3.3 Construction of the *I*-Measure μ*

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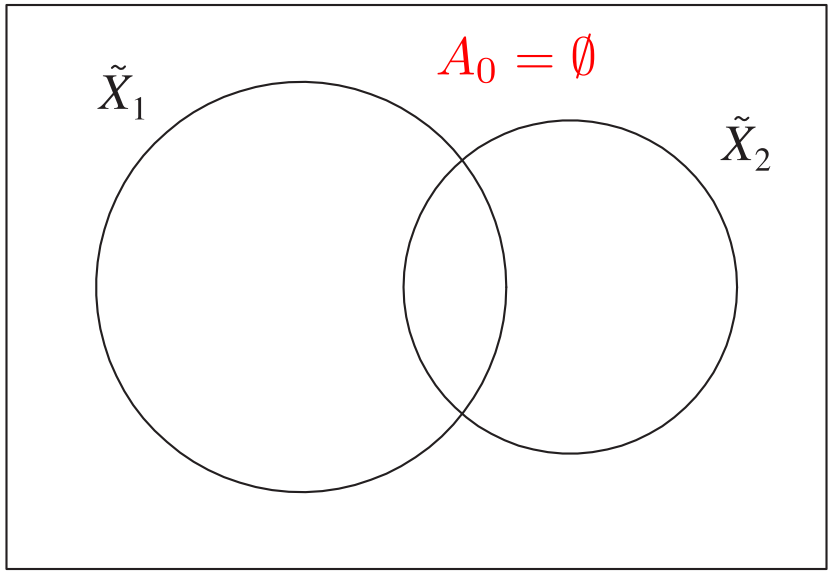
- A signed measure μ on $\mathcal{F}_n$ is completely specified by the values of μ on the atoms of $\mathcal{A}$, because $\mu(A_0)$ always vanishes.

Example For $n = 2$,

- $\Omega = \tilde{X}_1 \cup \tilde{X}_2$
- $A_0 = \tilde{X}_1^c \cap \tilde{X}_2^c = \emptyset$



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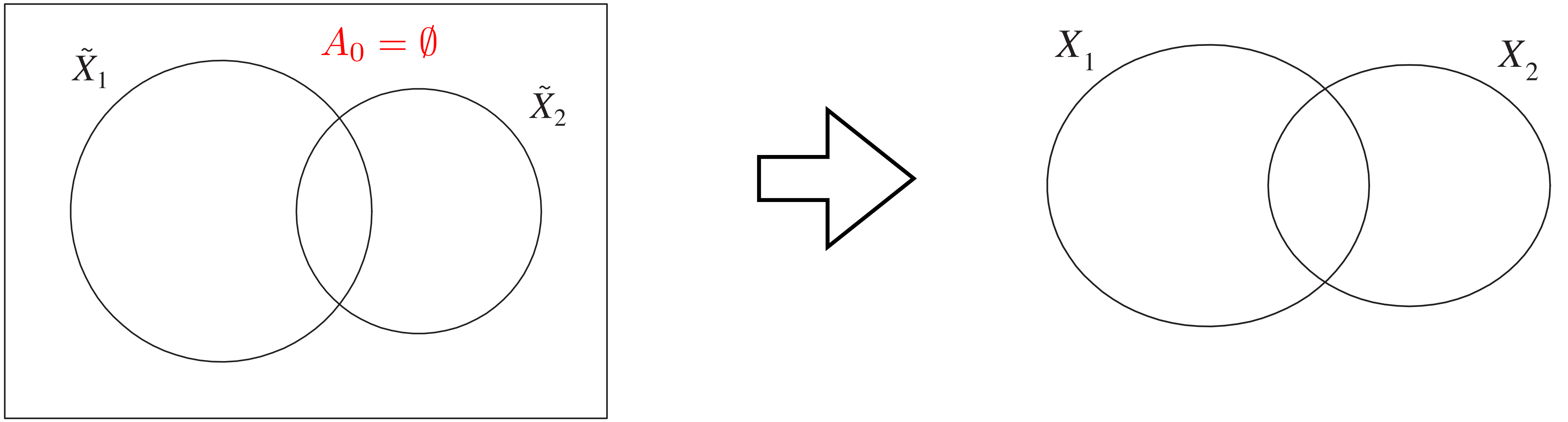

