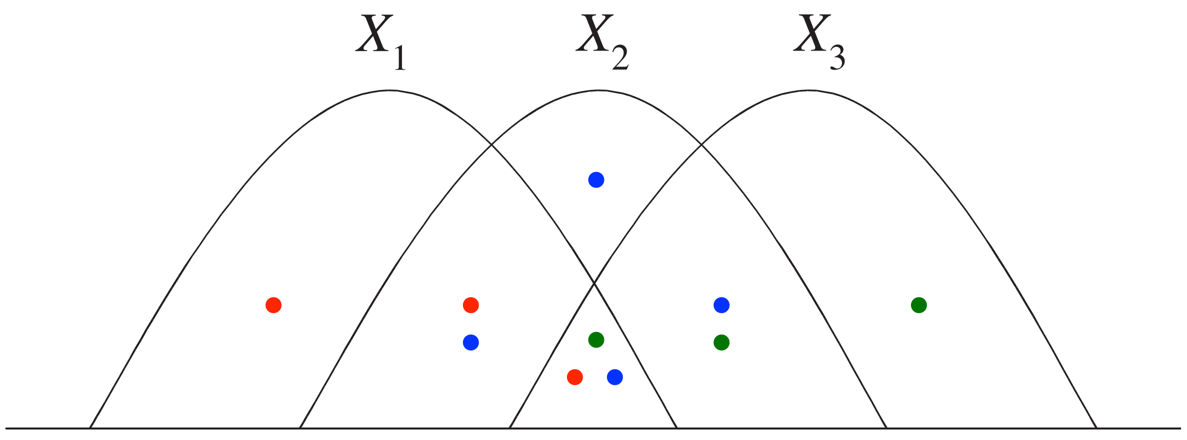

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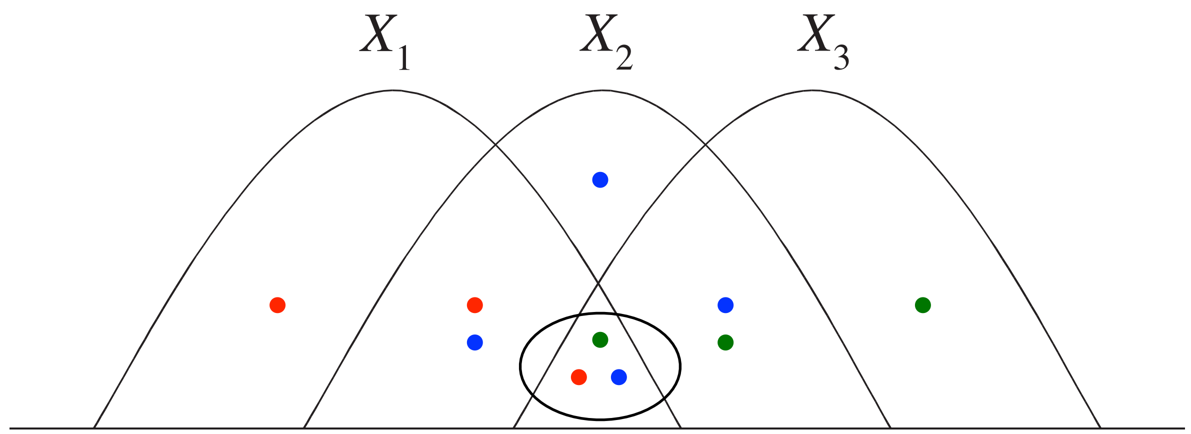

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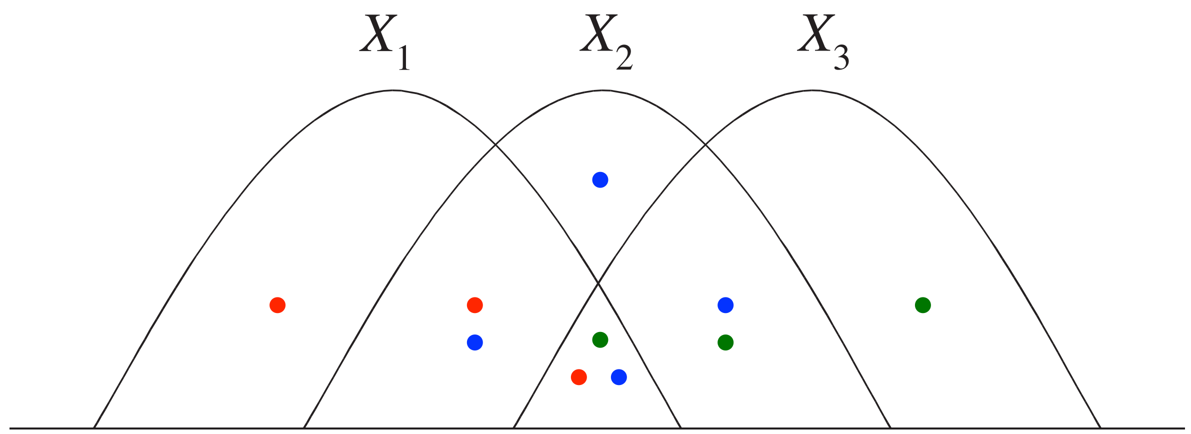

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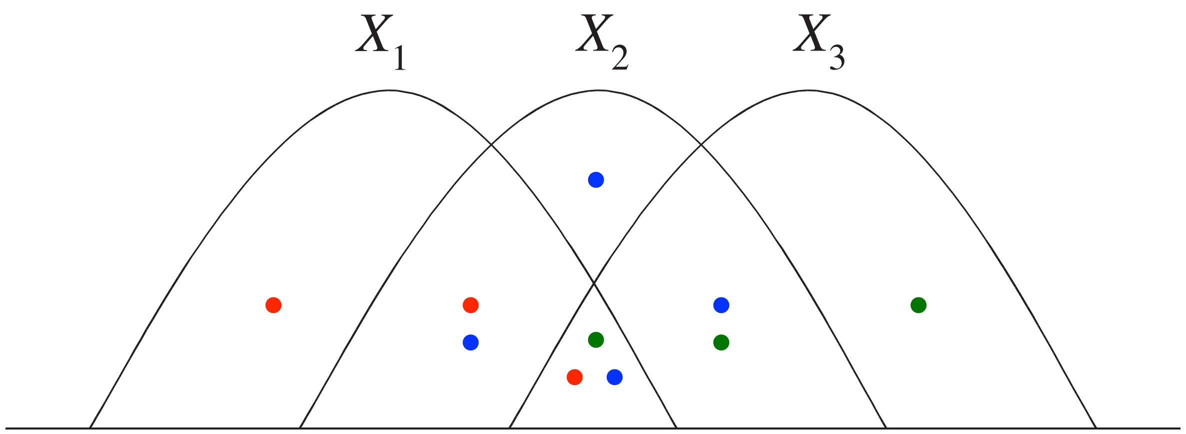

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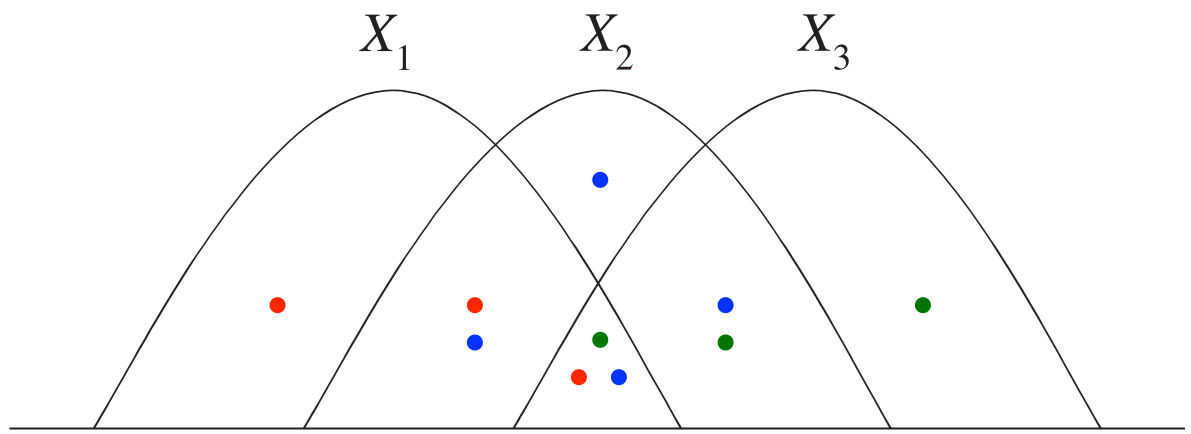

μ^* for $X_1 \to X_2 \to X_3$

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2. Also,

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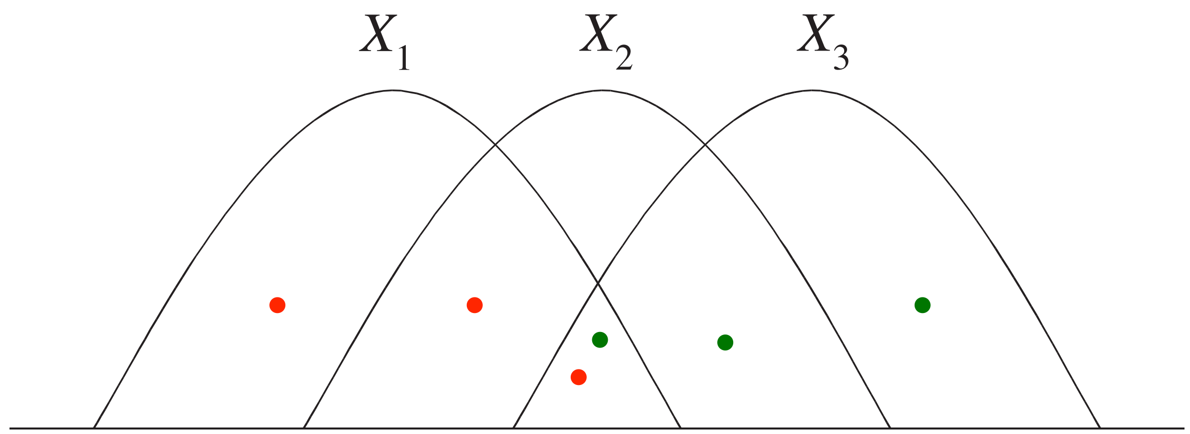

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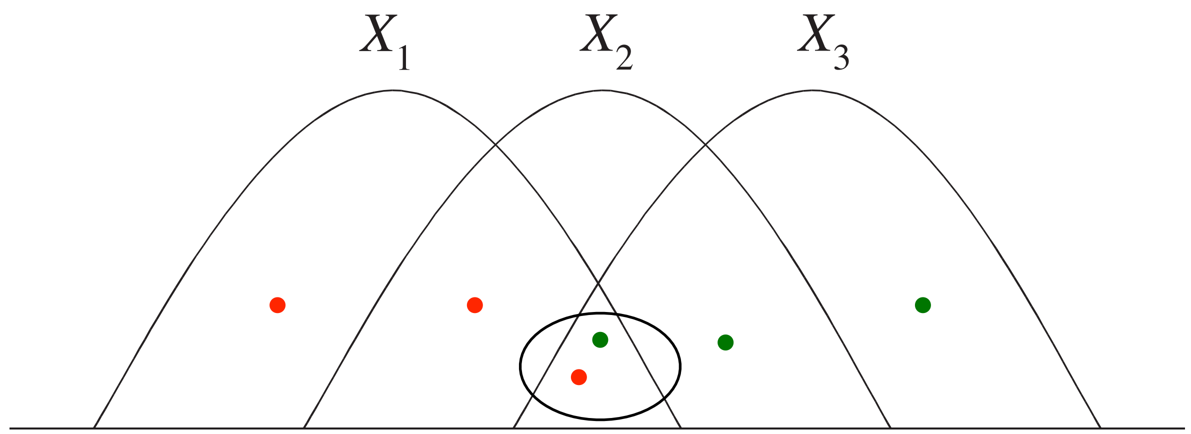

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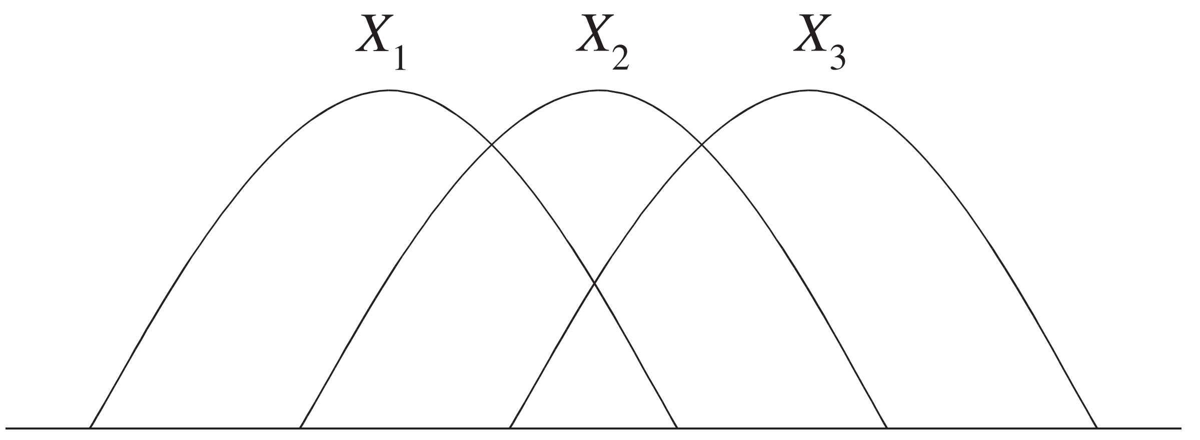

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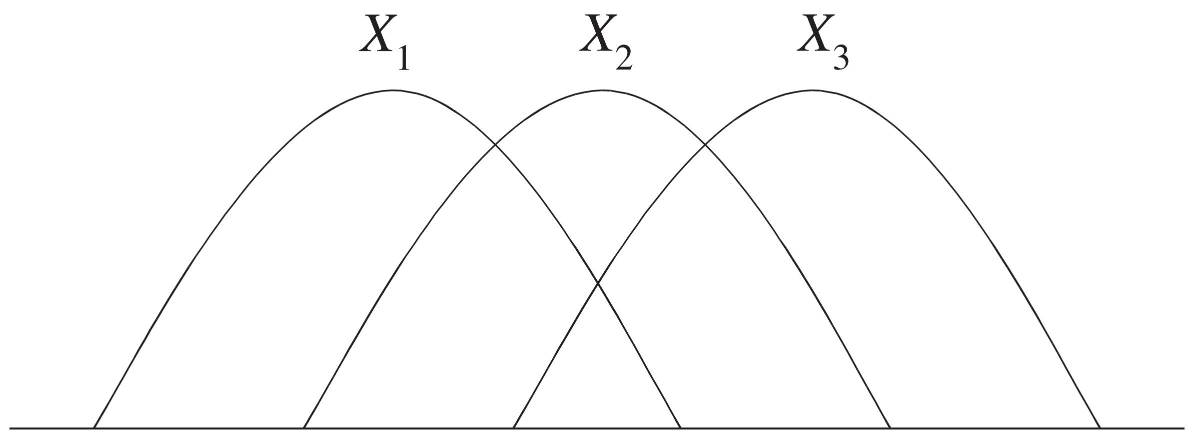
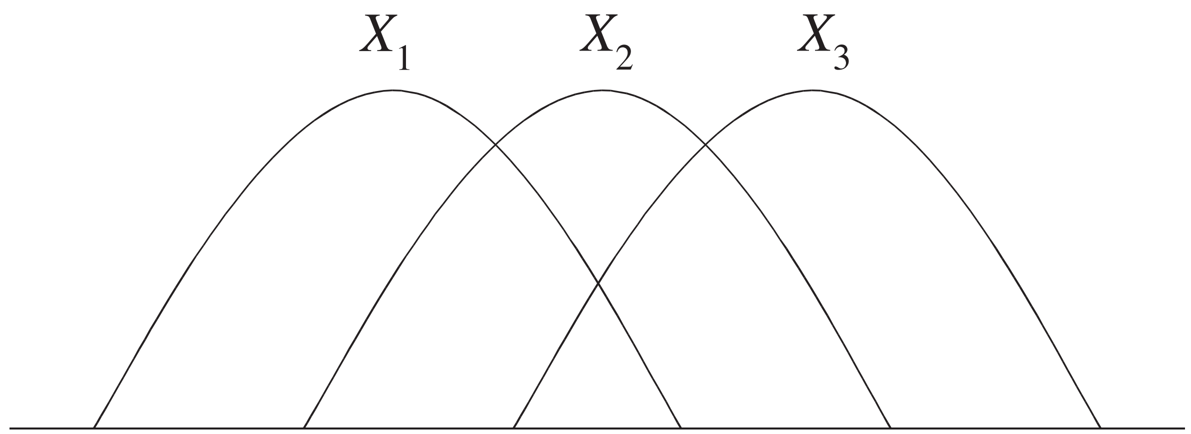
2. Also,

$$\begin{aligned} \mu^*(\tilde{X}_1 \cap \tilde{X}_2 \cap \tilde{X}_3) &= \mu^*(\tilde{X}_1 \cap \tilde{X}_3) \\ &= I(X_1;X_3) \\ &\geq 0. \end{aligned}$$



