香港中文大學
The Chinese University of Hong Kong

2.5 Informational Divergence

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- $D(\cdot\|\cdot)$ does not satisfy the triangular inequality.
- Also called *relative entropy* or the *Kullback-Leibler distance*.

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$$\ln a \leq a - 1$$

with equality if and only if $a = 1$.

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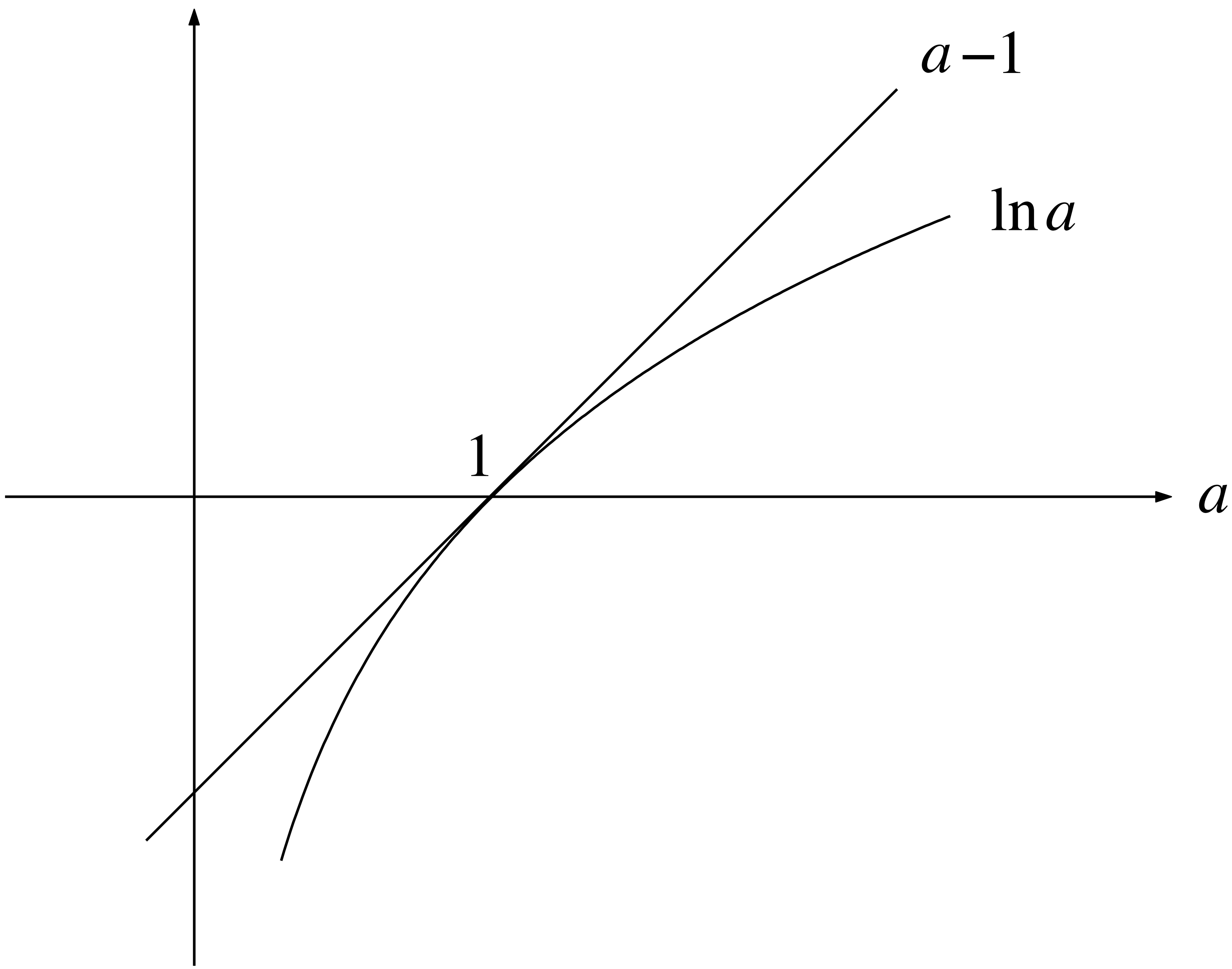

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with equality if and only if $a = 1$.

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$$D(p\|q) \geq 0 \tag{1}$$

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Example:

$$a_1 \log \frac{a_1}{b_1} + a_2 \log \frac{a_2}{b_2} \geq (a_1 + a_2) \log \frac{a_1 + a_2}{b_1 + b_2}.$$

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$$\sum_i a_i \log \frac{a_i}{b_i} \geq \left(\sum_i a_i\right) \log \frac{\sum_i a_i}{\sum_i b_i} \tag{1}$$

Moreover, equality holds if and only if $\frac{a_i}{b_i} = \mathit{constant}$ for all i.

Proof

1. Let $a'_i = a_i / \sum_j a_j$ and $b'_i = b_i / \sum_j b_j$. Then $\{a'_i\}$ and $\{b'_i\}$ are probability distributions.

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which implies (1).

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3. Equality holds if and only if for all i,

$$a_i' = b_i' \quad \text{or} \quad \frac{a_i}{b_i} = \mathit{constant}.$$

The theorem is proved.

Theorem 2.32 (Log-Sum Inequality) For positive numbers $a_1, a_2, \cdots$ and nonnegative numbers $b_1, b_2, \cdots$ such that $\sum_i a_i < \infty$ and $0 < \sum_i b_i < \infty$,

$$\sum_i a_i \log \frac{a_i}{b_i} \geq \left(\sum_i a_i\right) \log \frac{\sum_i a_i}{\sum_i b_i} \qquad (1)$$

Moreover, equality holds if and only if $\frac{a_i}{b_i} = constant$ for all i.

Proof

1. Let $a_i' = a_i / \sum_j a_j$ and $b_i' = b_i / \sum_j b_j$. Then $\{a_i'\}$ and $\{b_i'\}$ are probability distributions.

2. Using the divergence inequality, we have

$$\begin{aligned}
0 &\leq \sum_i a_i' \log \frac{a_i'}{b_i'} \\
&= \sum_i \frac{a_i}{\sum_j a_j} \log \frac{a_i / \sum_j a_j}{b_i / \sum_j b_j} \\
&= \frac{1}{\sum_j a_j} \left[\sum_i a_i \log \frac{a_i / \sum_j a_j}{b_i / \sum_j b_j}\right] \\
&= \frac{1}{\sum_j a_j} \left[\sum_i a_i \log \frac{a_i}{b_i} - \sum_i a_i \log \frac{\sum_j a_j}{\sum_j b_j}\right] \\
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\end{aligned}$$

which implies (1).

3. Equality holds if and only if for all i,

$$a_i' = b_i' \quad \text{or} \quad \frac{a_i}{b_i} = constant.$$

The theorem is proved.

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- The log-sum inequality also implies the divergence inequality. (Exercise)
- The two inequalities are equivalent.

Theorem 2.33 (Pinsker's Inequality)

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- For a sequence of probability distributions q_k, as $k \to \infty$, if $D(p\|q_k) \to 0$ or $D(q_k\|p) \to 0$, then $V(p,q_k) = V(q_k,p) \to 0$.

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- That is, "convergence in divergence" is a stronger notion than "convergence in variational distance."
- See Problems 23 and 24 for details.