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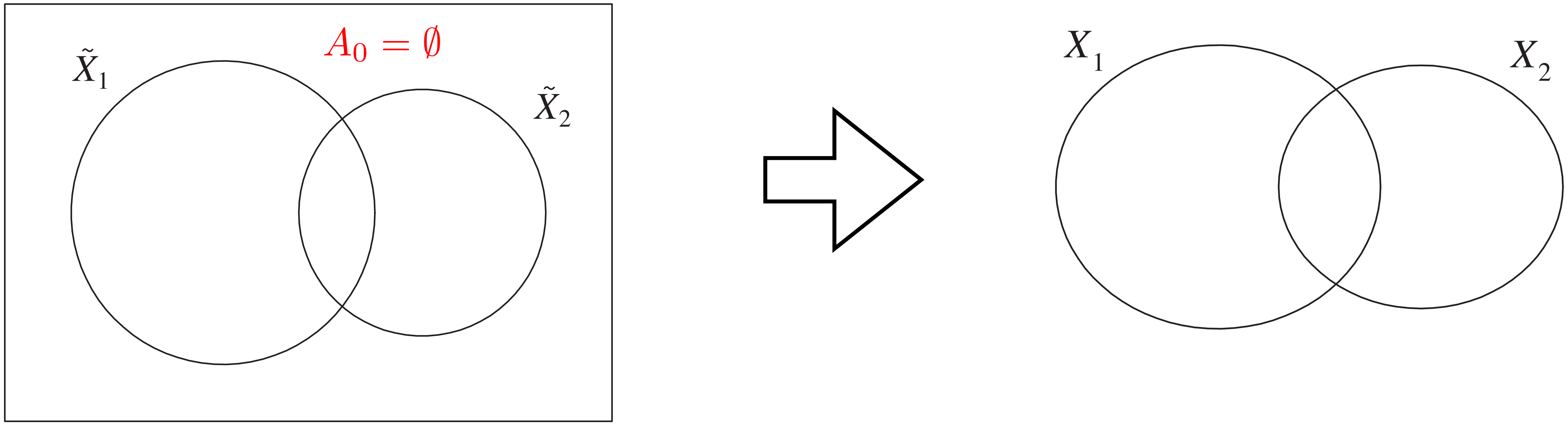

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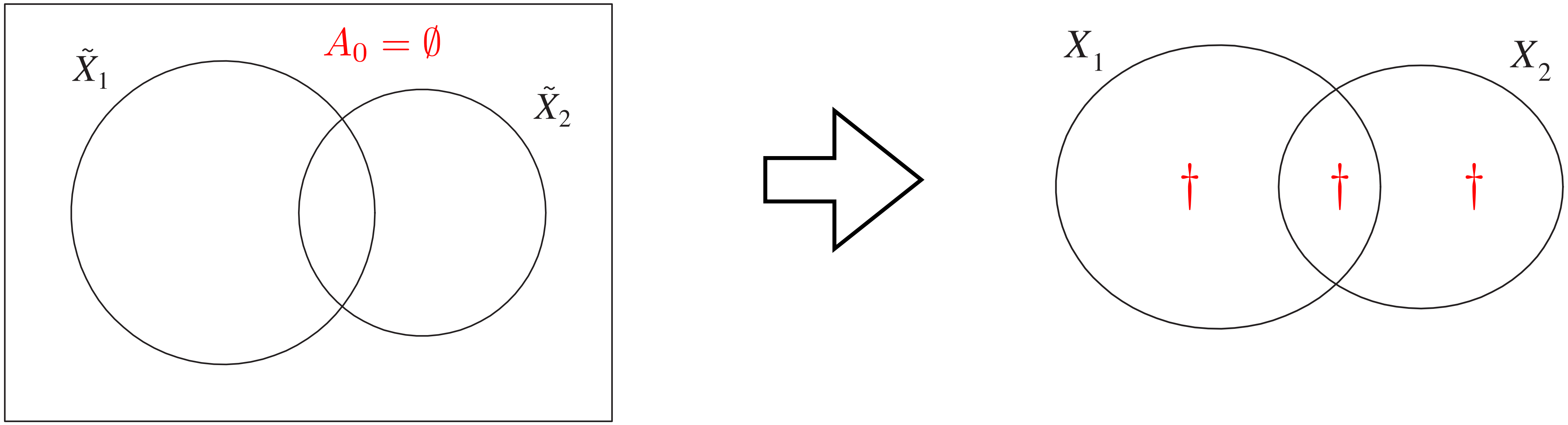