μ^* for $X_1 \to X_2 \to X_3$



1. In this information diagram,

$$\begin{aligned} I(X_1; X_3|X_2) &= \mu^*(\tilde{X}_1 \cap \tilde{X}_3 - \tilde{X}_2) \\ &= \mu^*(\emptyset) \\ &= 0. \end{aligned}$$

2. Also,

$$\begin{aligned} \mu^*(\tilde{X}_1 \cap \tilde{X}_2 \cap \tilde{X}_3) &= \mu^*(\tilde{X}_1 \cap \tilde{X}_3) \\ &= I(X_1; X_3) \\ &\geq 0. \end{aligned}$$

3. Since the values of μ^* on all the remaining atoms correspond to Shannon's information measures and hence are nonnegative, we conclude that μ^* is a measure.

- For $X_1 \to X_2 \to X_3 \to X_4$, μ^* vanishes on the following 5 atoms:

$$\tilde{X}_1 \cap \tilde{X}_2^c \cap \tilde{X}_3 \cap \tilde{X}_4^c$$
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- The information diagram can be displayed in two dimensions.

- For $X_1 \to X_2 \to X_3 \to X_4$, μ^* vanishes on the following 5 atoms:

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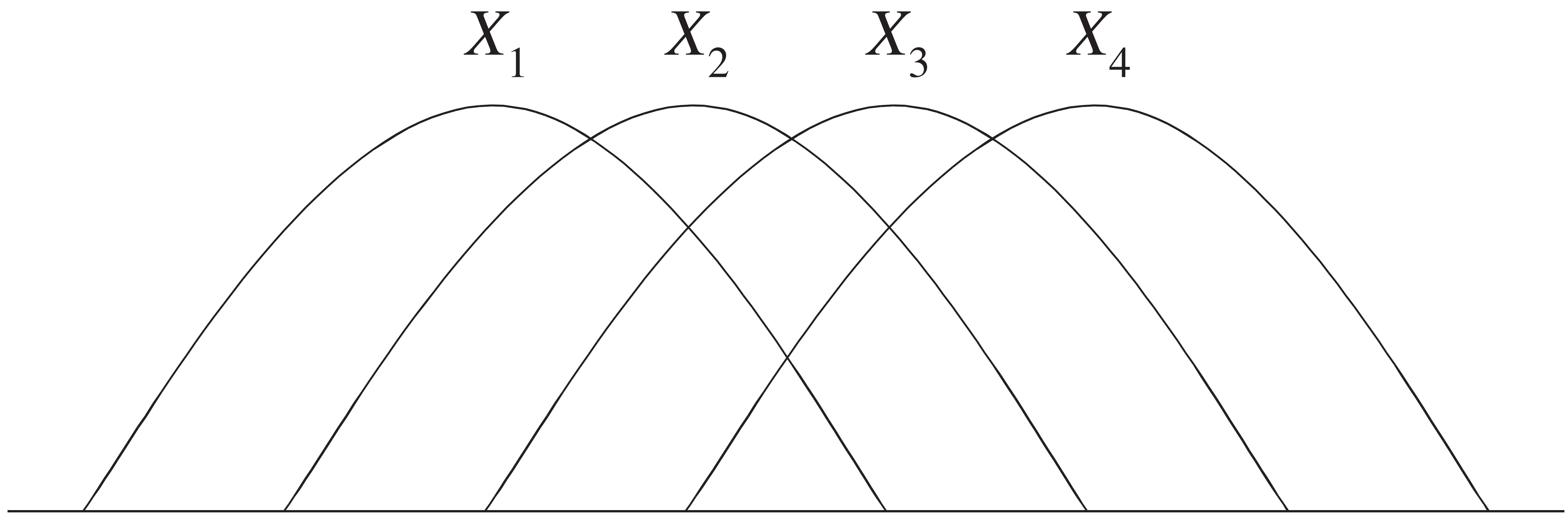
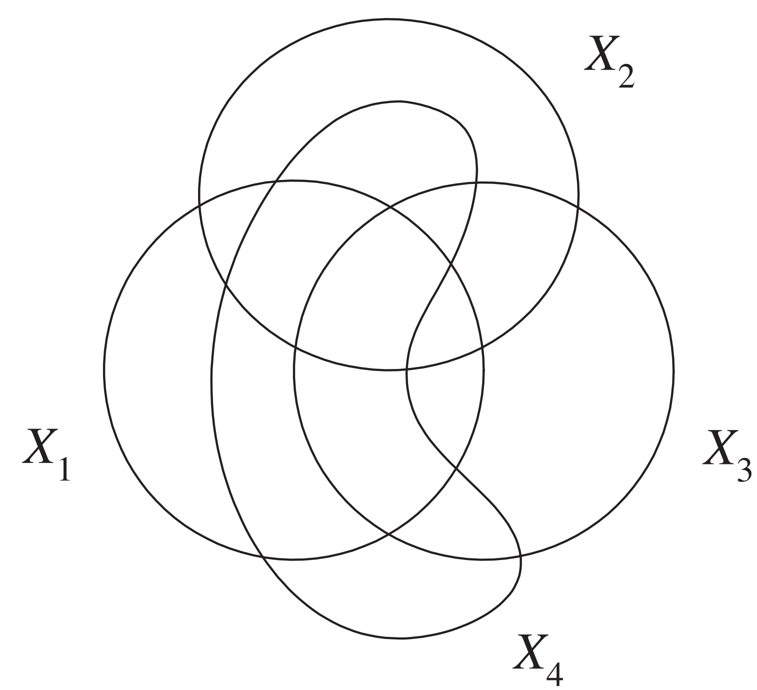


X1
X2
X3
X4

Structure of μ^* for $X_1 \to X_2 \to X_3 \to X_4$



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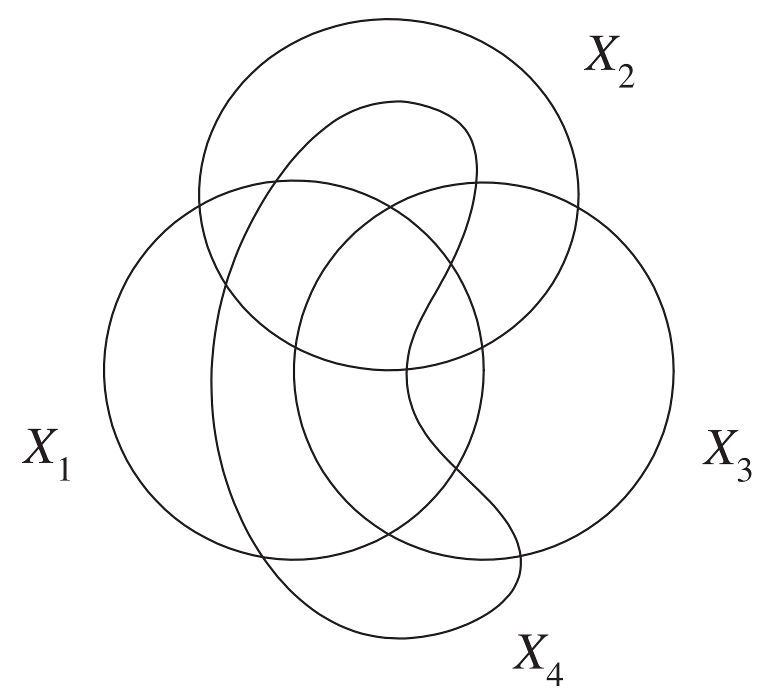

Structure of μ^* for $X_1 \to X_2 \to X_3 \to X_4$

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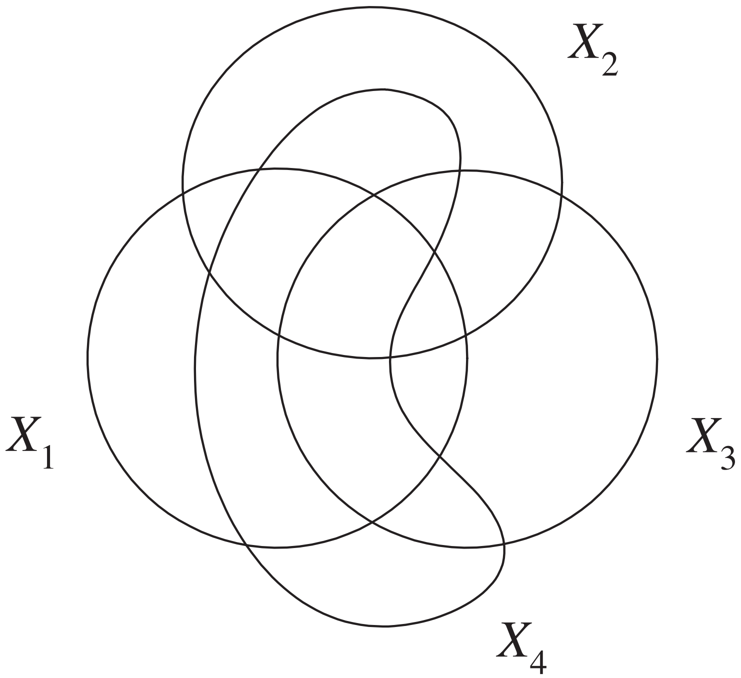

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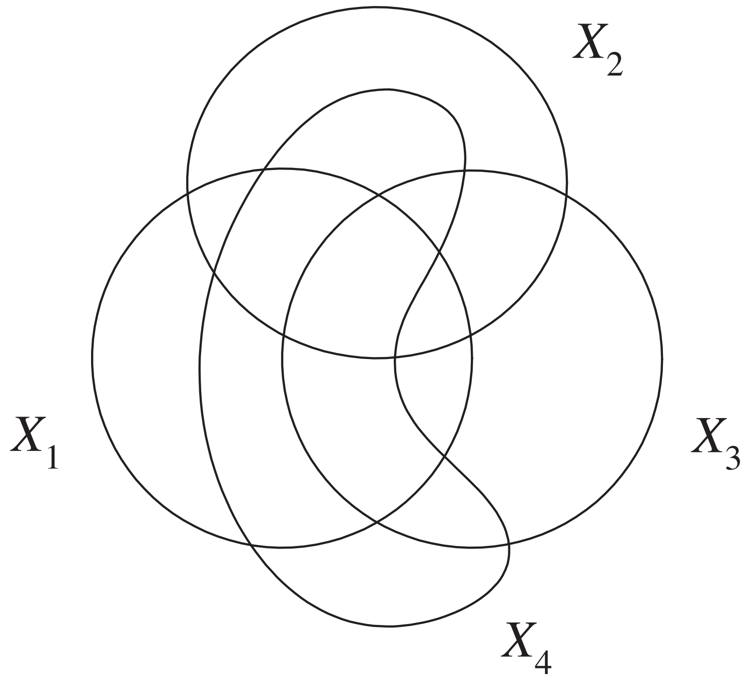

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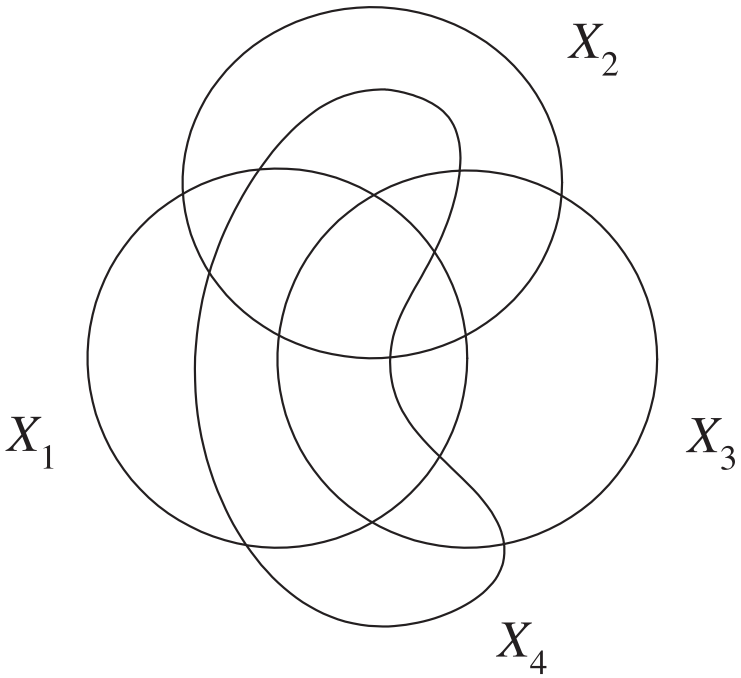

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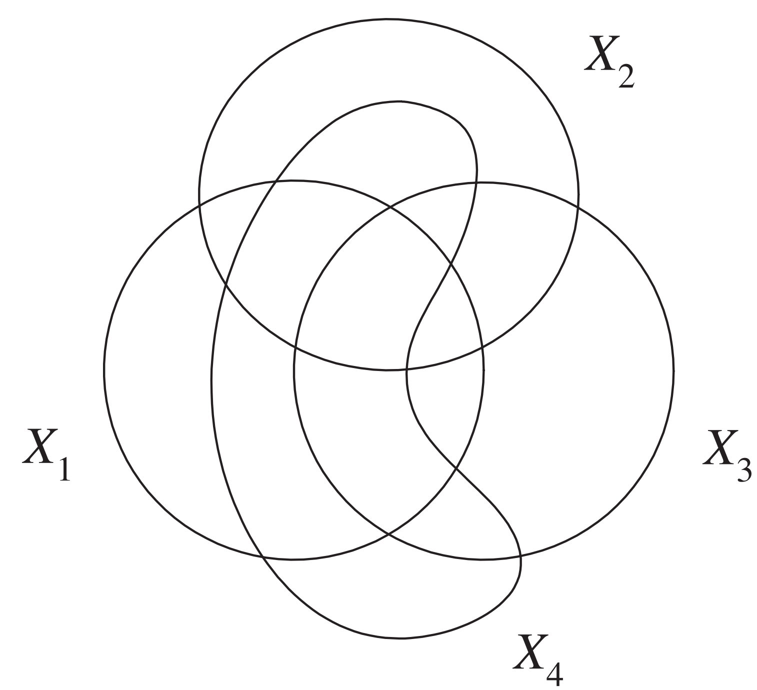

