香港中文大學
The Chinese University of Hong Kong

2.2 Shannon’s Information Measures

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- A bit in information theory is different from a bit in computer science.

Remark $H(X)$ depends only on the distribution of X but not on the actual values taken by X, hence also write $H(p_X)$.

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$$p_X(0) = 0.3,\ p_X(1) = 0.7$$

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- In probability theory, when $Eg(X)$ is considered, usually $g(x)$ depends only on the value of x but not on $p(x)$.

Binary Entropy Function

- For $0 \leq \gamma \leq 1$, define the binary entropy function

$$h_b(\gamma) = -\gamma \log \gamma - (1-\gamma) \log(1-\gamma)$$

with the convention $0 \log 0 = 0$, as by L'Hopital's rule,

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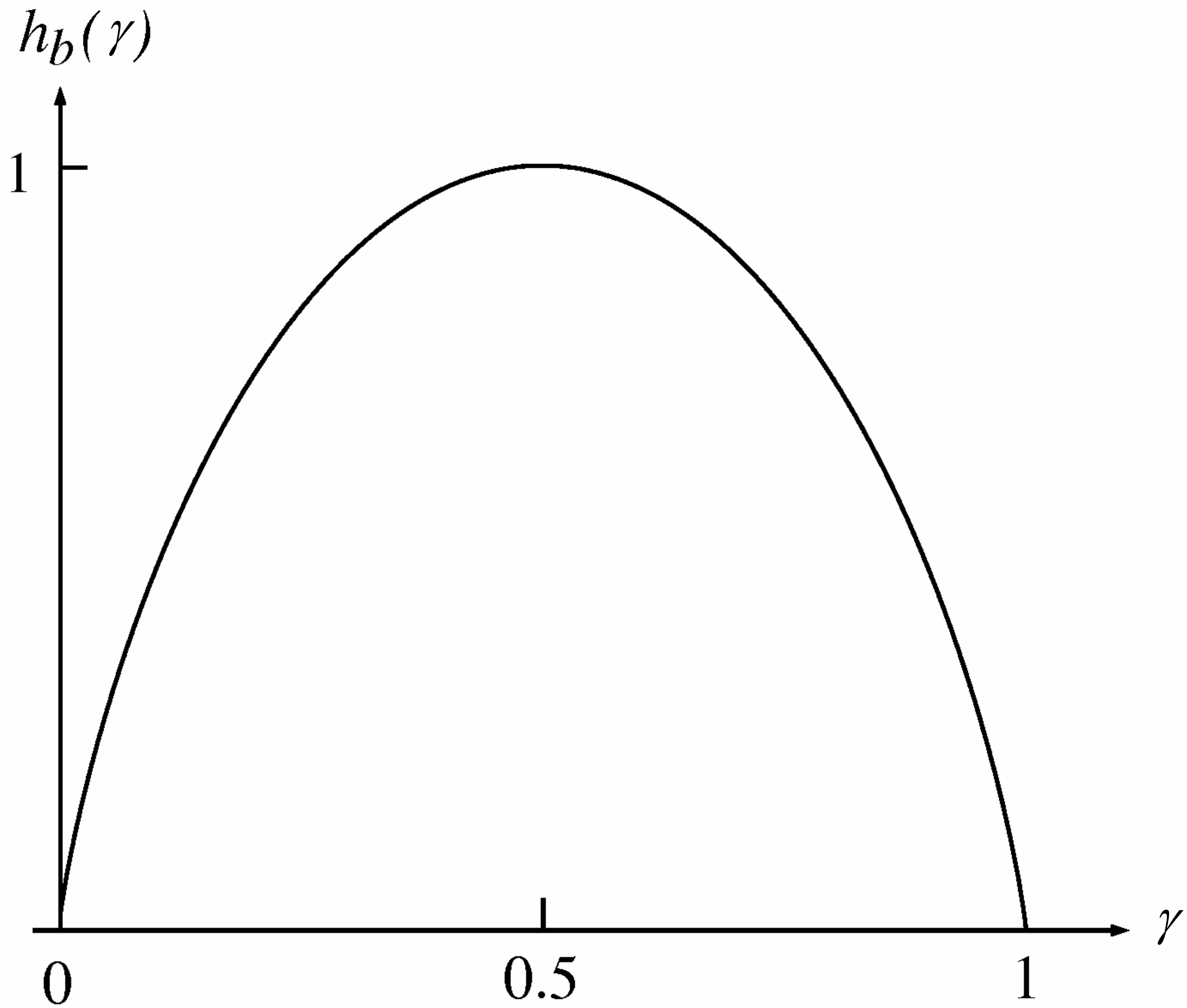
- $h_b(\gamma)$ achieves the maximum value 1 when $\gamma = \frac{1}{2}$.