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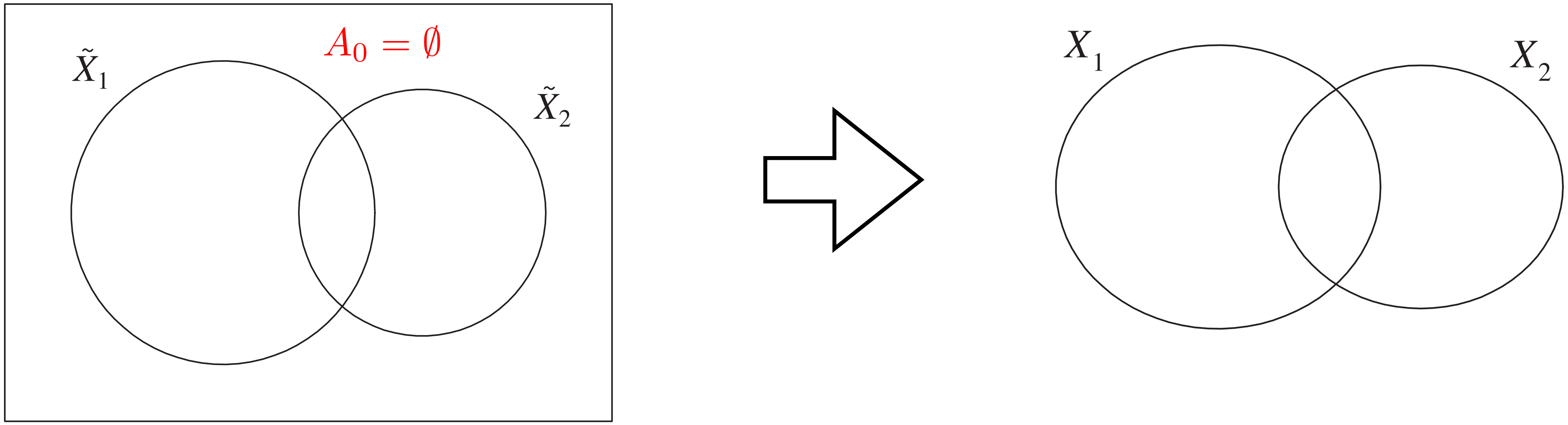

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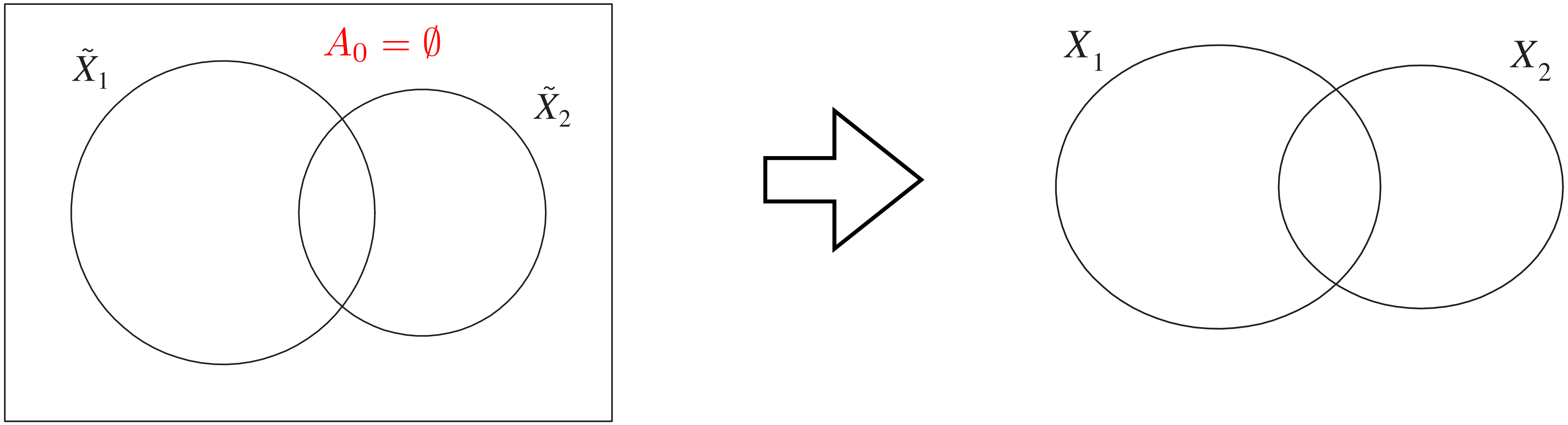