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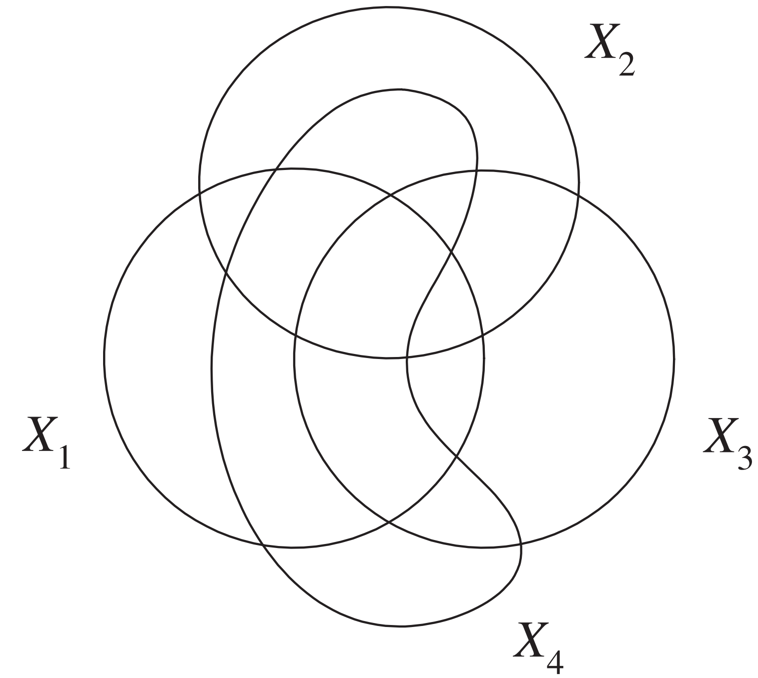

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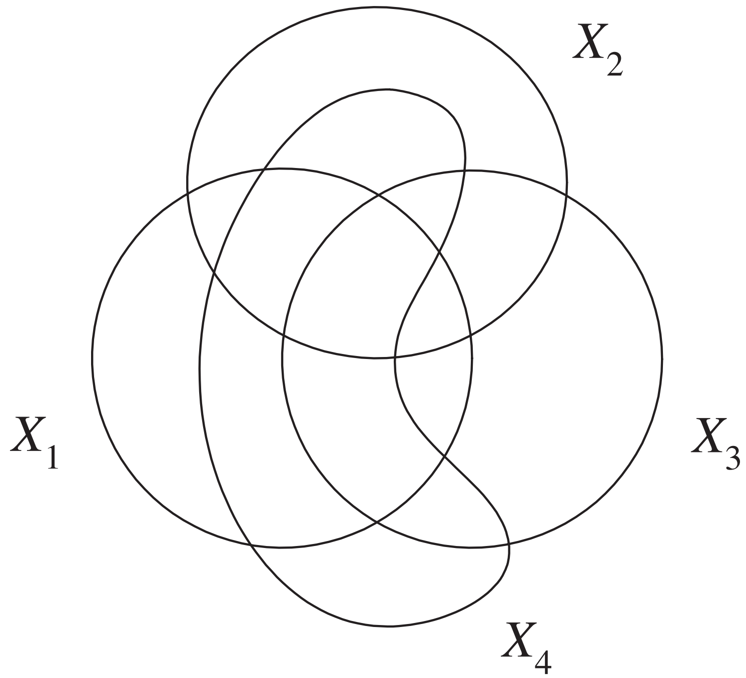

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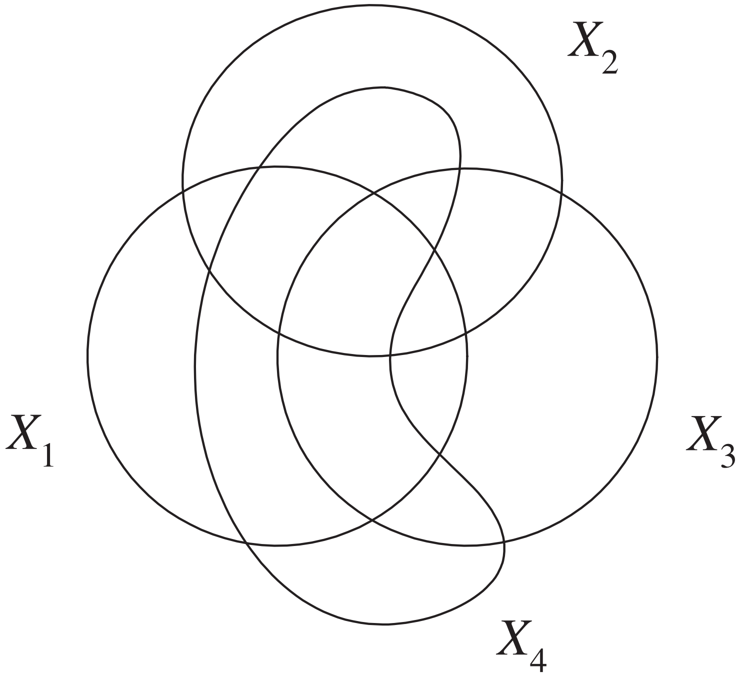

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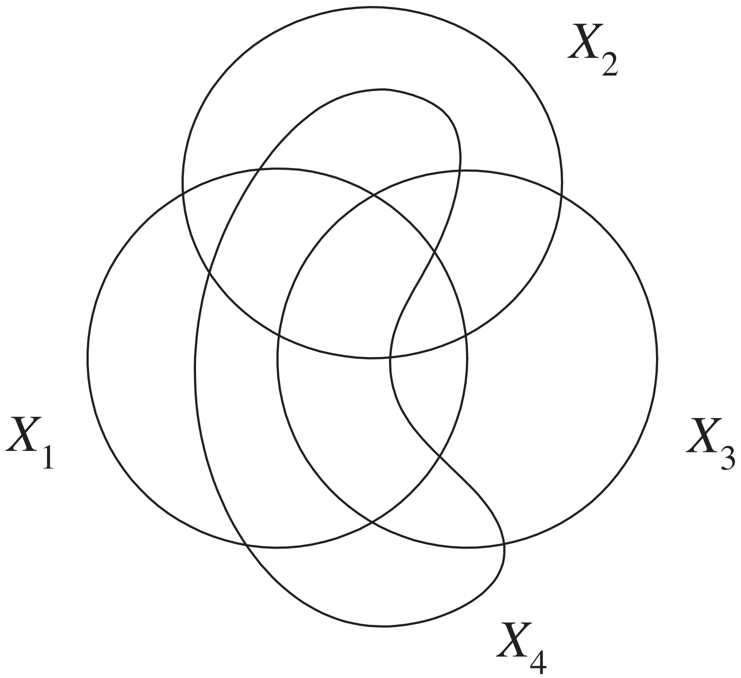

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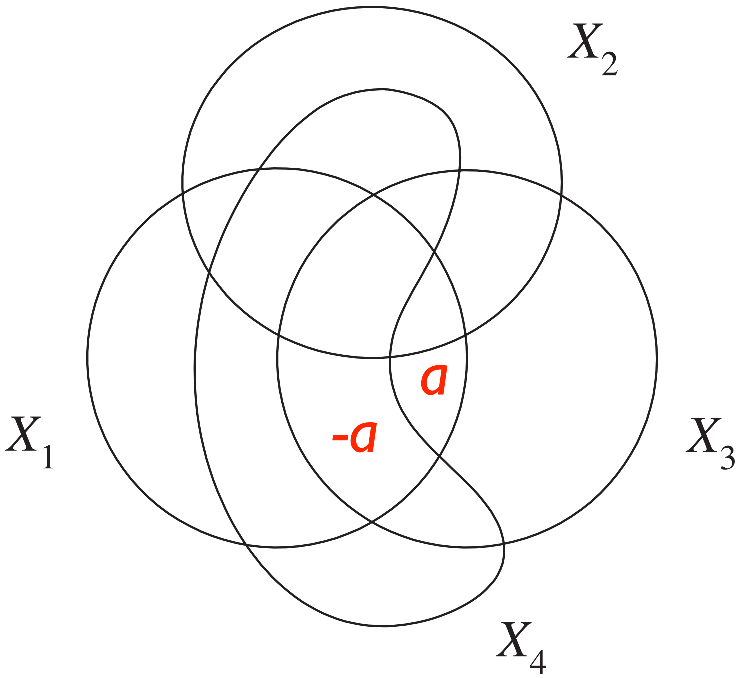

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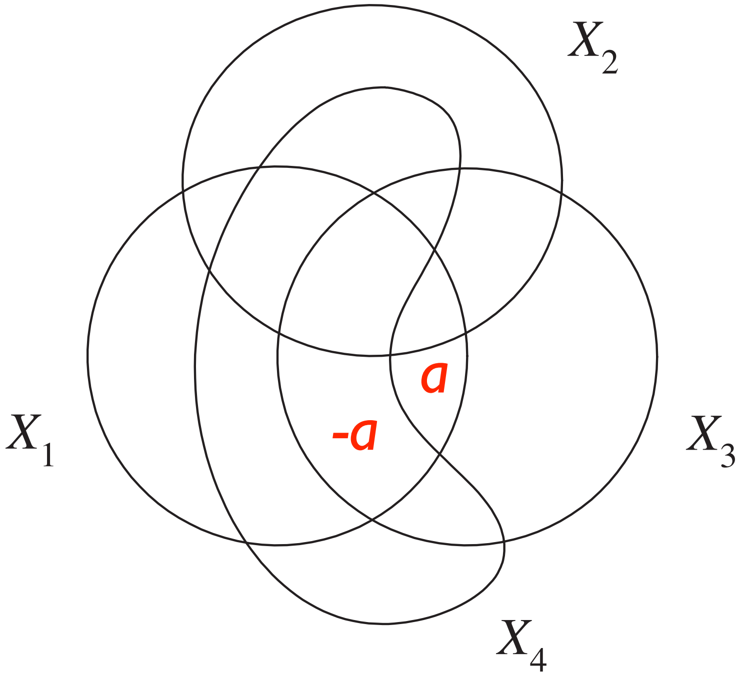

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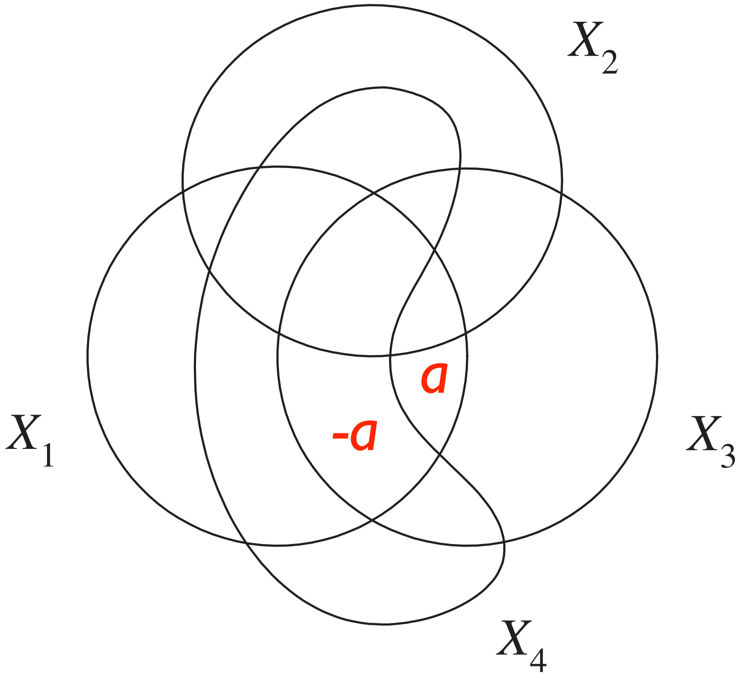

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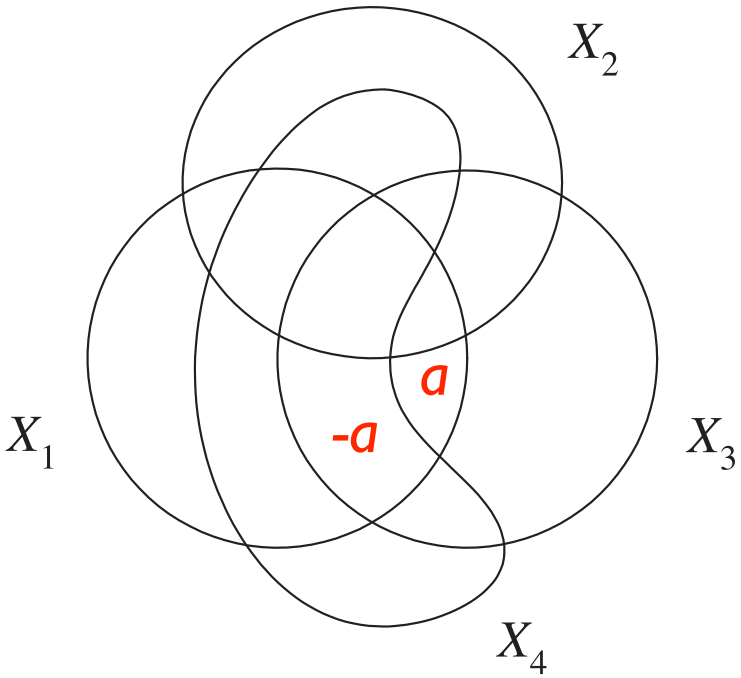

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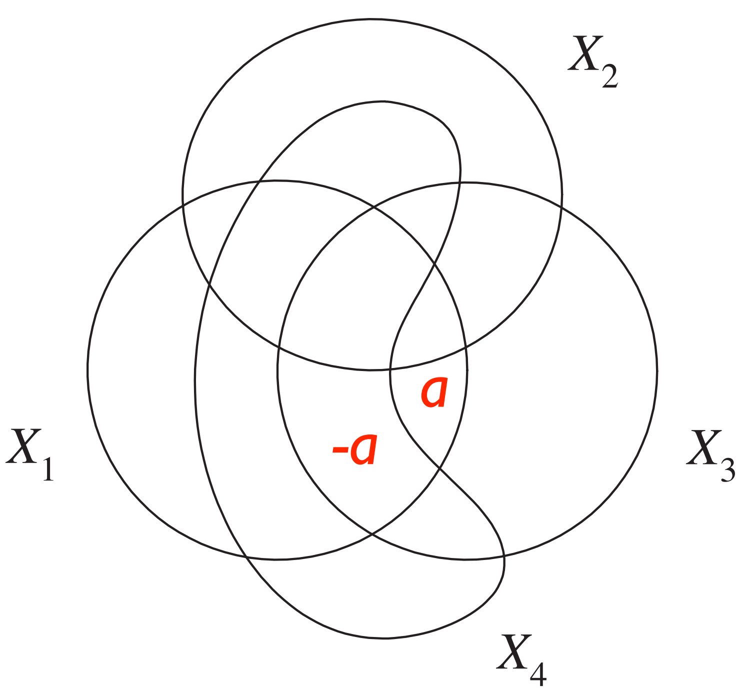

