3.4 μ* can be Negative

For $n = 2$,

- μ^* is always nonnegative.

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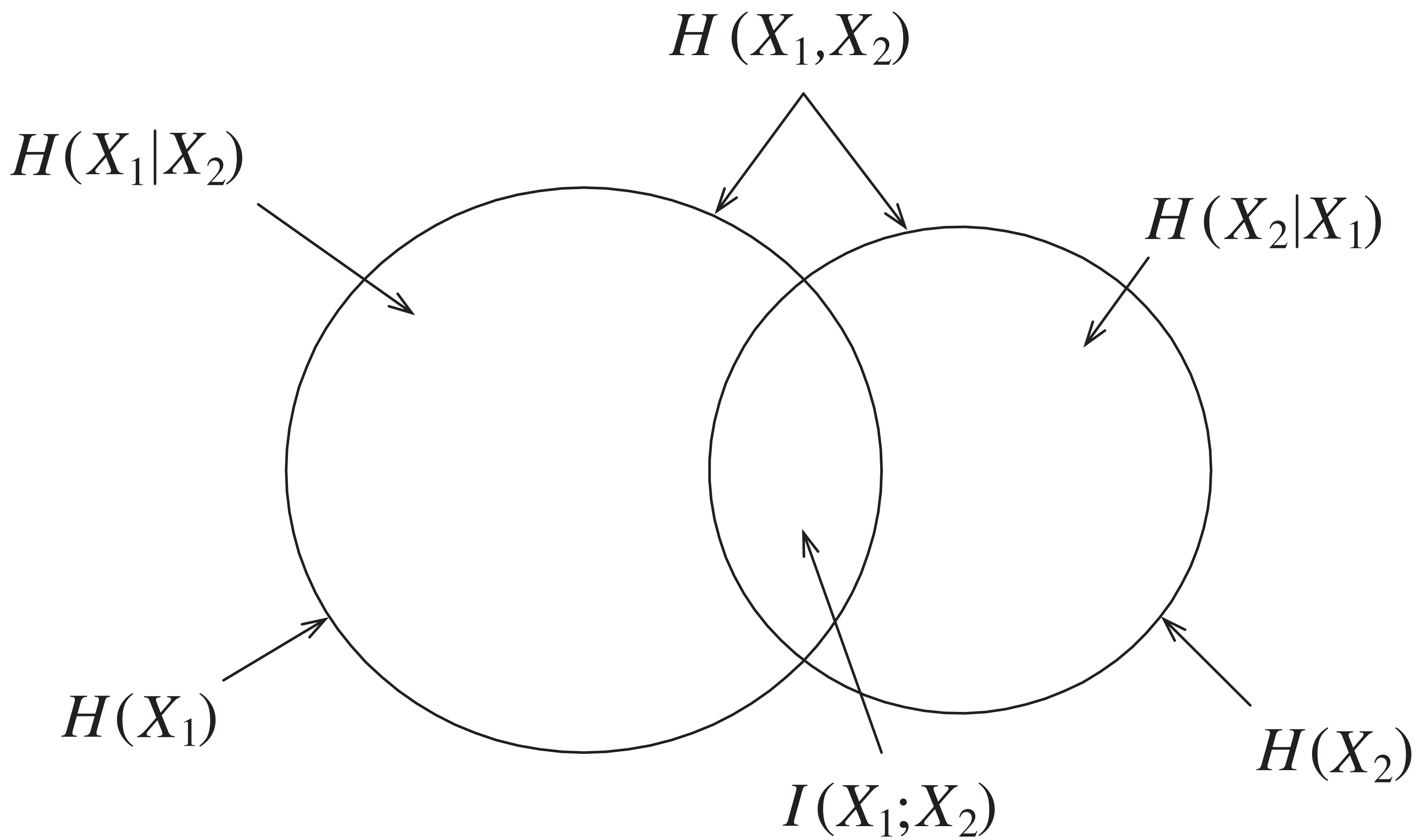
For $n = 3$,