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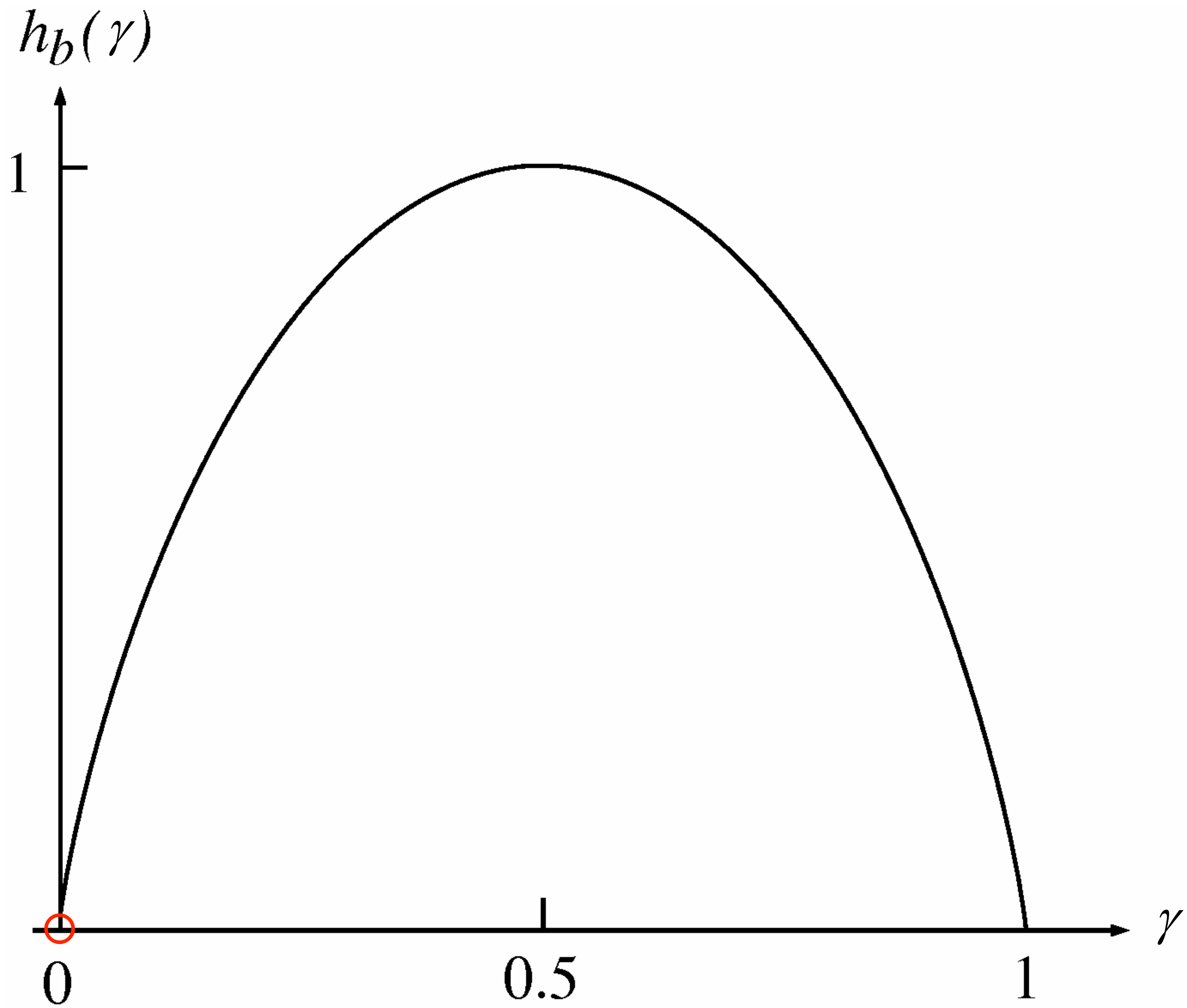
γ