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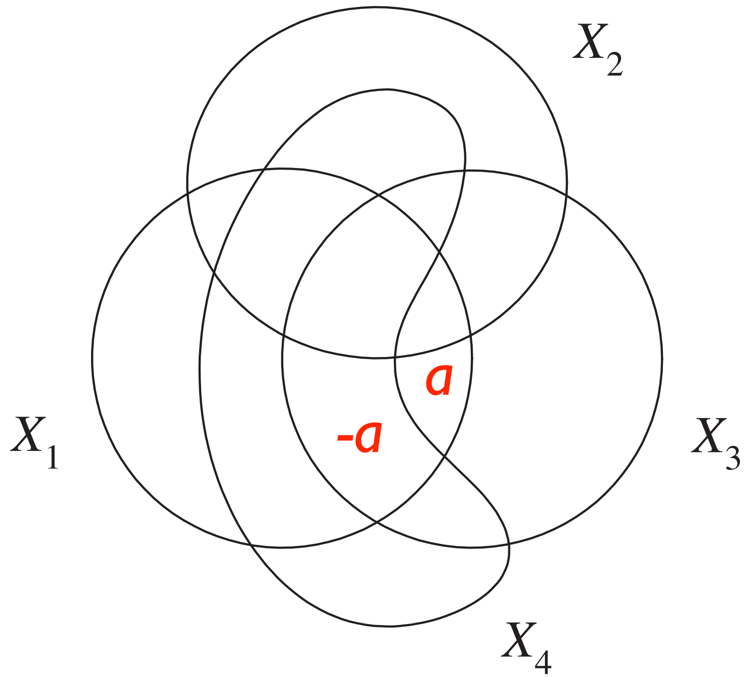

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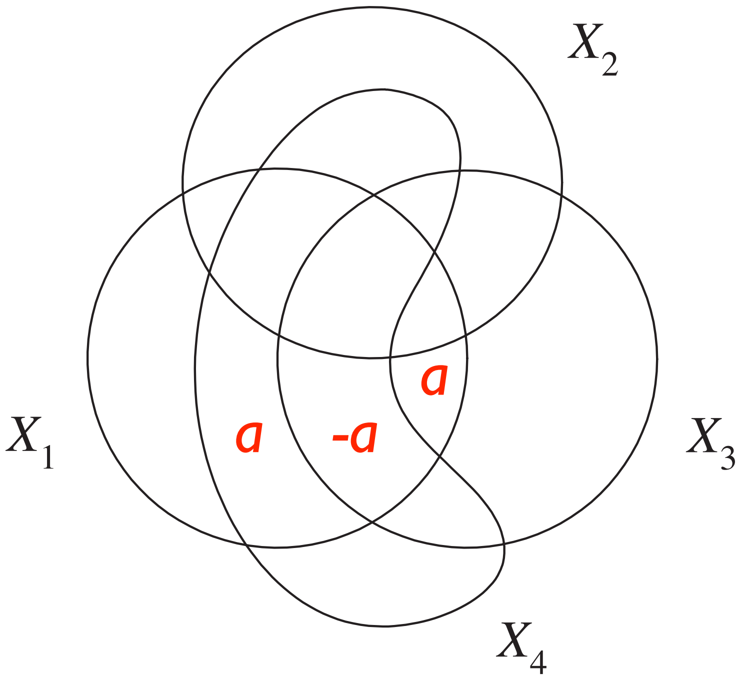

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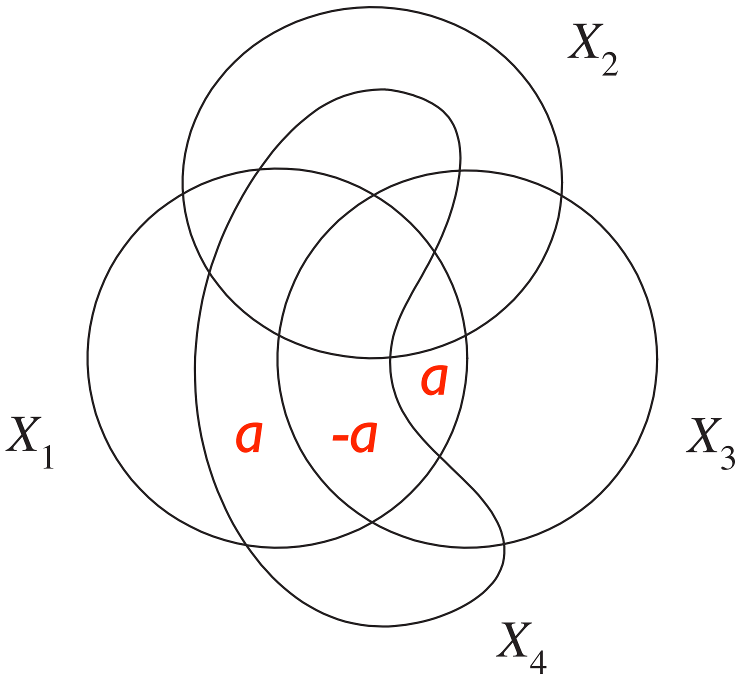

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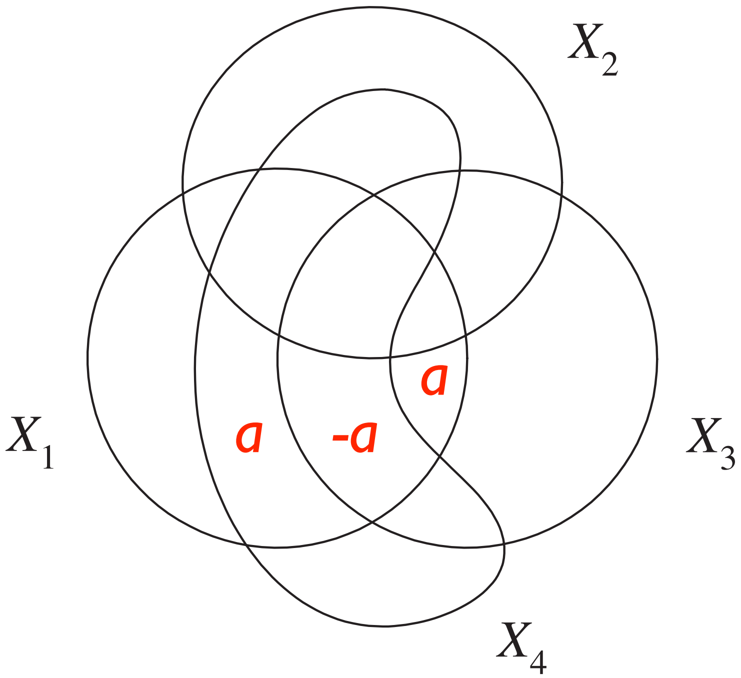

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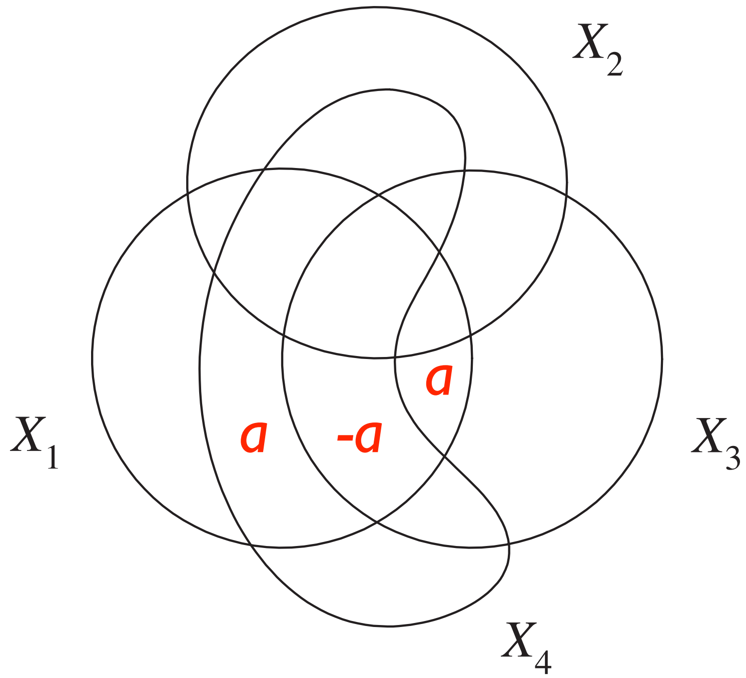

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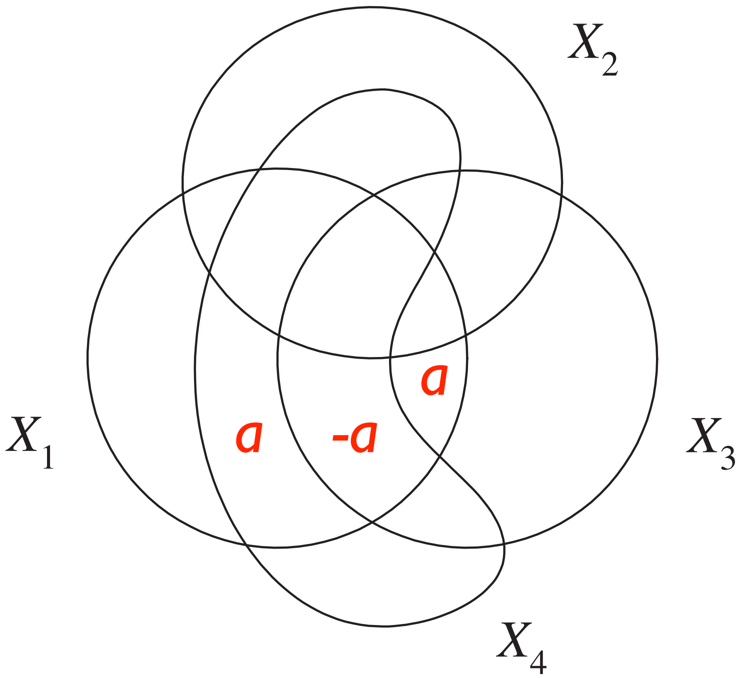

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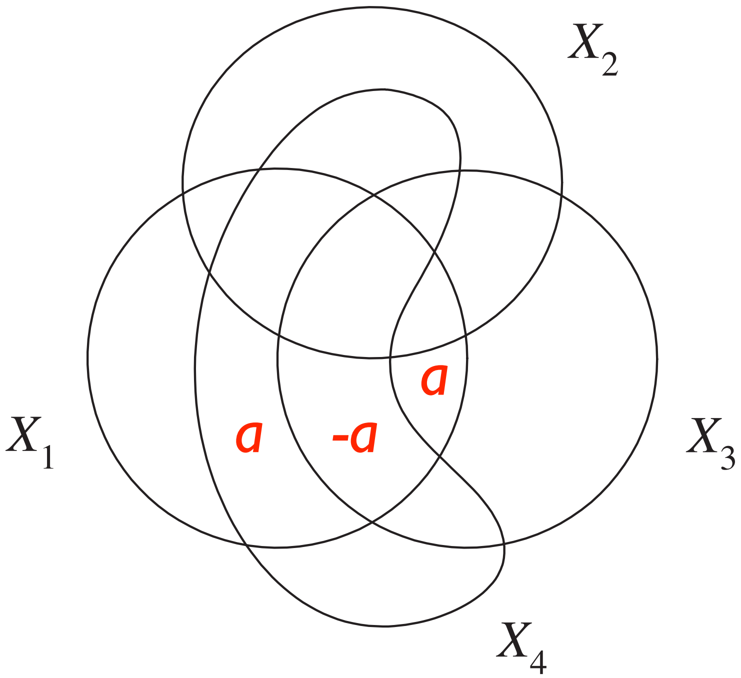

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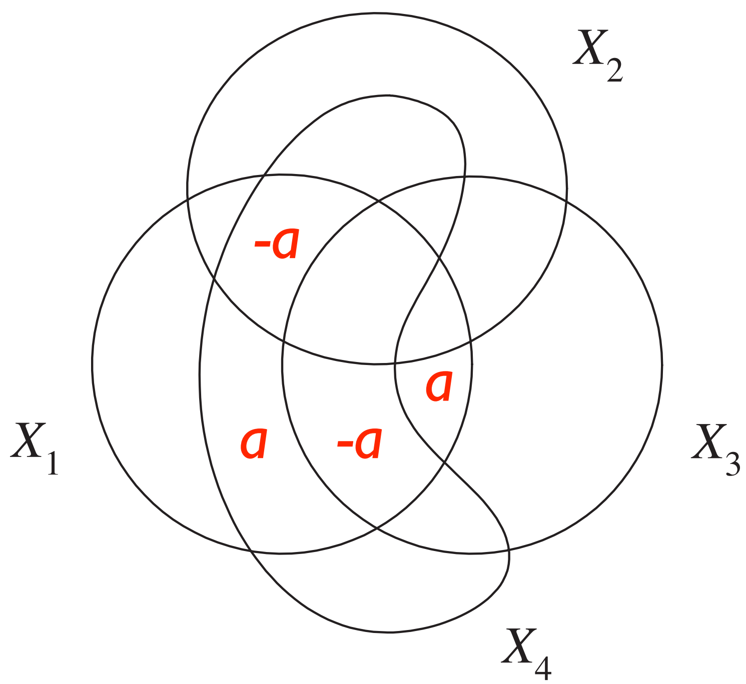

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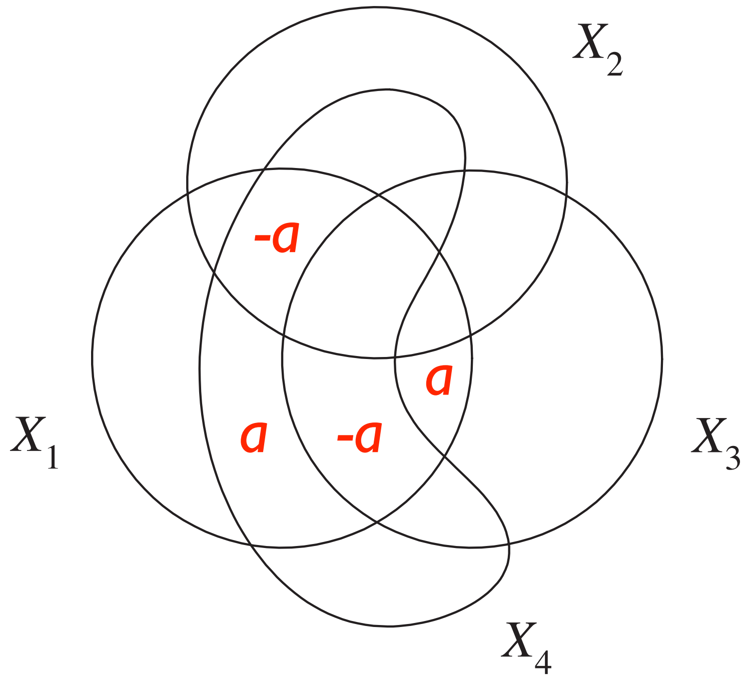

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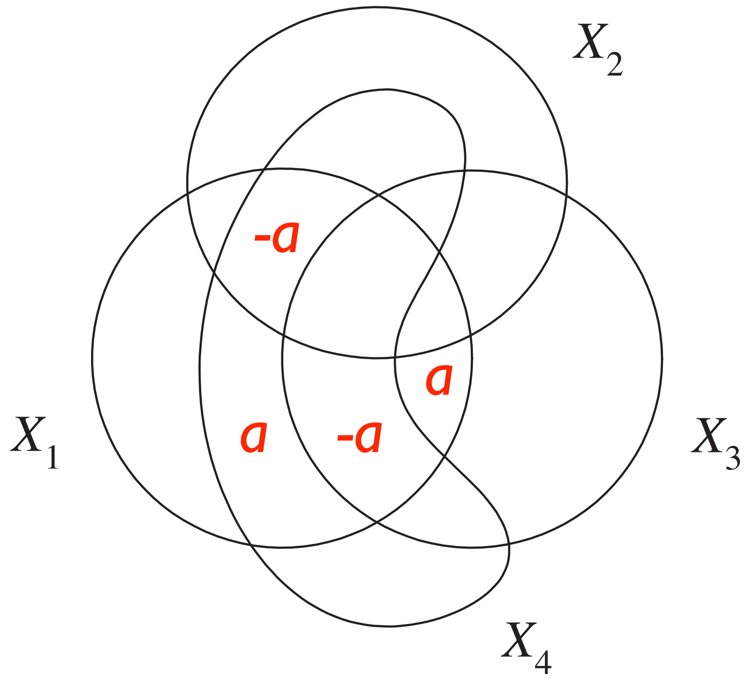