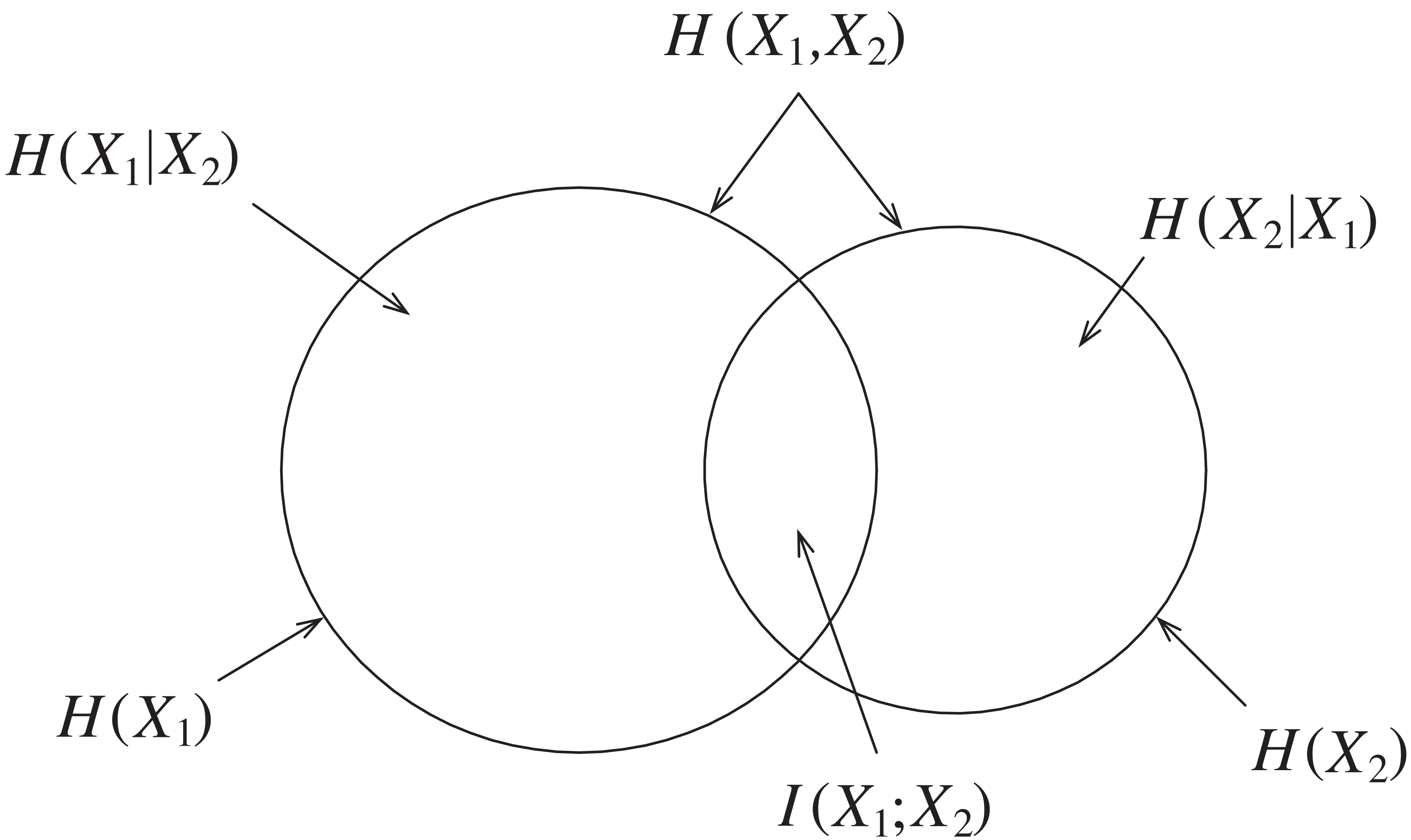
- μ^* is not always nonnegative because

$$\mu^*(\tilde{X}_1 \cap \tilde{X}_2 \cap \tilde{X}_3) = I(X_1; X_2; X_3)$$

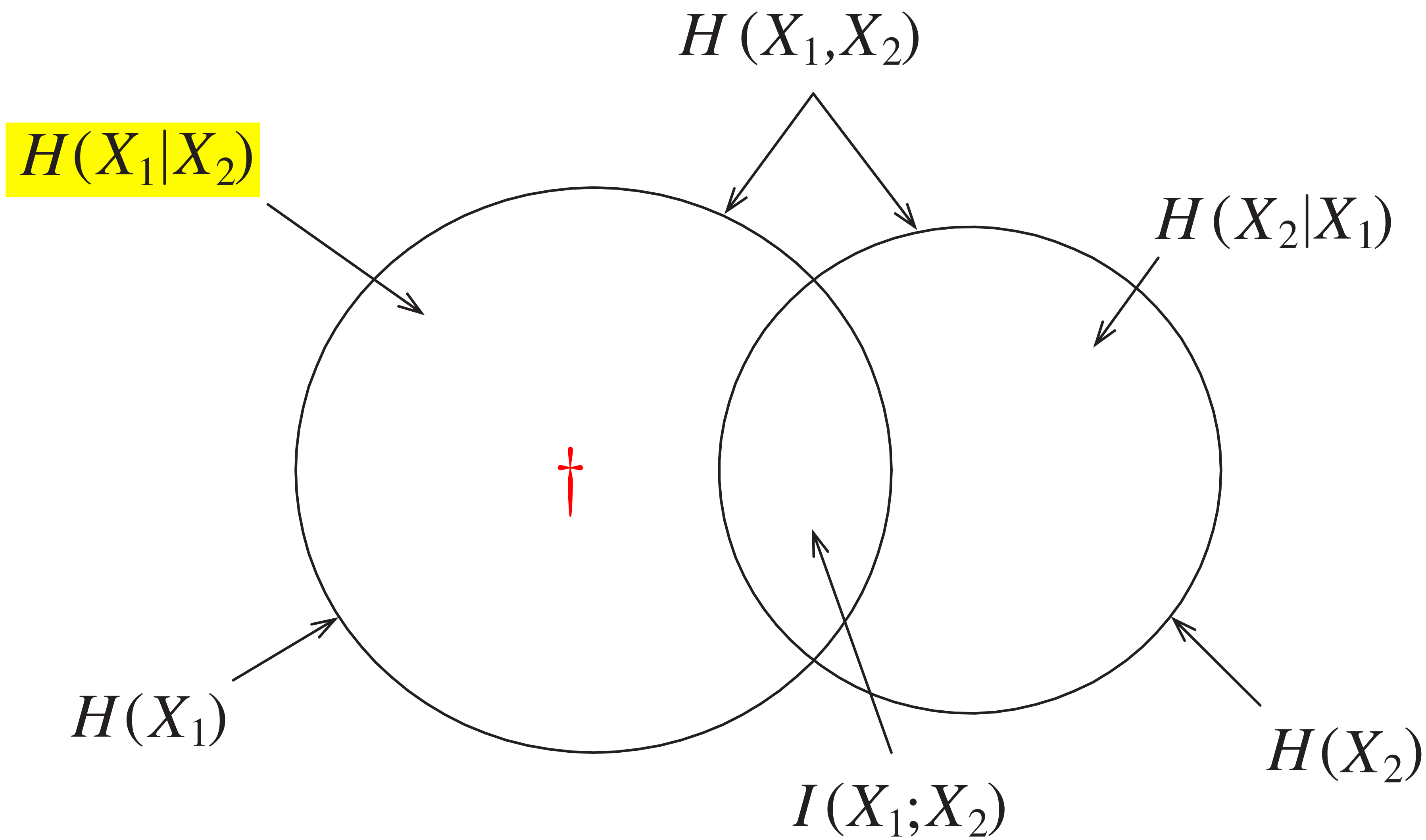
  can be negative.



$H(X_1,X_2)$
$H(X_1|X_2)$
$H(X_2|X_1)$
$H(X_1)$
$H(X_2)$
$I(X_1;X_2)$



- For $n = 2$, the values of μ^* on the nonempty atoms of $\mathcal{F}_2$ are all Shannon's information measures which are nonnegative.



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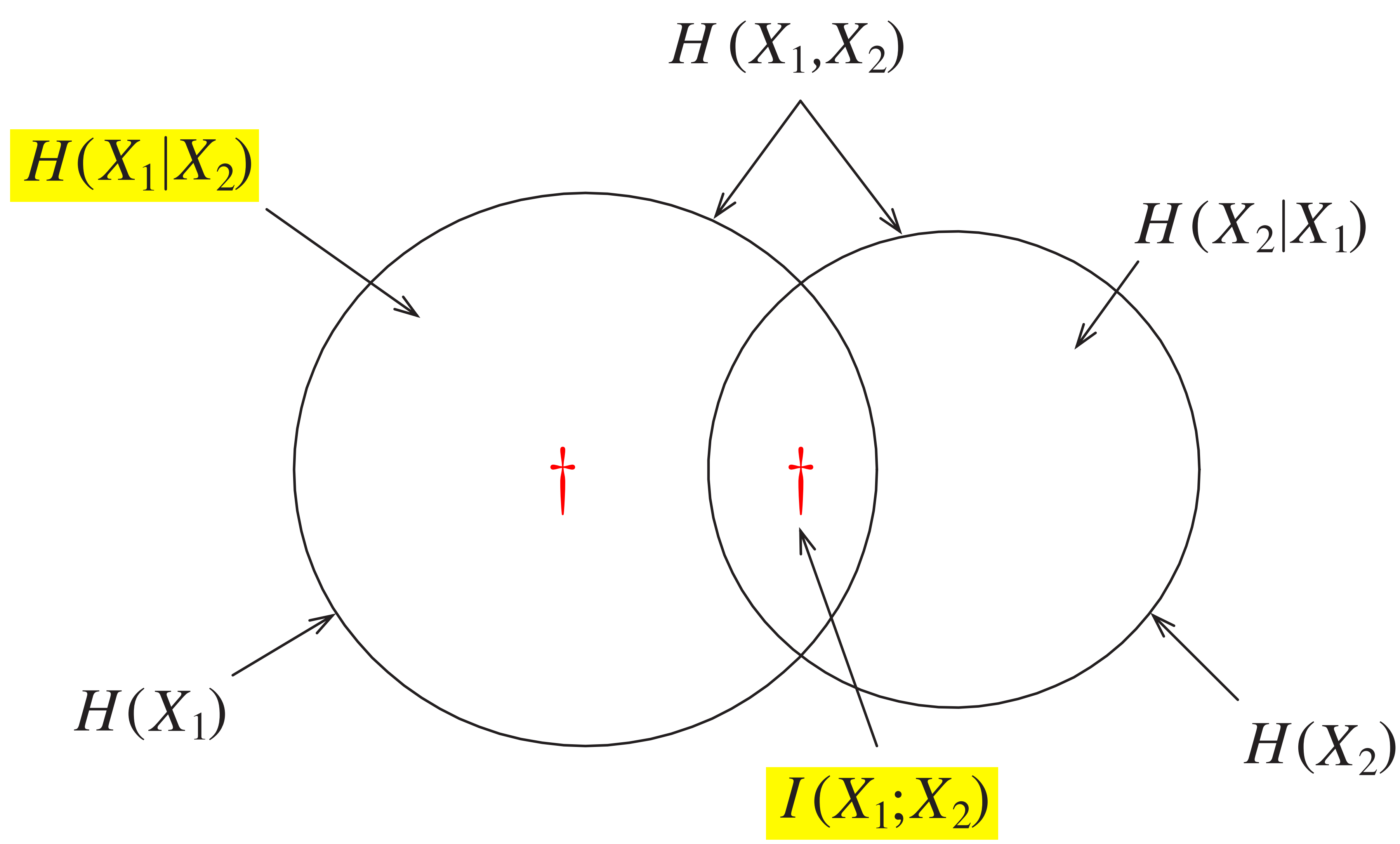