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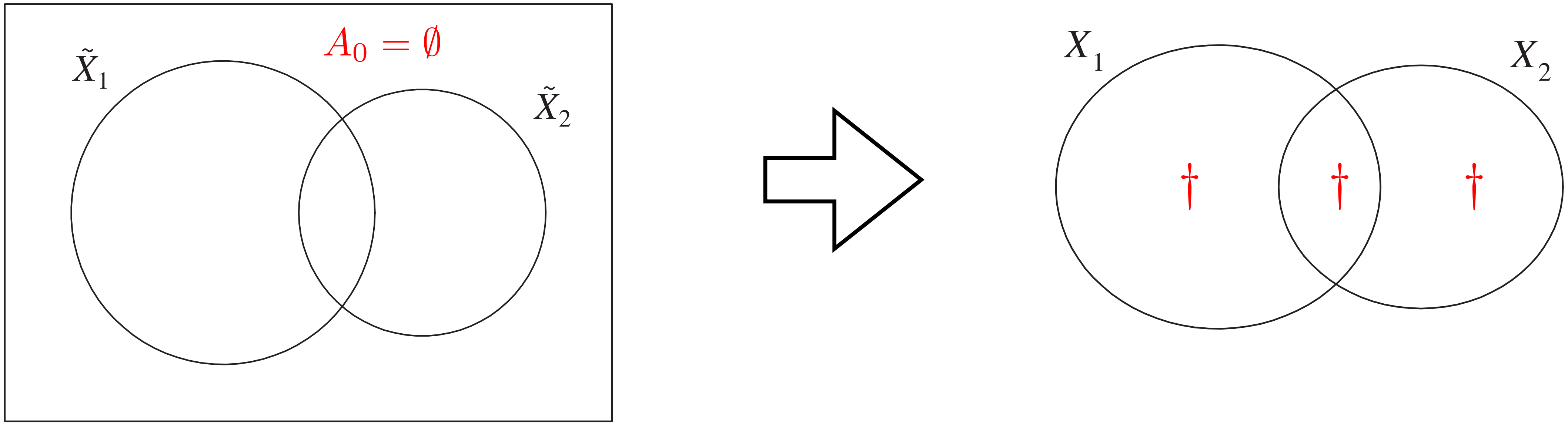

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$$\mathcal{B} = \left\{ \tilde{X}_G : \; G \text{ is a nonempty subset of } \mathcal{N}_n \right\}.$$

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Remark We have seen that a signed measure μ on $\mathcal{F}_n$ is completely specified by $\{\mu(A), A \in \mathcal{A}\}$, the set of values of μ on the nonempty atoms. This theorem says that μ can instead be specified by $\{\mu(B), B \in \mathcal{B}\}$, the set of values of μ on the unions.

Appendix 3.A

In this appendix, we show that for any $A \in \mathcal{A}$, the set of non-empty atoms of $\mathcal{F}_n$, $\mu(A)$ can be expressed as a linear combination of the values of μ on the unions of $\tilde{X}_i$'s.