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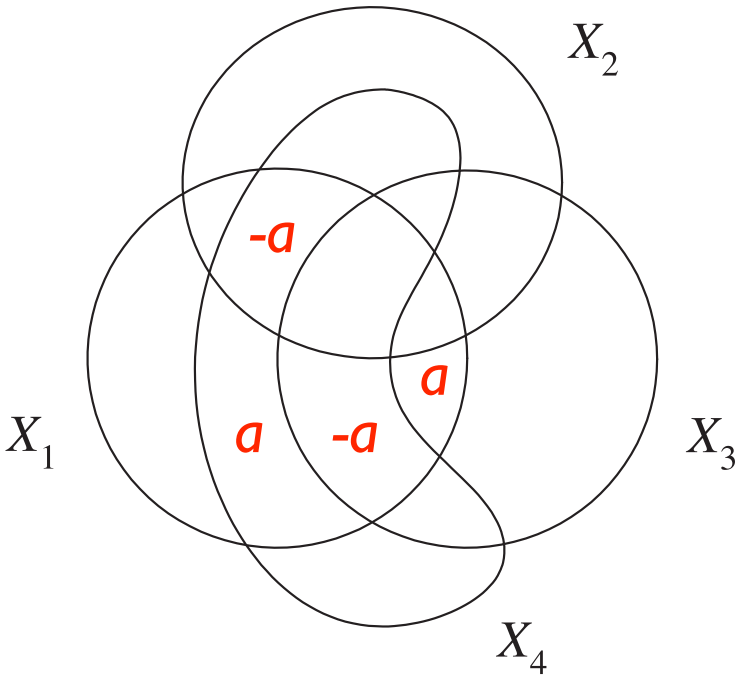

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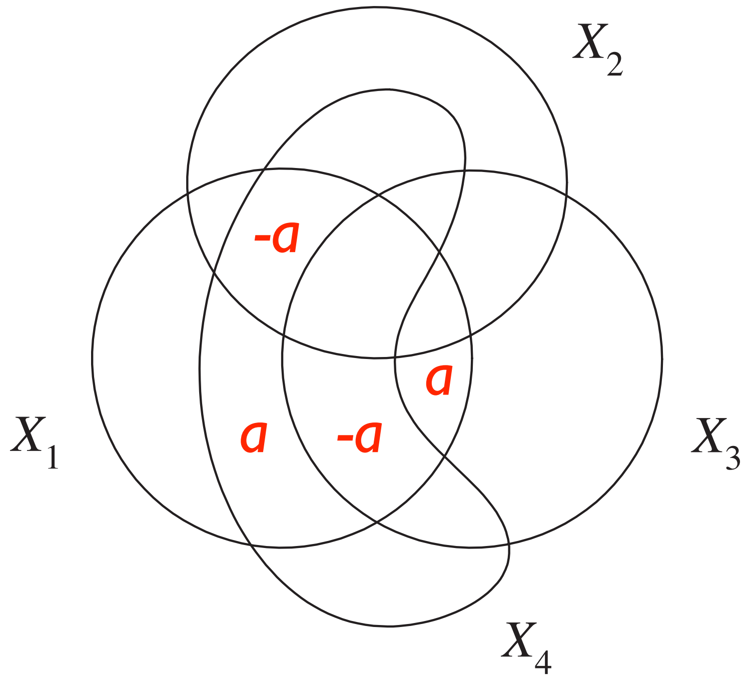

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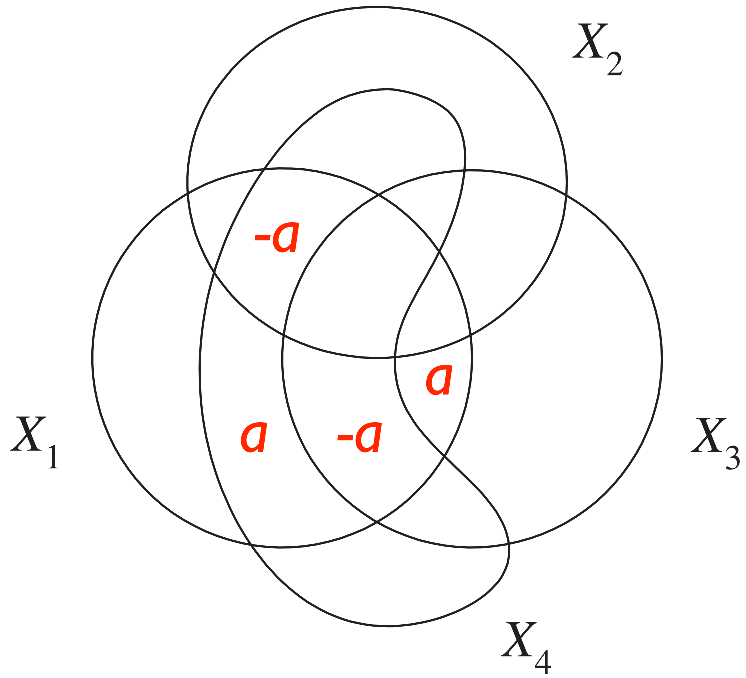

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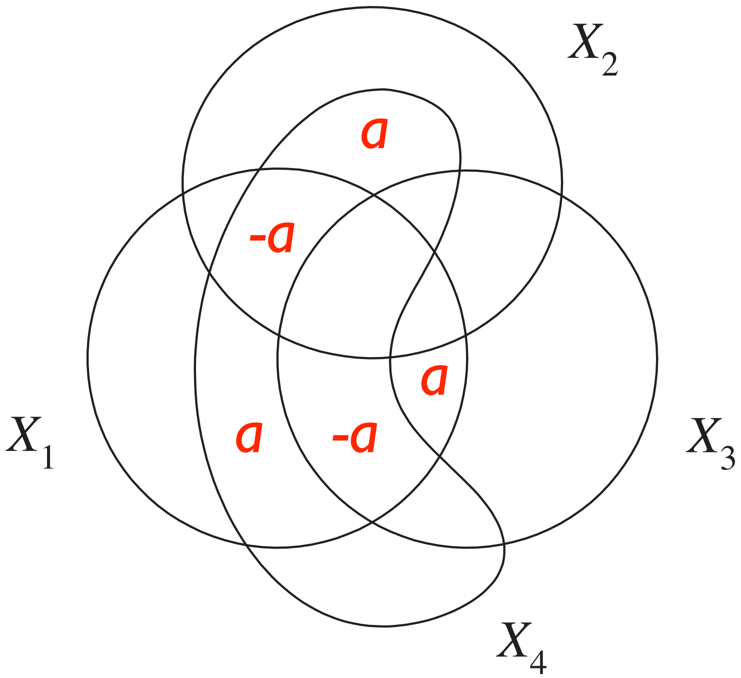

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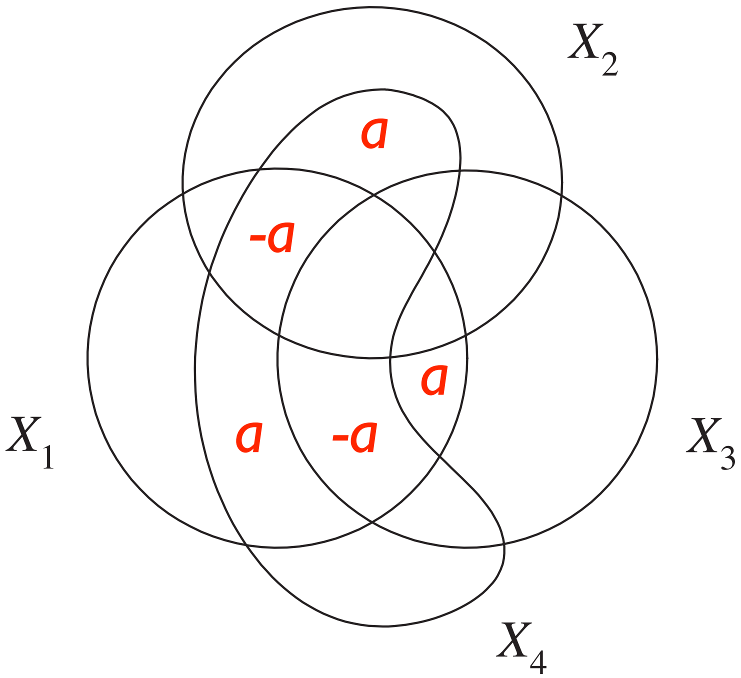

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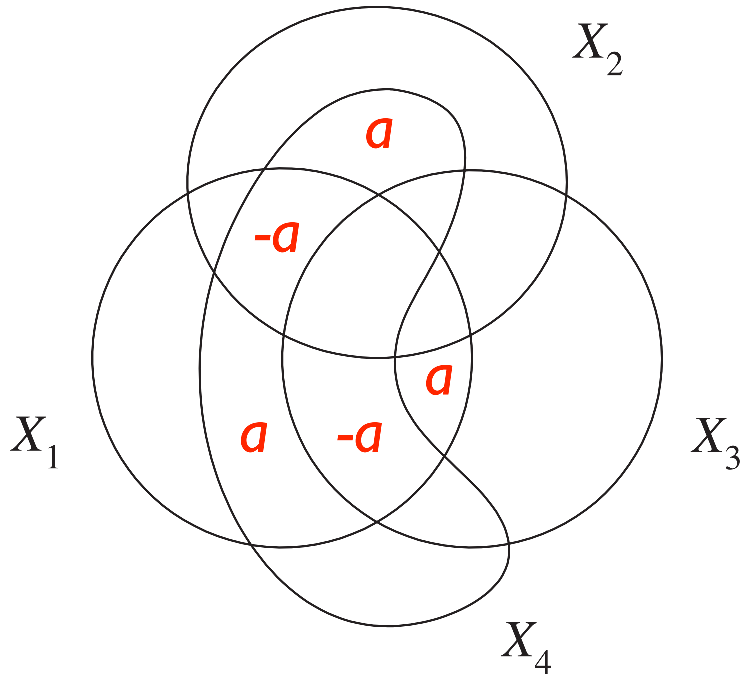

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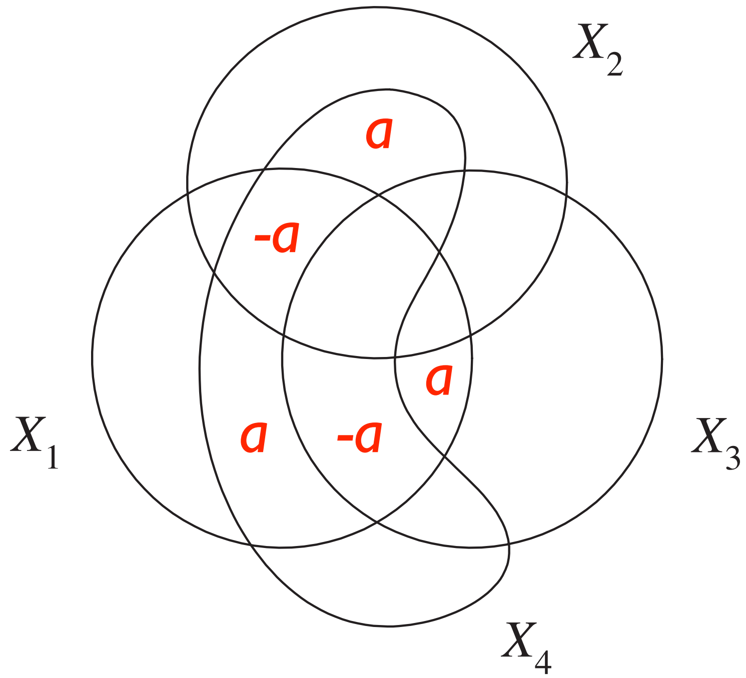

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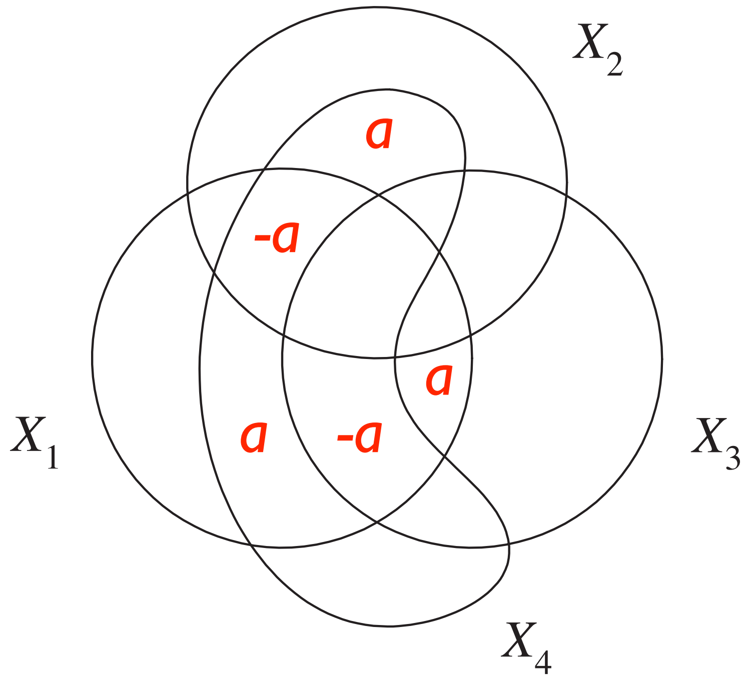

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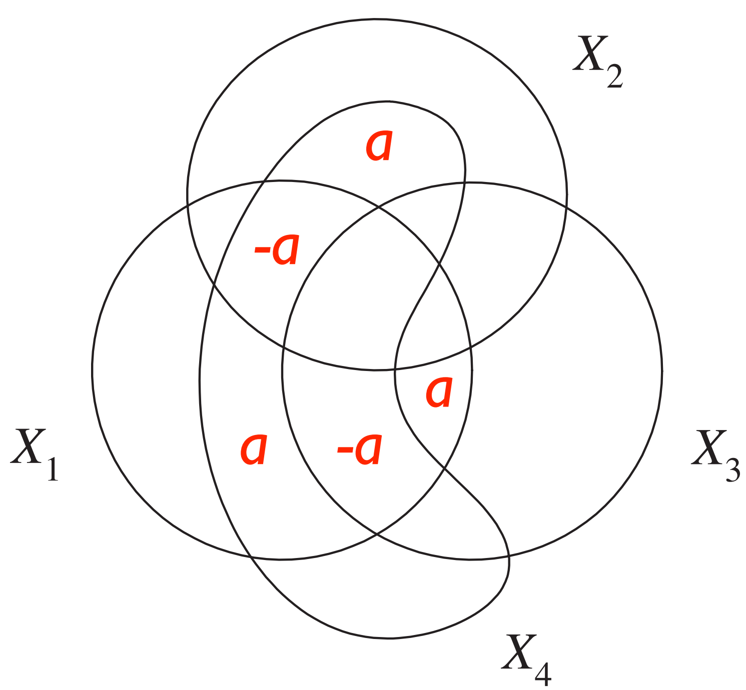