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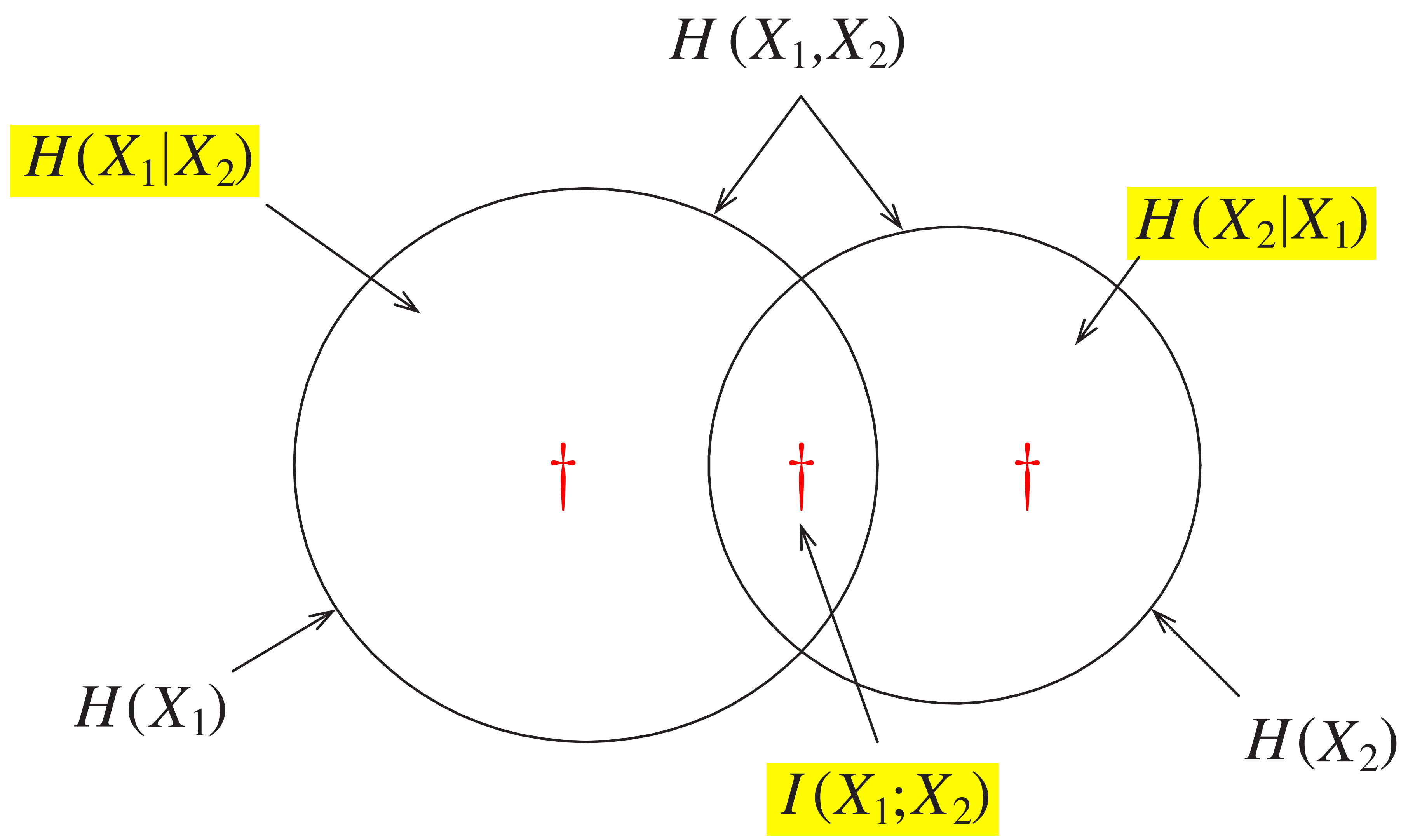

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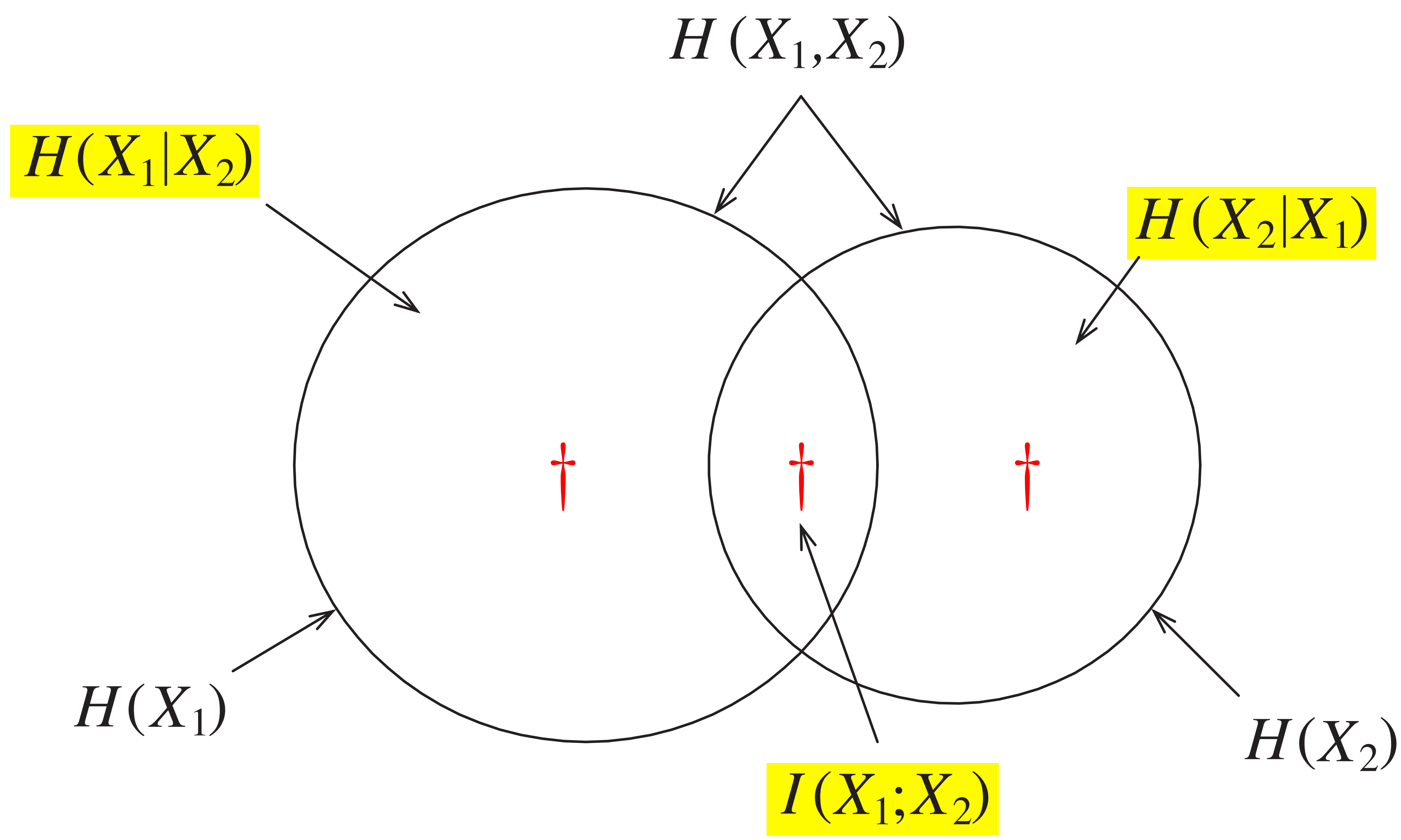
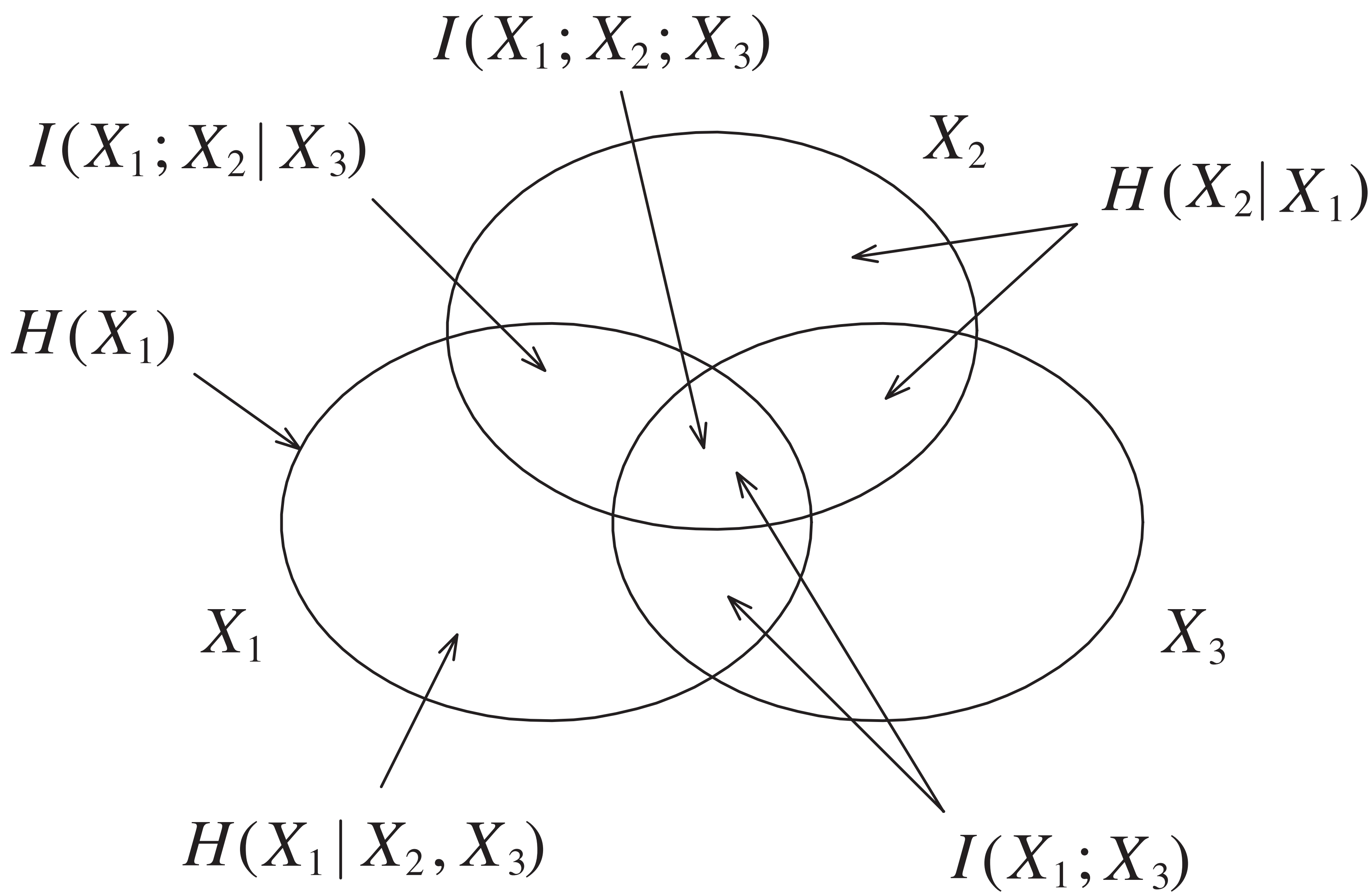