- It is easy to check that for a set-additive function μ,

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2) \tag{1}$$

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- Note that (1) is a special case ($m = 2$) of the Inclusion-Exclusion Formula:

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{m} A_k\right) &= \sum_{1 \le i \le m} \mu(A_i) - \sum_{1 \le i < j \le m} \mu(A_i \cap A_j) + \cdots \\ &\quad + (-1)^{m+1} \mu(A_1 \cap A_2 \cap \cdots \cap A_m). \end{aligned}$$

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Theorem 3.19 (Variation of the Inclusion-Exclusion Formula)

$$\begin{aligned}\mu\left(\bigcap_{k=1}^{m} A_k - B\right) &= \sum_{1\le i\le m}\mu(A_i - B) - \sum_{1\le i<j\le m}\mu(A_i\cup A_j - B) + \cdots \\ &\quad +(-1)^{m+1}\mu(A_1\cup A_2\cup\cdots\cup A_m - B).\end{aligned}$$

Proof See textbook.

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7. Thus we have expressed $\mu(A)$ of a nonempty atom A as a linear combination of the values of μ on the unions of $\tilde{X}_i$'s.

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Theorem 3.6 Let

$$\mathcal{B} = \left\{ \tilde{X}_G : \ G \text{ is a nonempty subset of } \mathcal{N}_n \right\}.$$

Then a signed measure μ on $\mathcal{F}_n$ is completely specified by $\{\mu(B), B \in \mathcal{B}\}$, which can be any set of real numbers.

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$\mathbf{u}$ column k-vector of $\mu(A)$, $A \in \mathcal{A}$

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8. Hence, $\mu(B), B \in \mathcal{B}$ completely specifies a signed measure μ on $\mathcal{F}_n$.

Two Lemmas

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Construction of the I-Measure μ^* on $\mathcal{F}_n$

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Theorem 3.6 Let

$$\mathcal{B} = \left\{ \tilde{X}_G : \ G \text{ is a nonempty subset of } \mathcal{N}_n \right\}.$$

Then a signed measure μ on $\mathcal{F}_n$ is completely specified by $\{\mu(B), B \in \mathcal{B}\}$, which can be any set of real numbers.

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- That is, the value of μ^* on the union of a collection G of set variables is equal to the joint entropy of the collection G of random variables.

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- μ^* is meaningful if it is consistent with all Shannon's information measures via the substitution of symbols, i.e., the following must hold for all (not necessarily disjoint) subsets G, G', G'' of $\mathcal{N}_n$ where G and G' are nonempty:

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Implications

- Can formally regard Shannon's information measures for n r.v.'s as the unique signed measure μ^* defined on $\mathcal{F}_n$.

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Implications

- Can formally regard Shannon's information measures for n r.v.'s as the unique signed measure μ^* defined on $\mathcal{F}_n$.
- Can employ set-theoretic tools to manipulate expressions of Shannon's information measures.

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2. In order for any signed measure μ to be consistent with all Shannon's information measures, it must be consistent with all entropies, i.e.,

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2. In order for any signed measure μ to be consistent with all Shannon's information measures, it must be consistent with all entropies, i.e.,

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3. Therefore μ^* is the unique signed measure that is consistent with all Shannon's information measures. It remains to show that μ^* is consistent with all Shannon's information measures.

4. This is shown by considering for all subsets G, G', G'' of $\mathcal{N}_n$,

$$\mu^*(\tilde{X}_G \cap \tilde{X}_{G'} - \tilde{X}_{G''})$$

Lemma 3.7

$$\mu(A \cap B - C) = \mu(A \cup C) + \mu(B \cup C) - \mu(A \cup B \cup C) - \mu(C).$$

Theorem 3.9 μ^* is the unique signed measure on $\mathcal{F}_n$ which is consistent with all Shannon's information measures.

Proof

1. Recall that μ^* is defined by

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$$\mu(A\cap B-C) = \mu(A\cup C)+\mu(B\cup C)-\mu(A\cup B\cup C)-\mu(C).$$

Lemma 3.8

$$I(X;Y|Z) = H(X,Z)+H(Y,Z)-H(X,Y,Z)-H(Z).$$

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&\quad = H(X_{G\cup G''}) + H(X_{G'\cup G''}) - H(X_{G\cup G'\cup G''}) - H(X_{G''}) \\
&\quad = I(X_G; X_{G'} | X_{G''}).
\end{aligned}$$

Lemma 3.7

$$\mu(A\cap B - C) = \mu(A\cup C) + \mu(B\cup C) - \mu(A\cup B\cup C) - \mu(C).$$

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