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4. The Markov subchain $X_2 \to X_3 \to X_4$ implies

$$0 = I(X_2; X_4|X_3) = I(X_1; X_2; X_4|X_3) + I(X_2; X_4|X_1, X_3).$$

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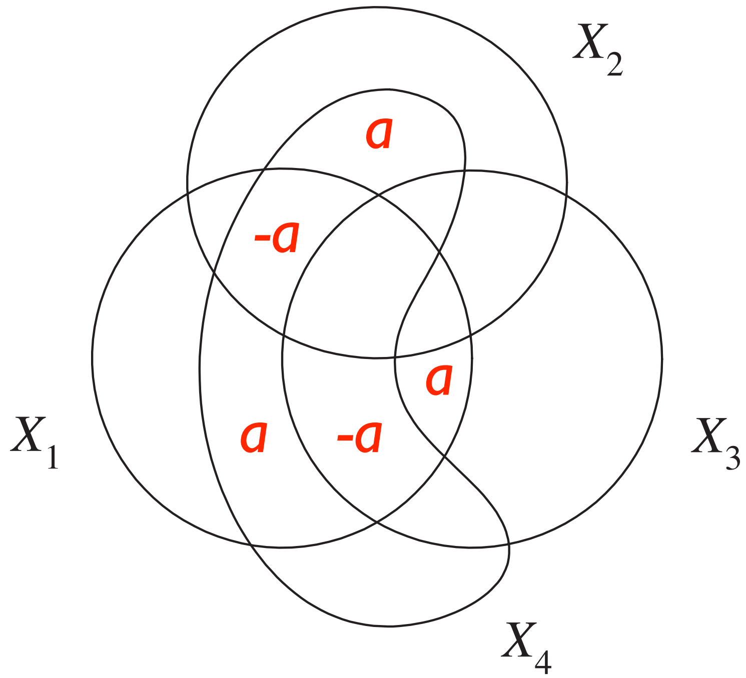
$$0 = I(X_1, X_2; X_4|X_3) = I(X_1; X_4|X_2, X_3) + I(X_1; X_2; X_4|X_3) + I(X_2; X_4|X_1, X_3).$$



Structure of μ^* for $X_1 \to X_2 \to X_3 \to X_4$

1. The Markov subchain $X_1 \to X_2 \to X_3$ implies

$$0 = I(X_1; X_3|X_2) = I(X_1; X_3; X_4|X_2) + I(X_1; X_3|X_2, X_4).$$

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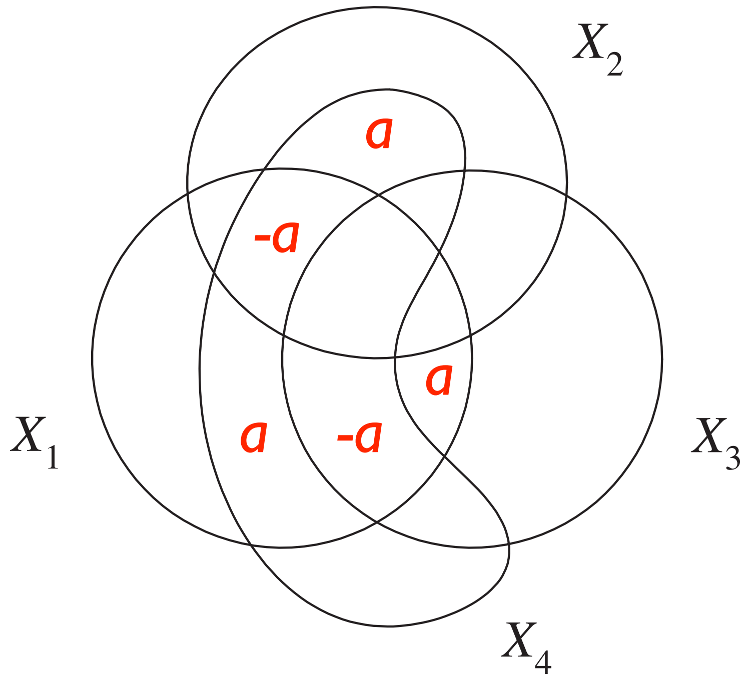

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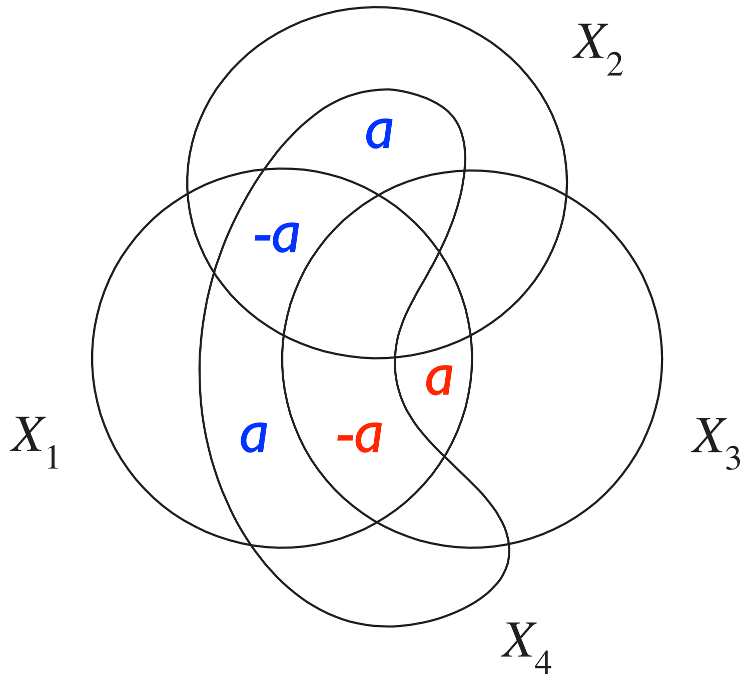

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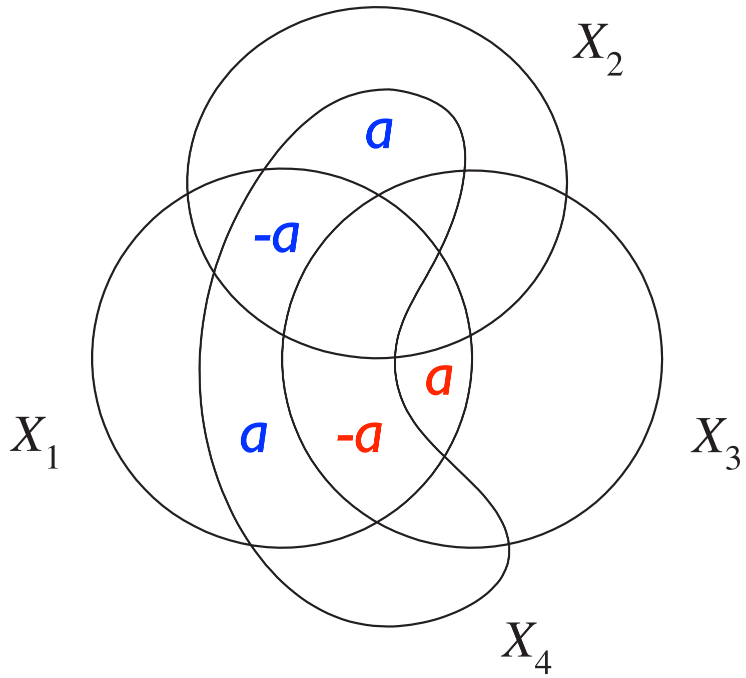
$$0 = a - a + a = a.$$



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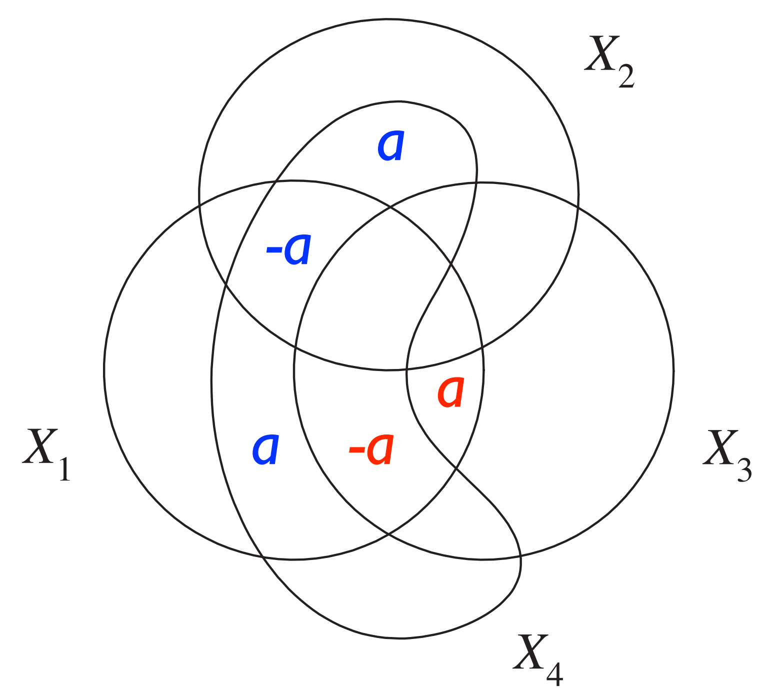
Therefore $a = 0$, and so μ^* vanishes on the corresponding 5 atoms as shown in the information diagram.



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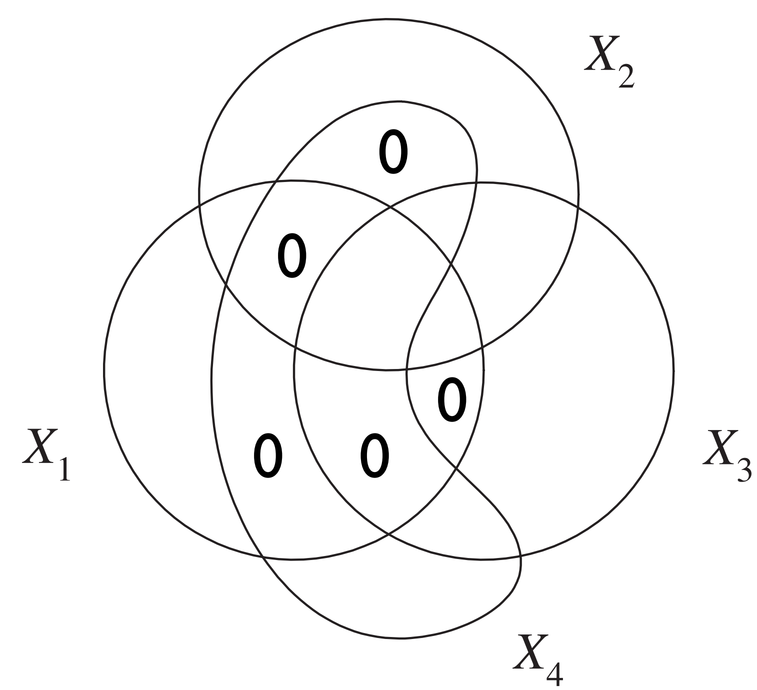
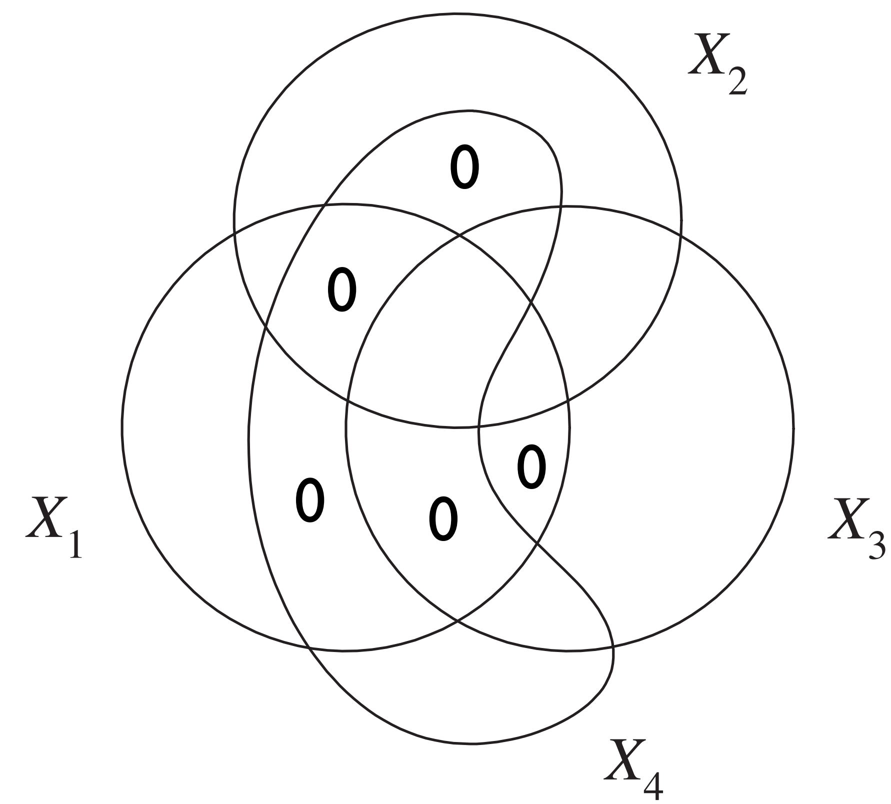


Nonnegatvity of μ^* for $X_1 \to X_2 \to X_3 \to X_4$



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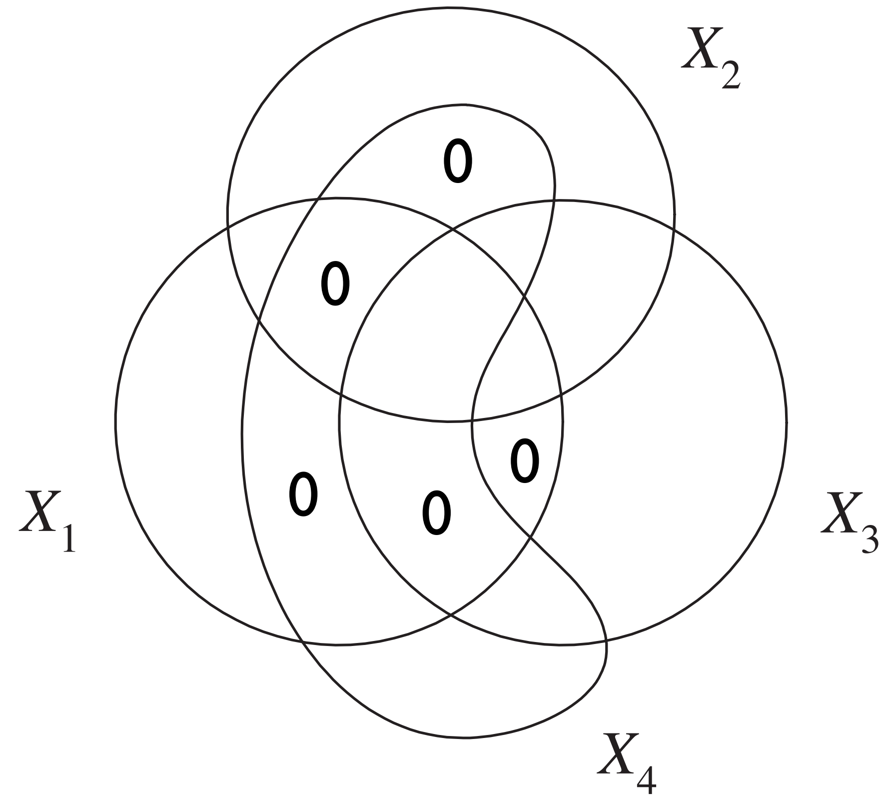

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Nonnegatvity of μ^* for $X_1 \to X_2 \to X_3 \to X_4$

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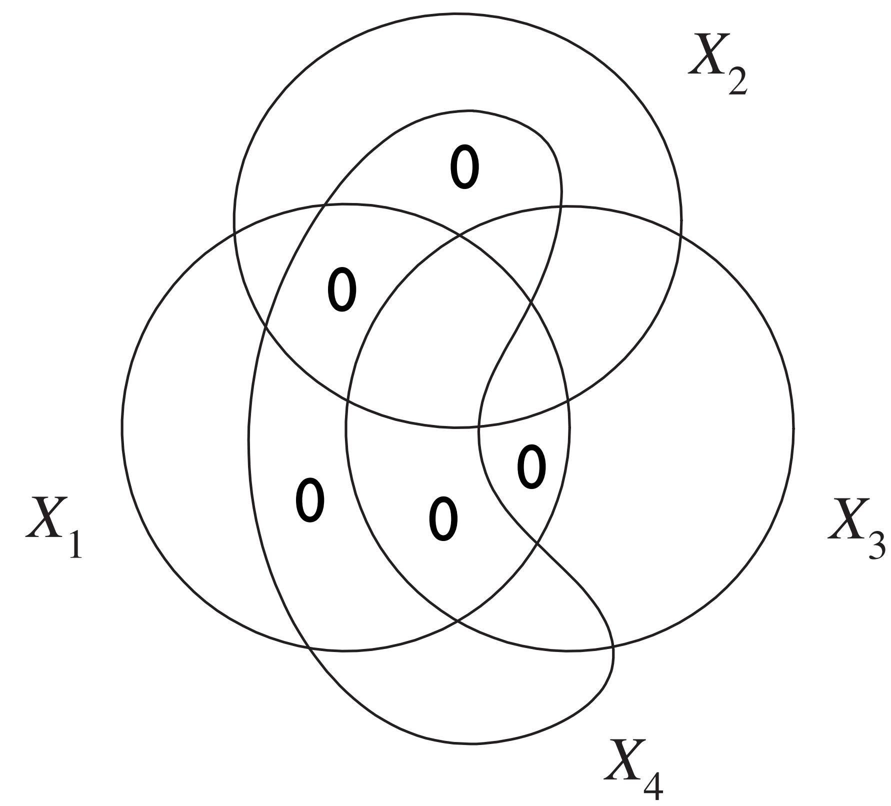
2. Suppress these atoms by setting them to $\emptyset$ to obtain the information diagram below.