- For $n = 2$, the values of μ^* on the nonempty atoms of $\mathcal{F}_2$ are all Shannon's information measures which are nonnegative.

- Therefore, μ^* is always nonnegative.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$I(X_1;X_2|X_3)$
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X_3
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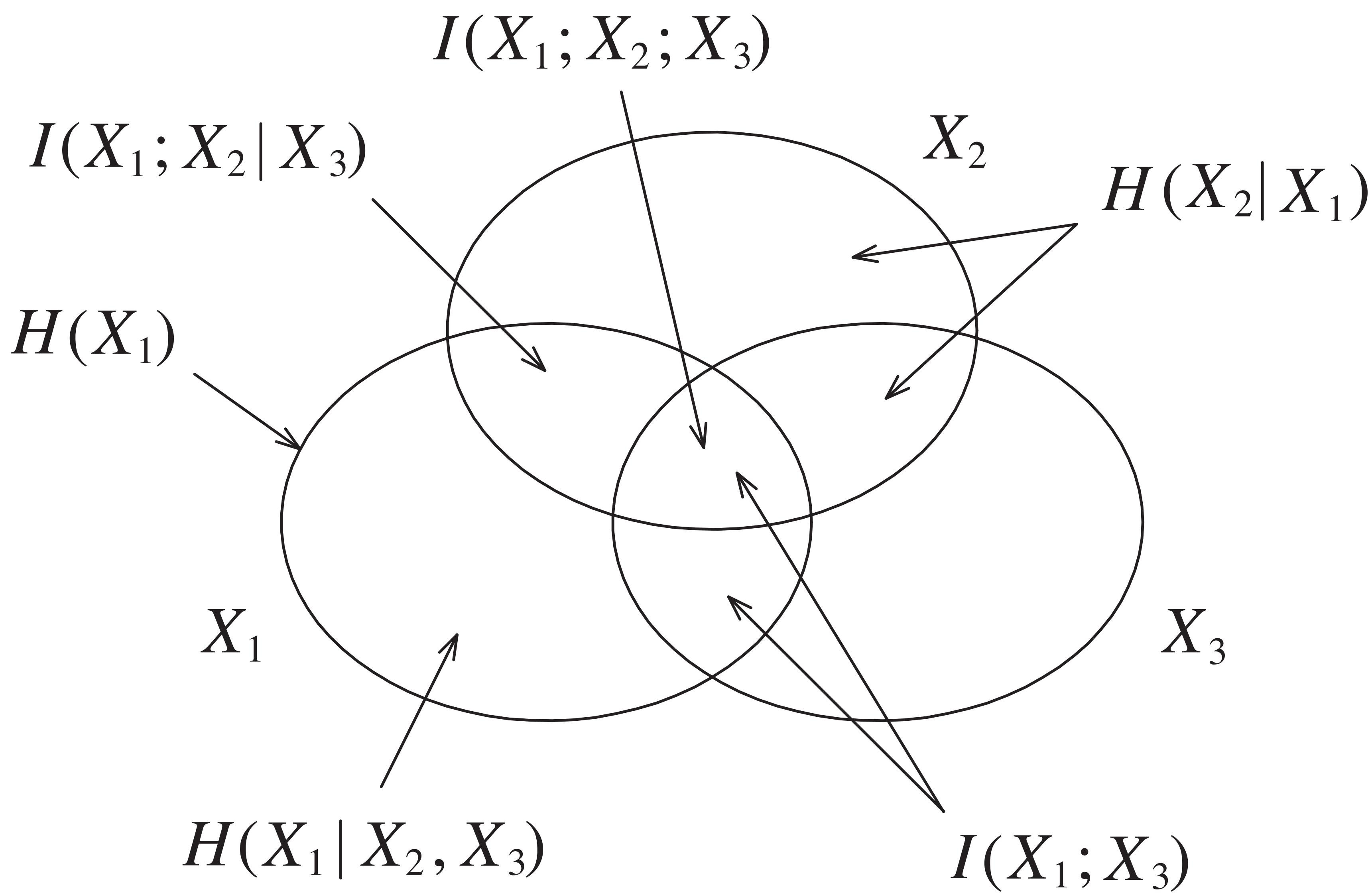
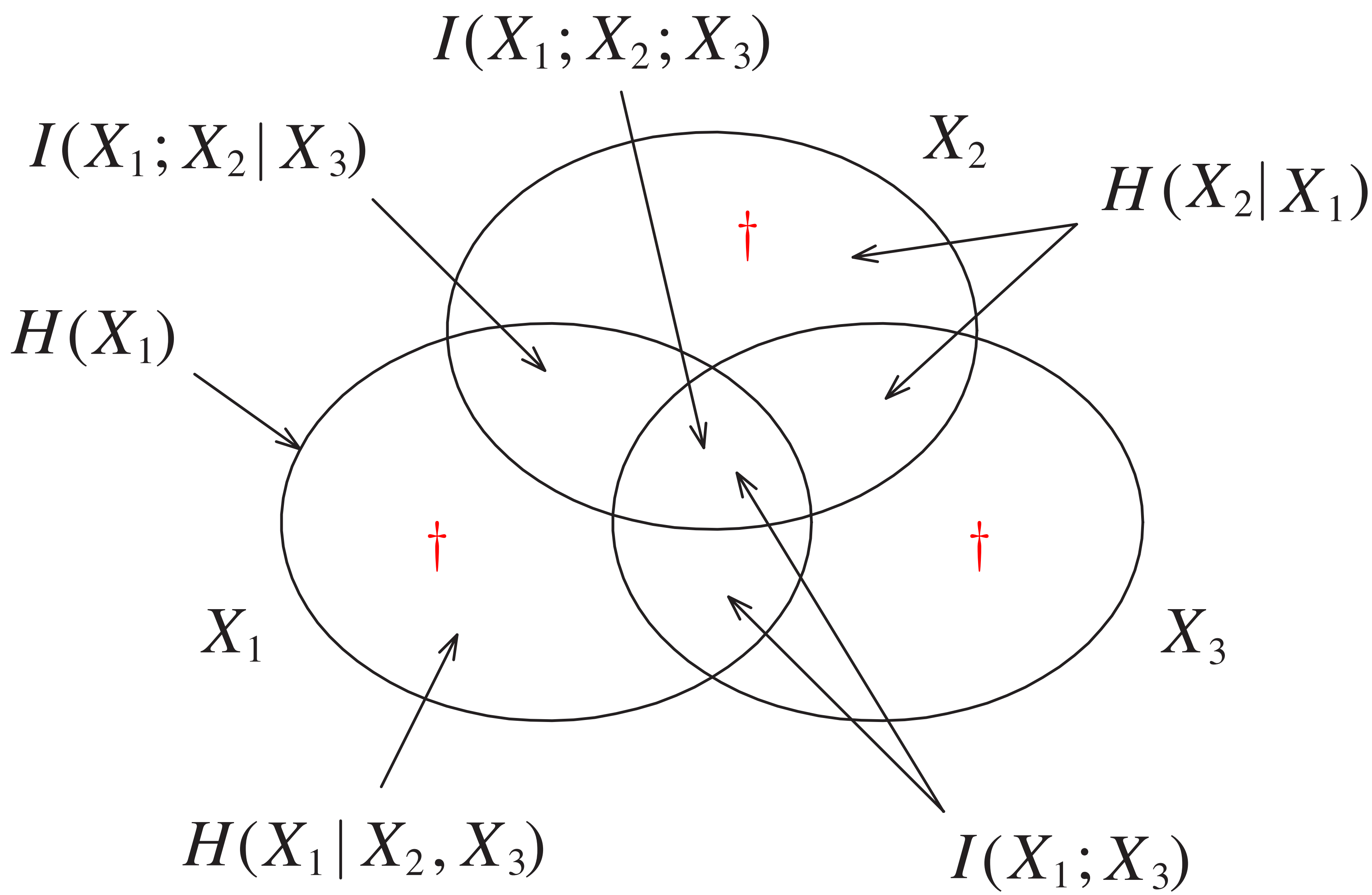