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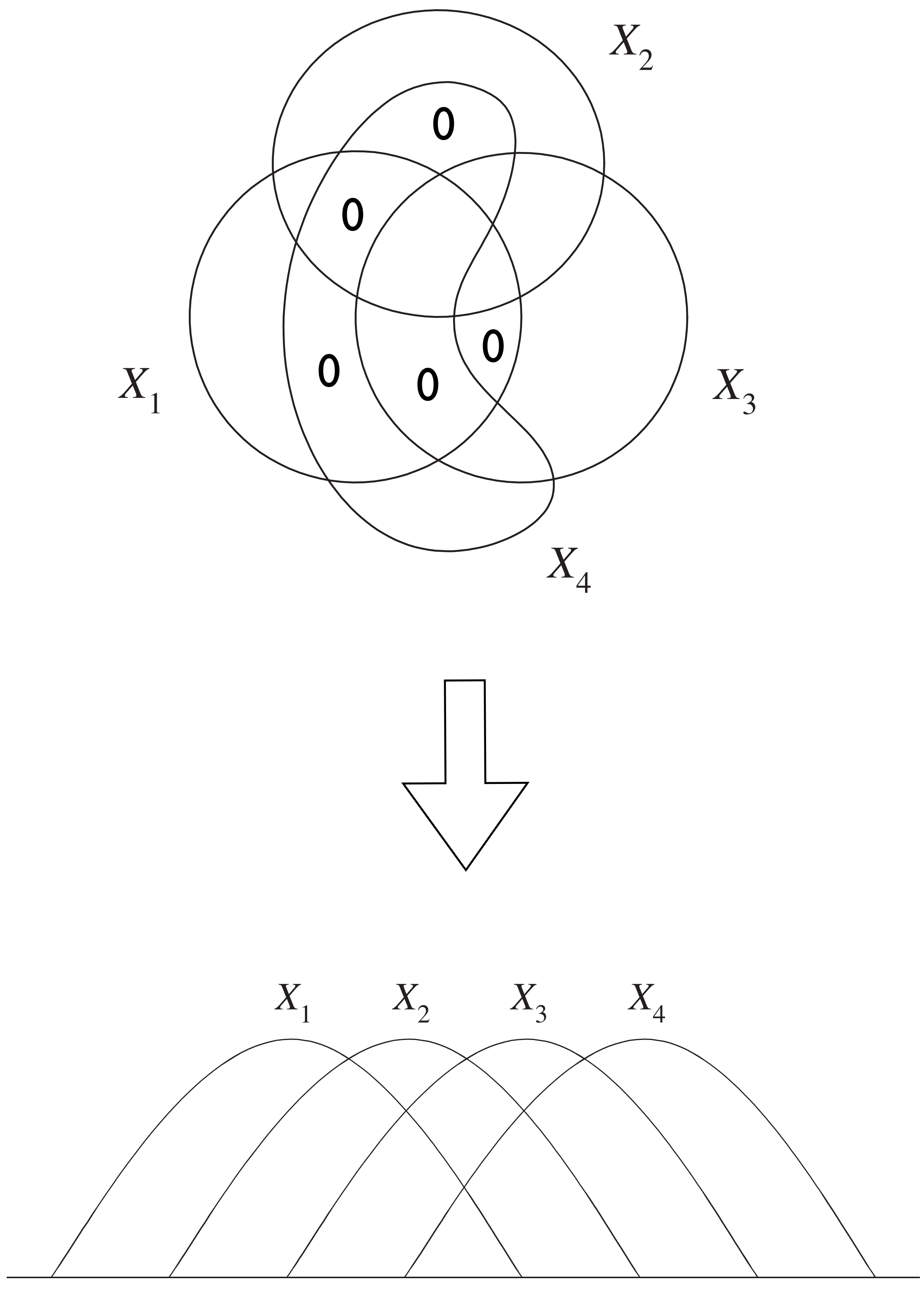

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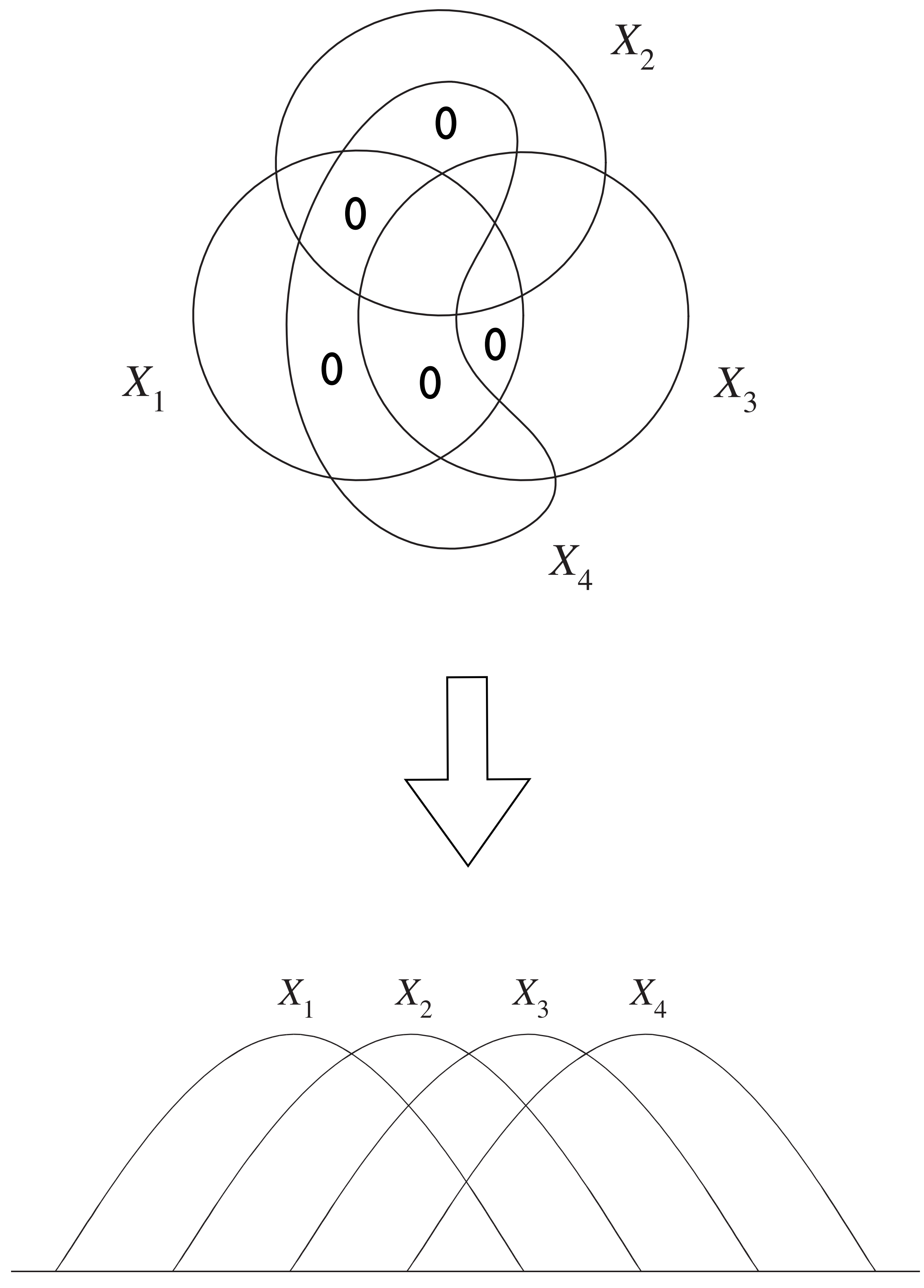

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$$(2^4 - 1) - 5 = 10$$

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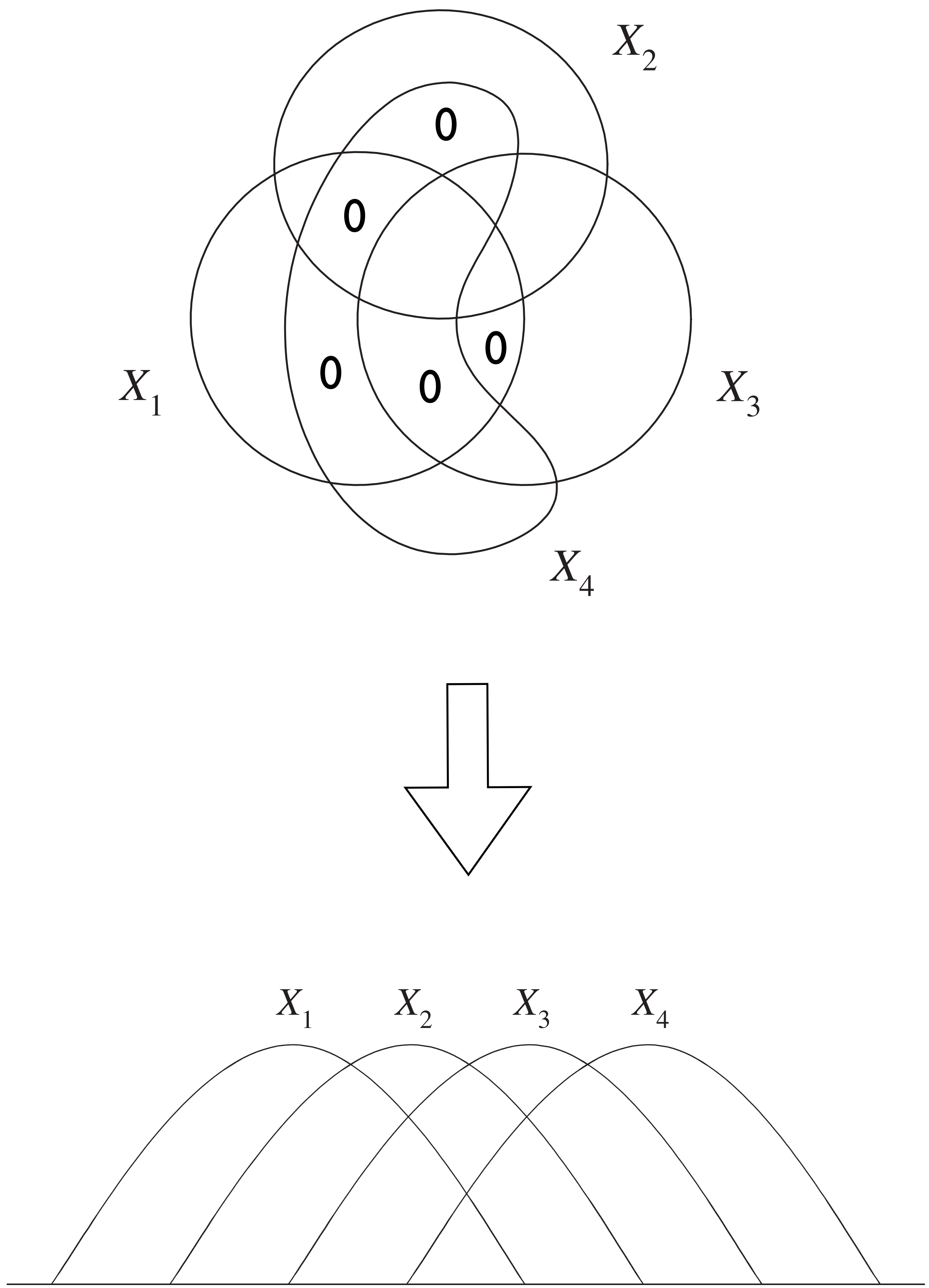

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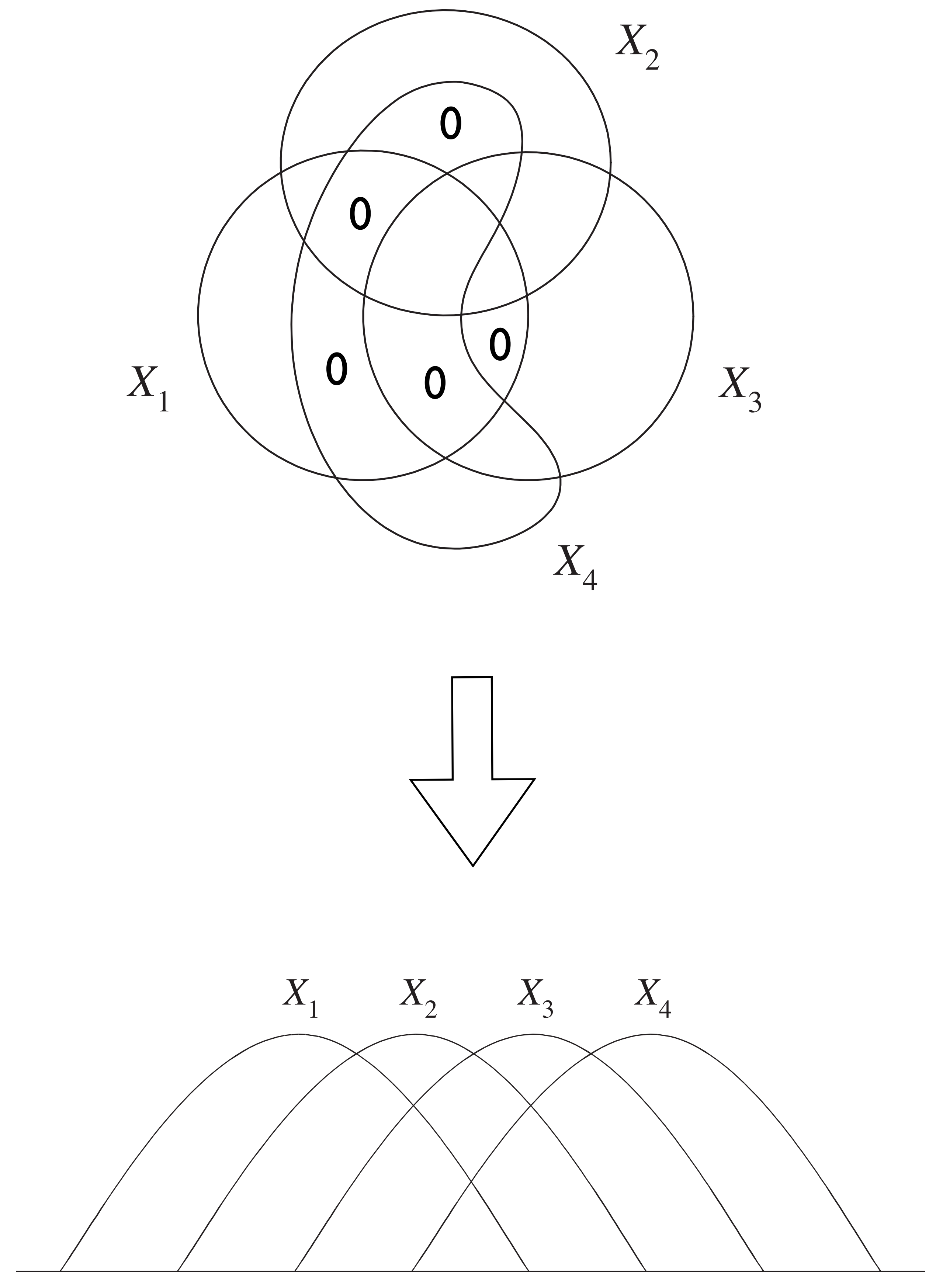