- For $n = 3$, the values of μ^* on the nonempty atoms of $\mathcal{F}_3$ all correspond to Shannon's information measures,



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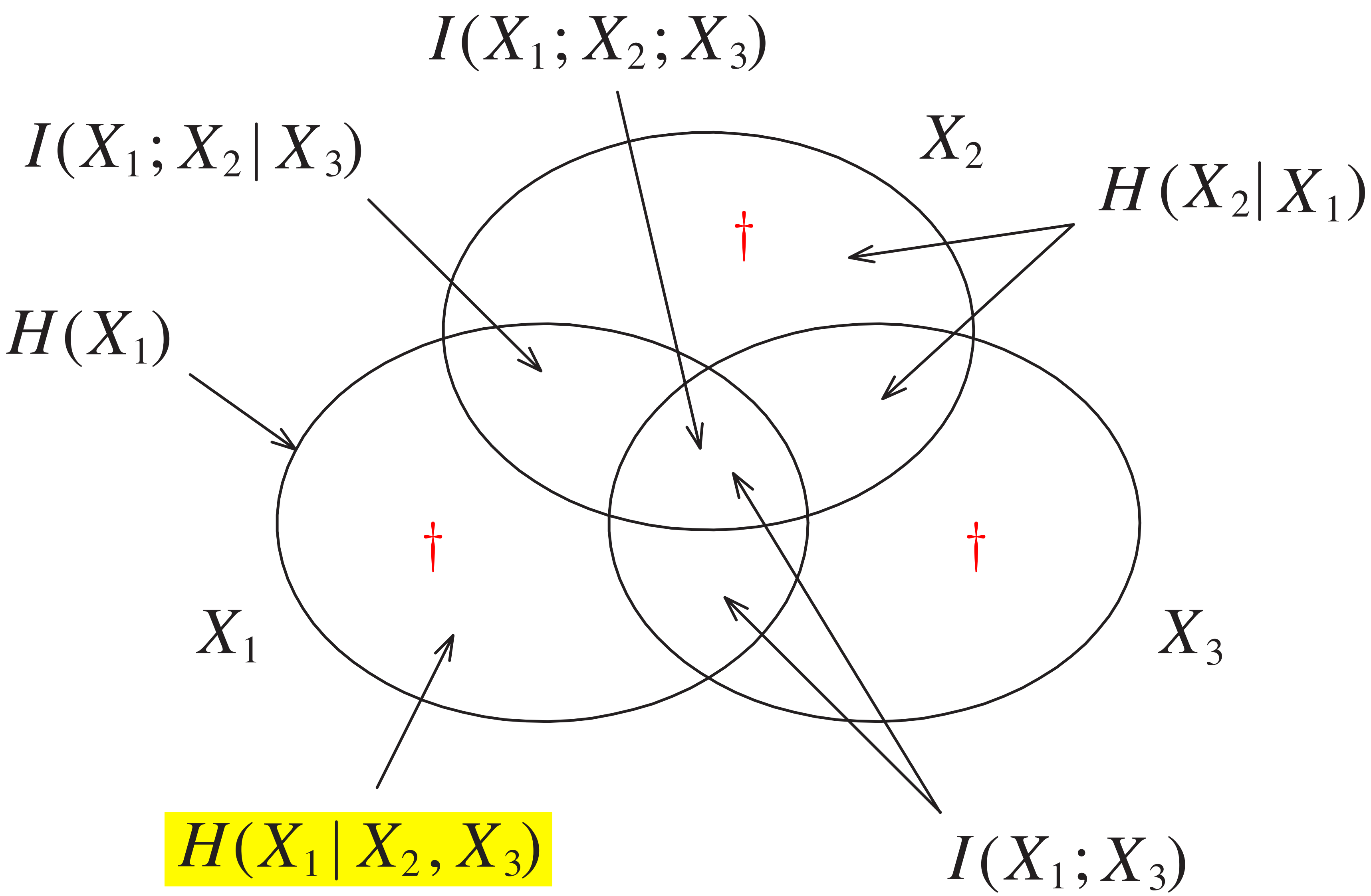

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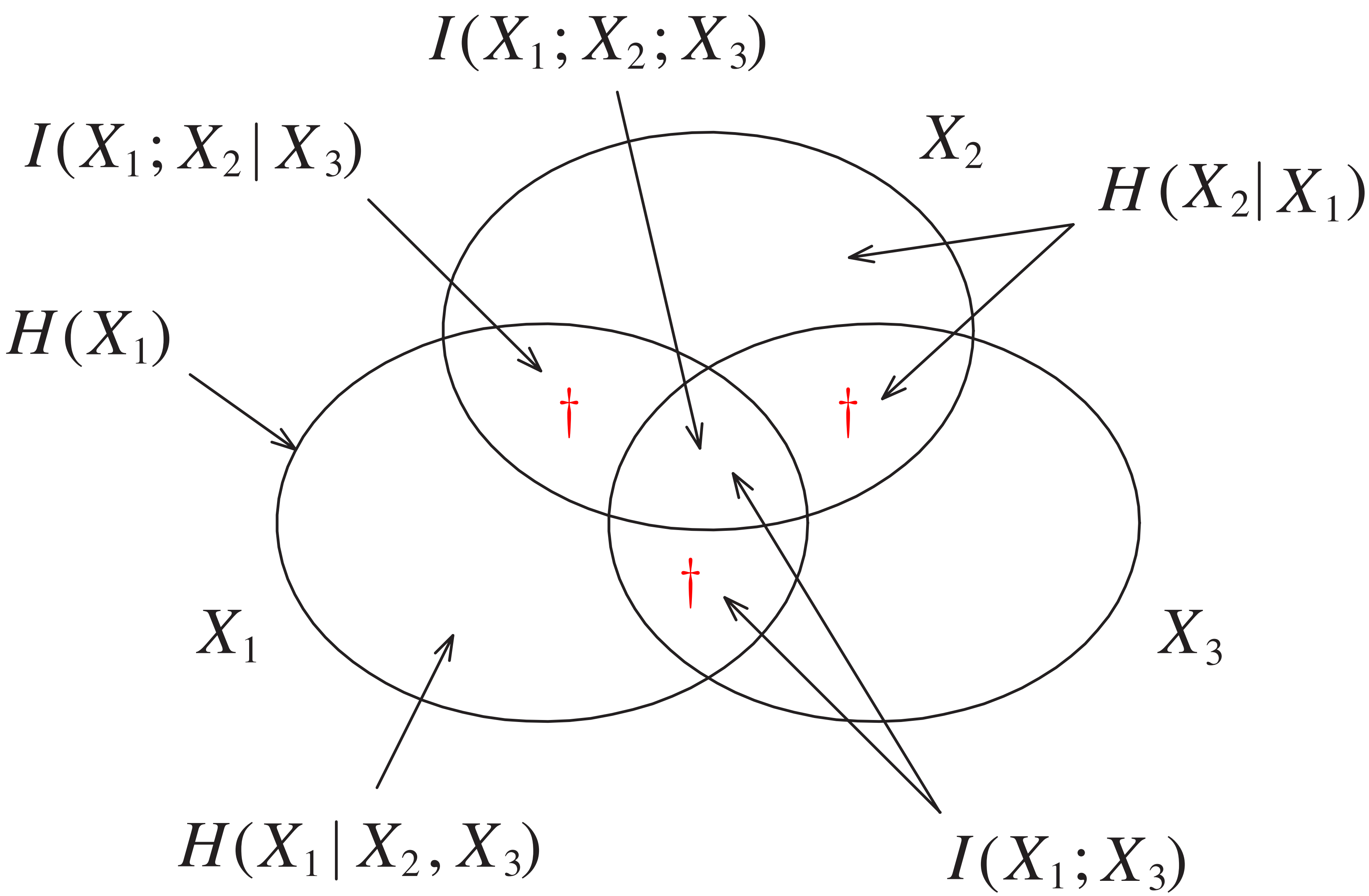

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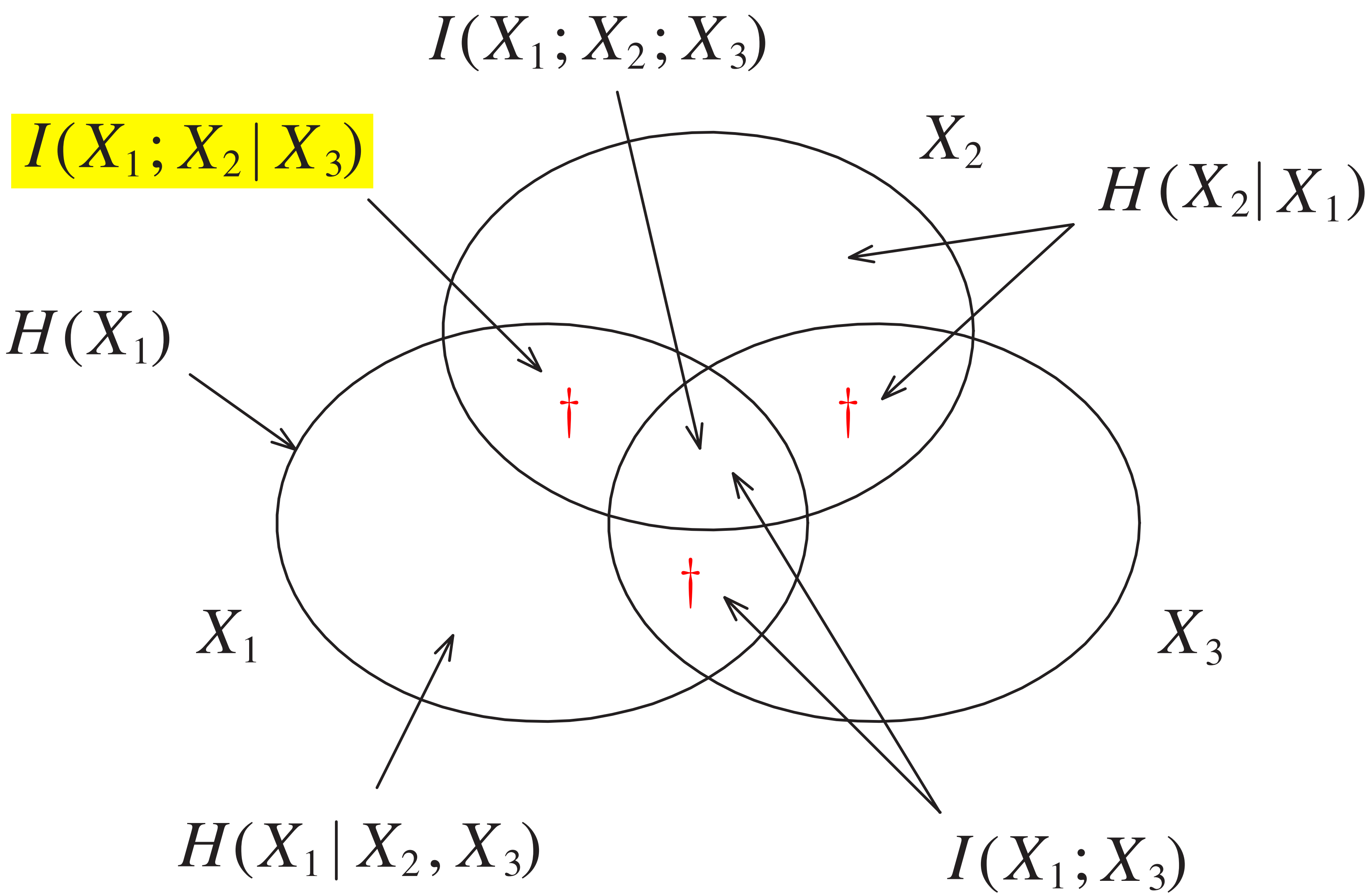