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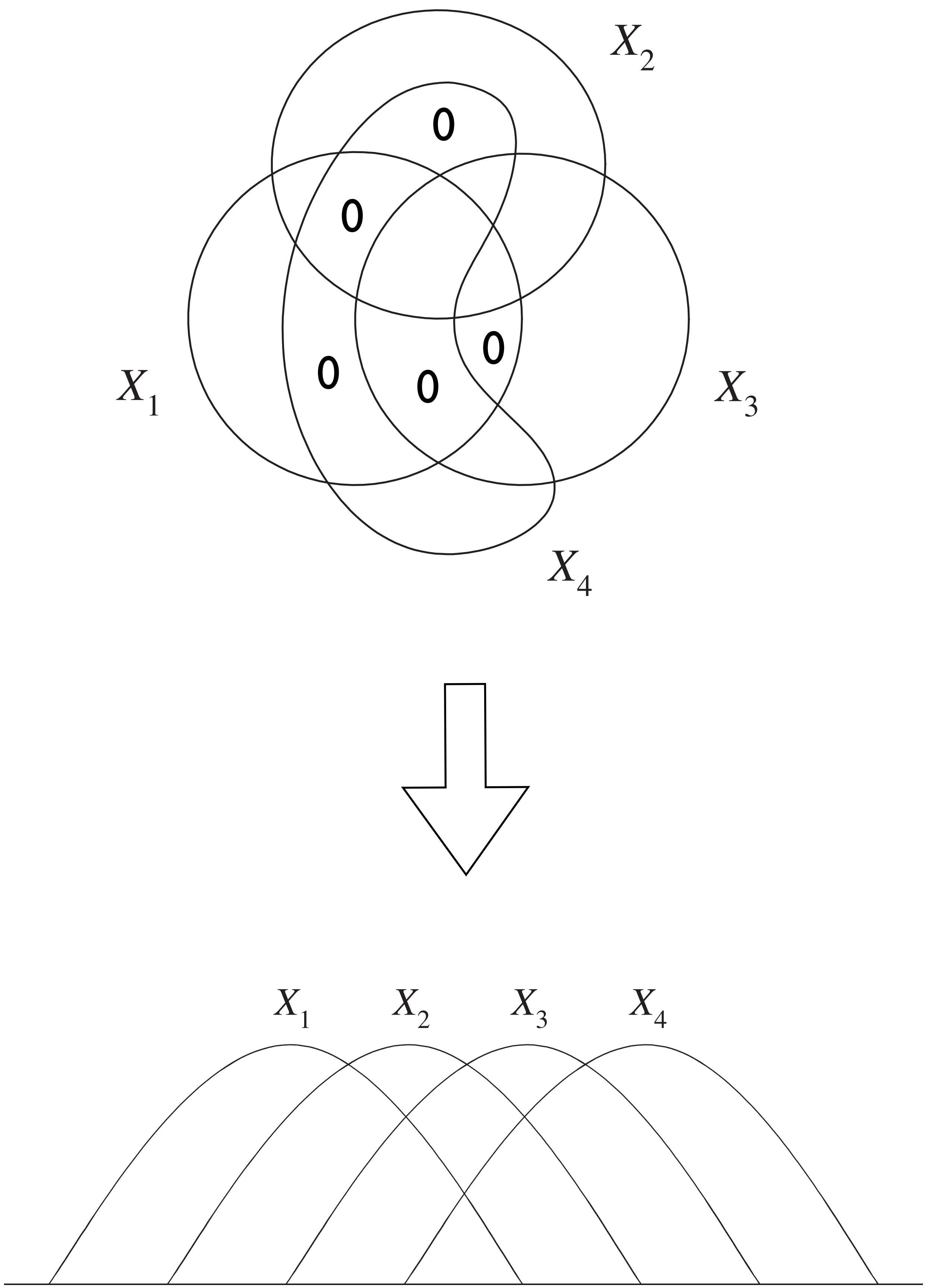
3. There are all Shannon's information measures which are always nonnegative. Therefore, μ^* is a measure.



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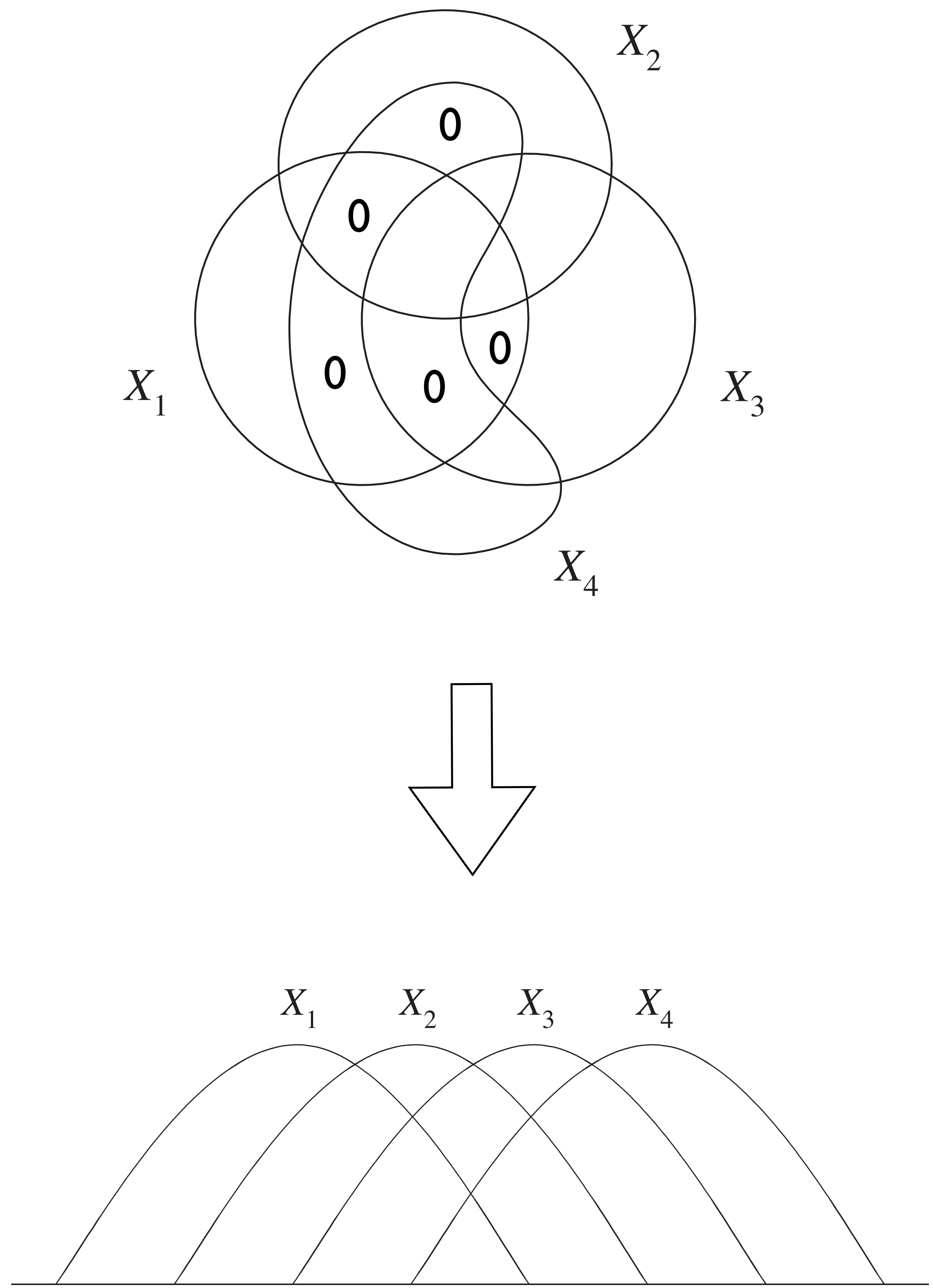
Exercise: Identify these 10 atoms in the information diagram at the bottom.



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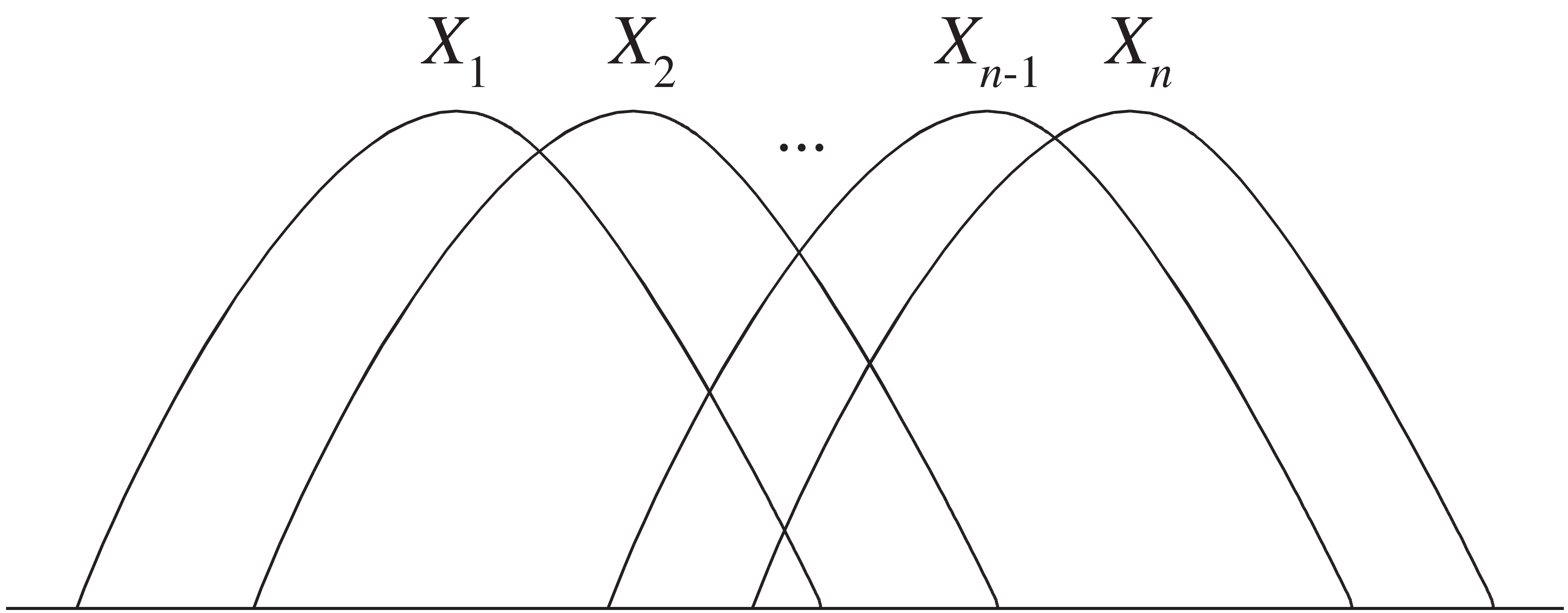

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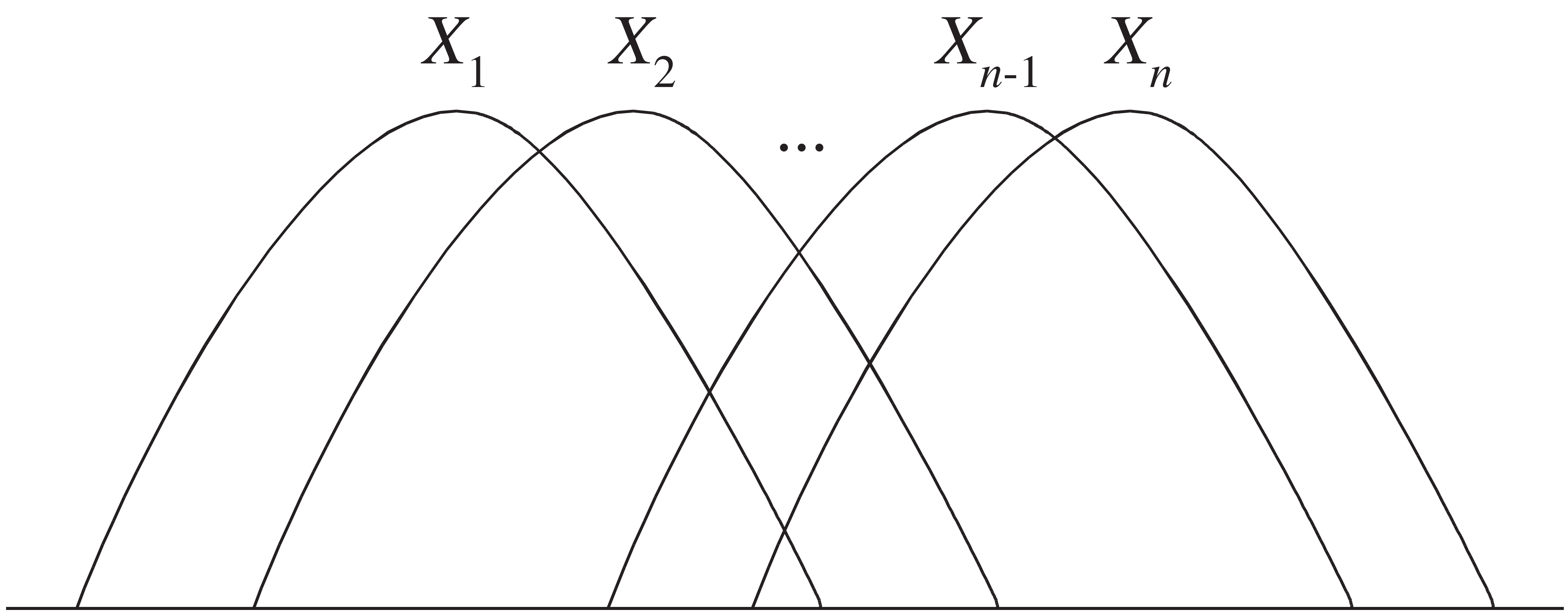

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- See Ch. 12 for a detailed discussion in the context of Markov random field.