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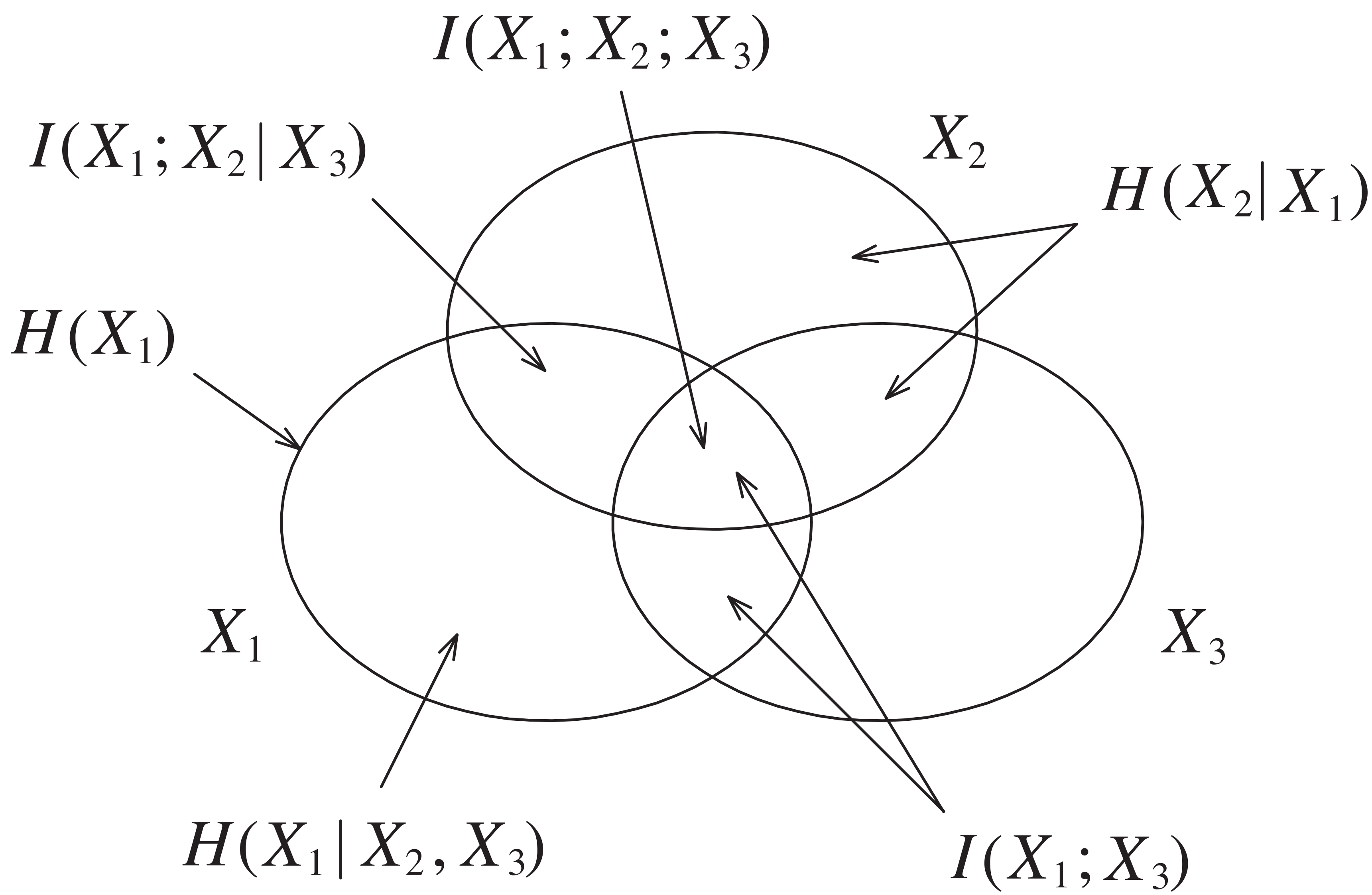

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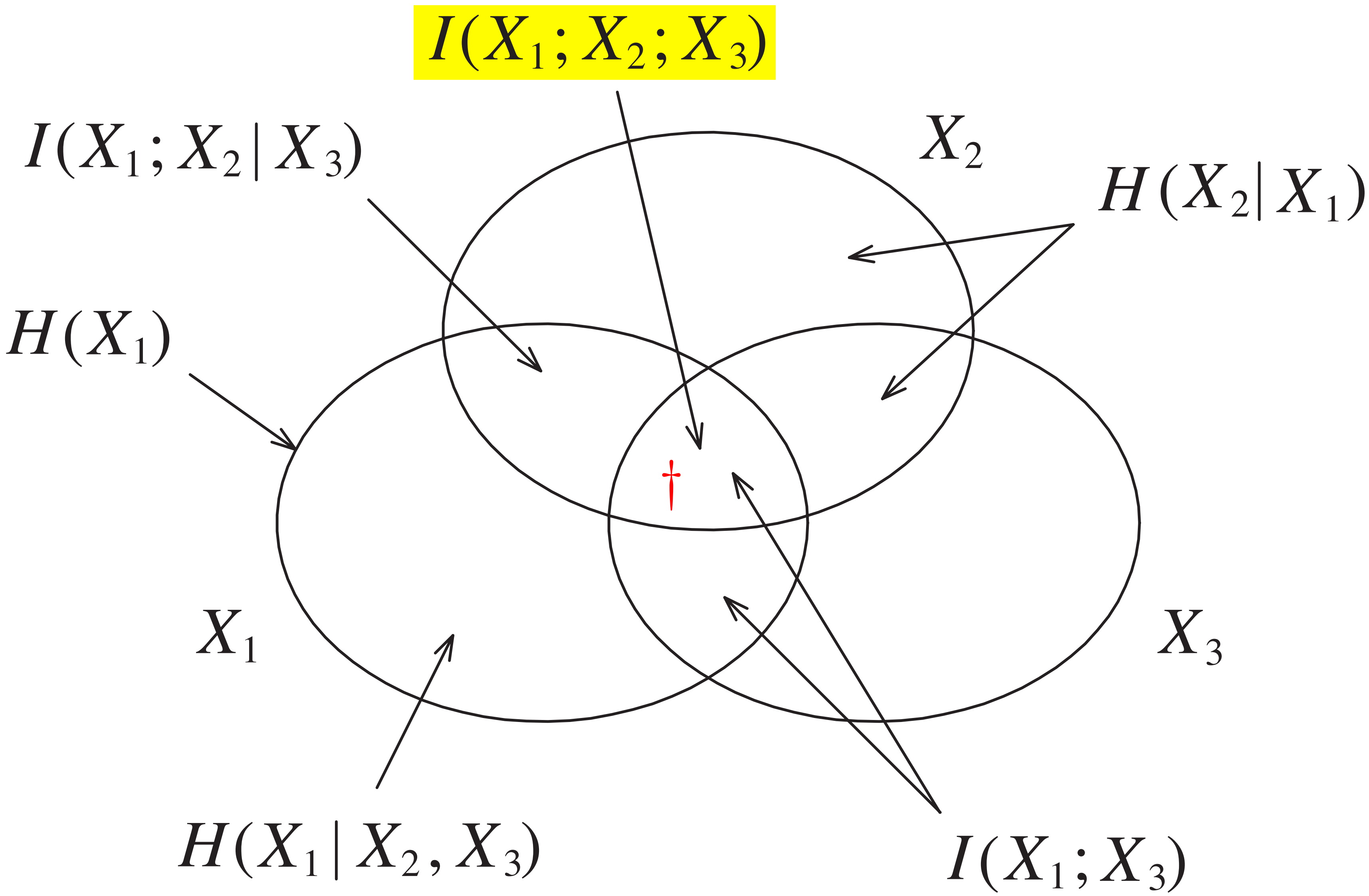

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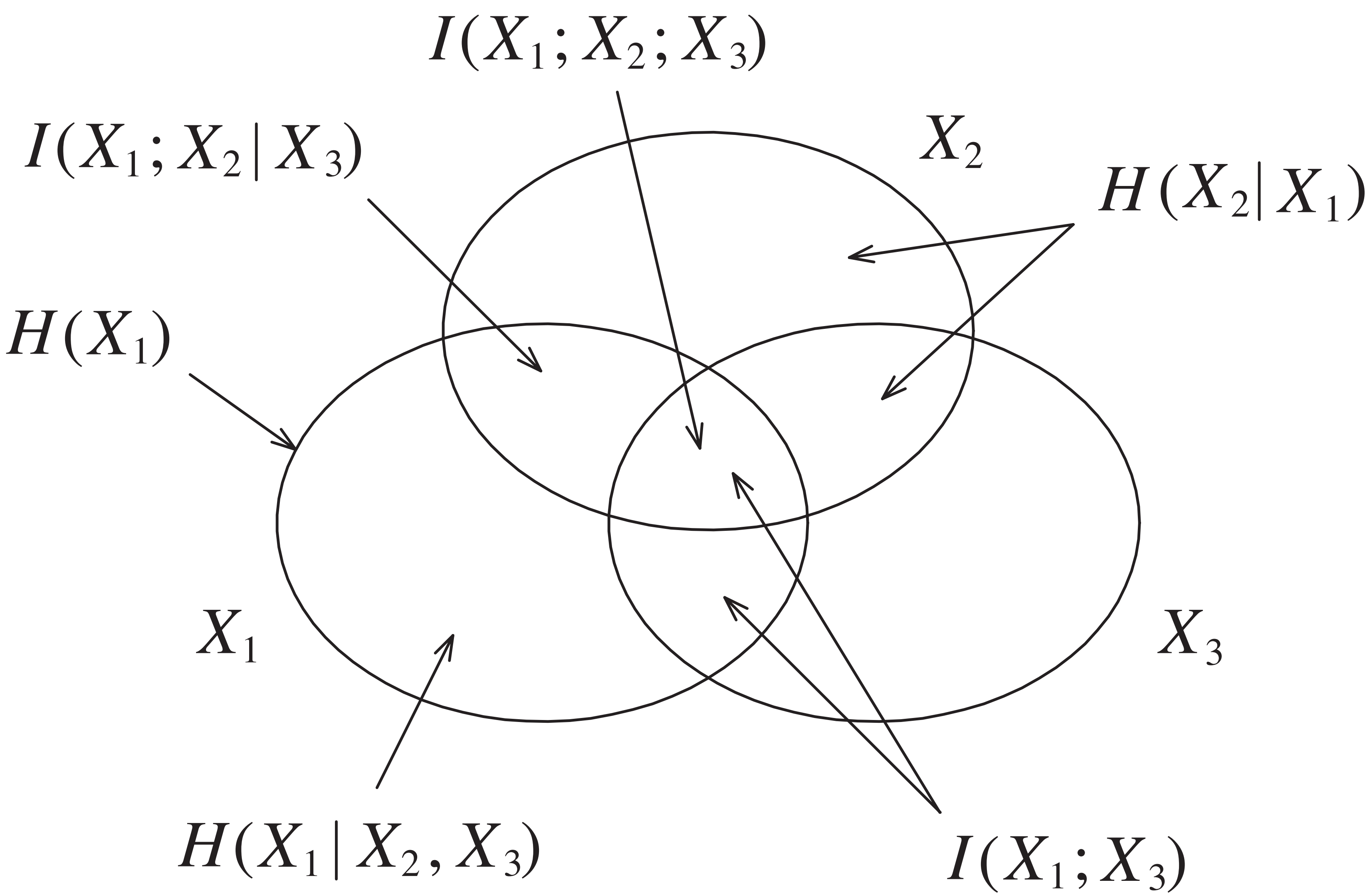

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- We will show that it is possible to construct r.v.'s X_1, X_2, and X_3 such that $\mu^*(\tilde{X}_1 \cap \tilde{X}_2 \cap \tilde{X}_3) < 0$.

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6. Now for distinct $1 \leq i, j, k \leq 3$,

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Lemma 3.8

$$I(X; Y|Z) = H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z).$$

Example 3.10

1. In this example, all entropies are in the base 2.

2. Let X_1 and X_2 be independent binary random variables with uniform distribution, i.e.,

$$\Pr\{X_i = 0\} = \Pr\{X_i = 1\} = 0.5,$$

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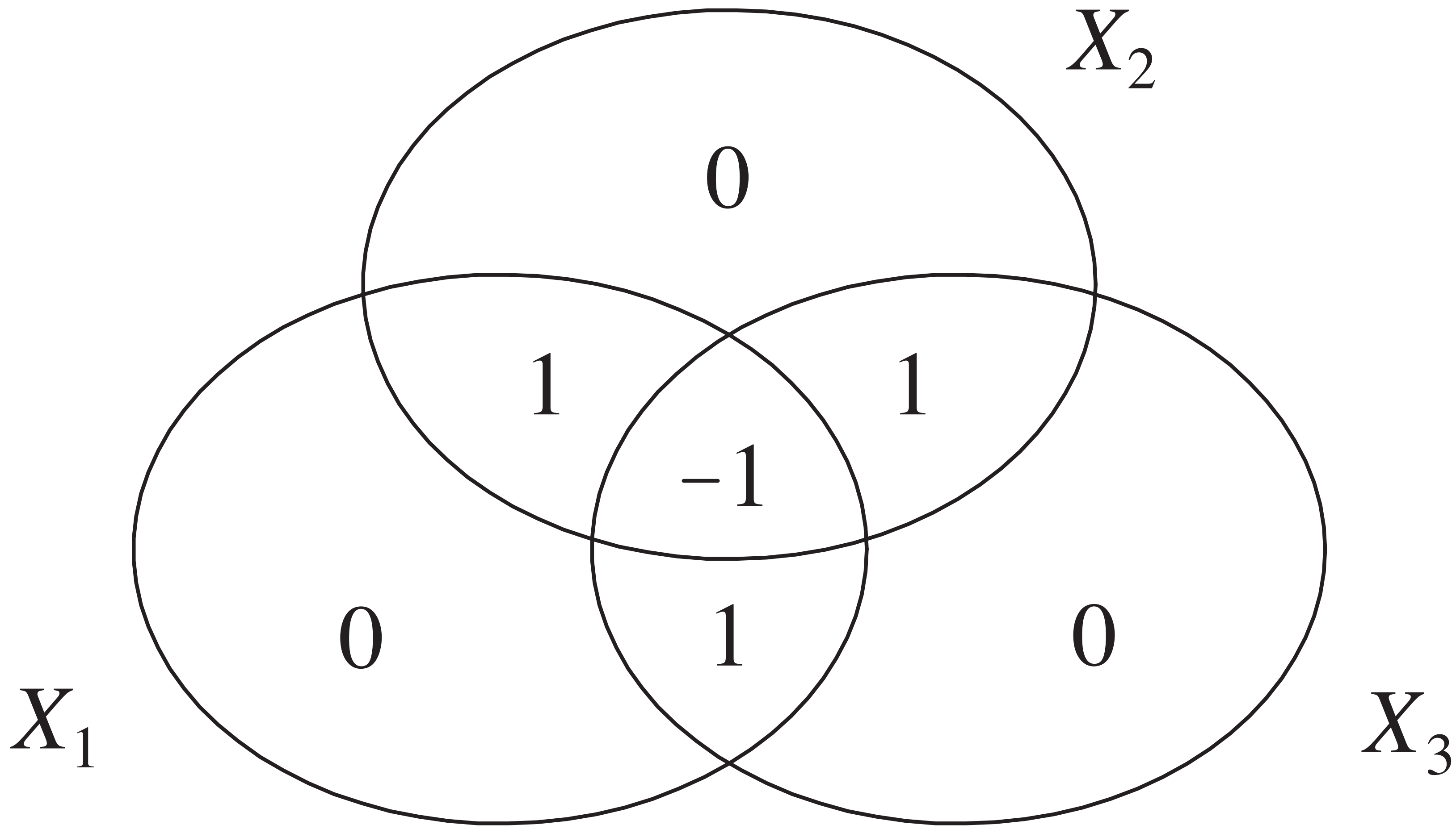

The information diagram for Example 3.10