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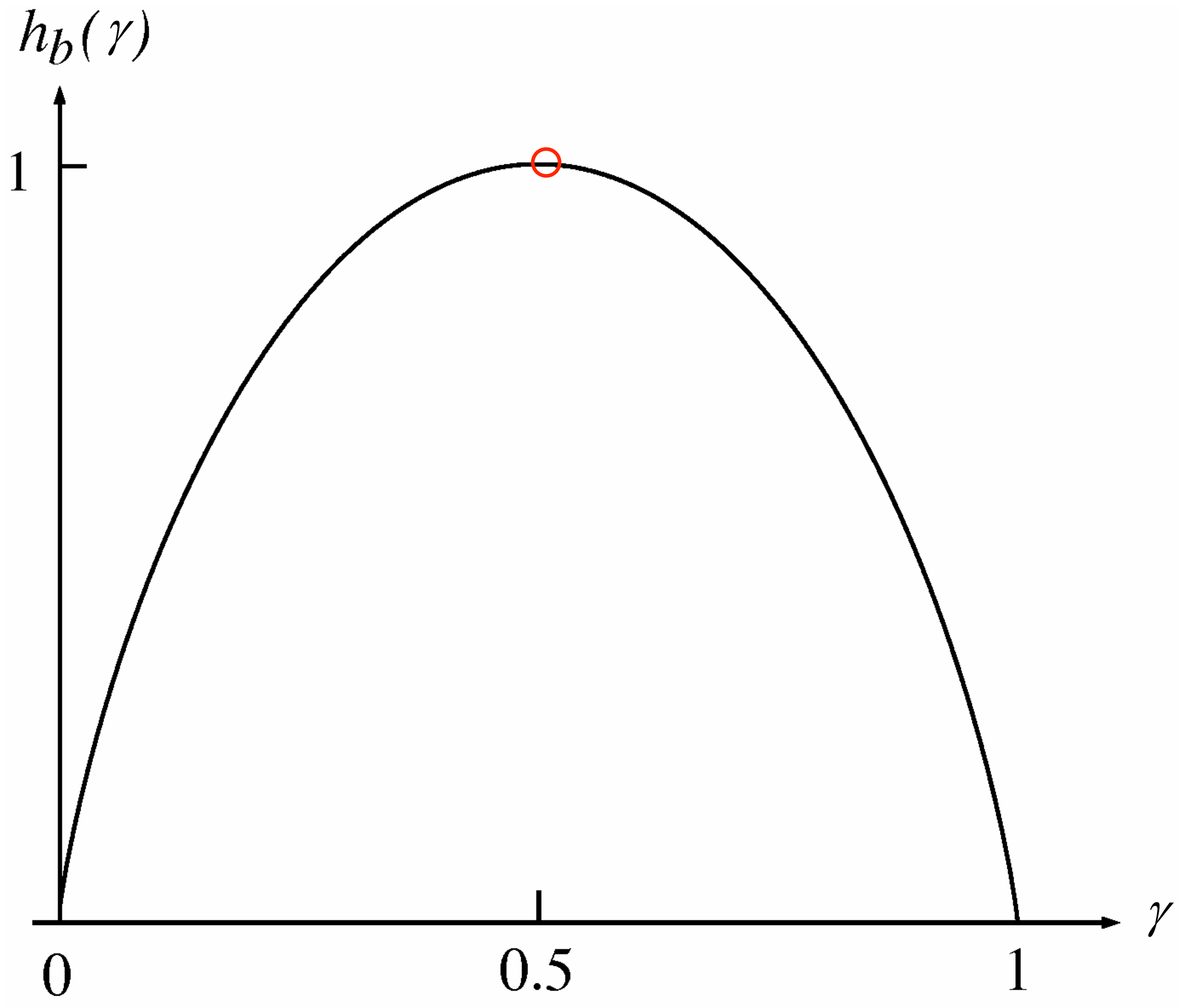
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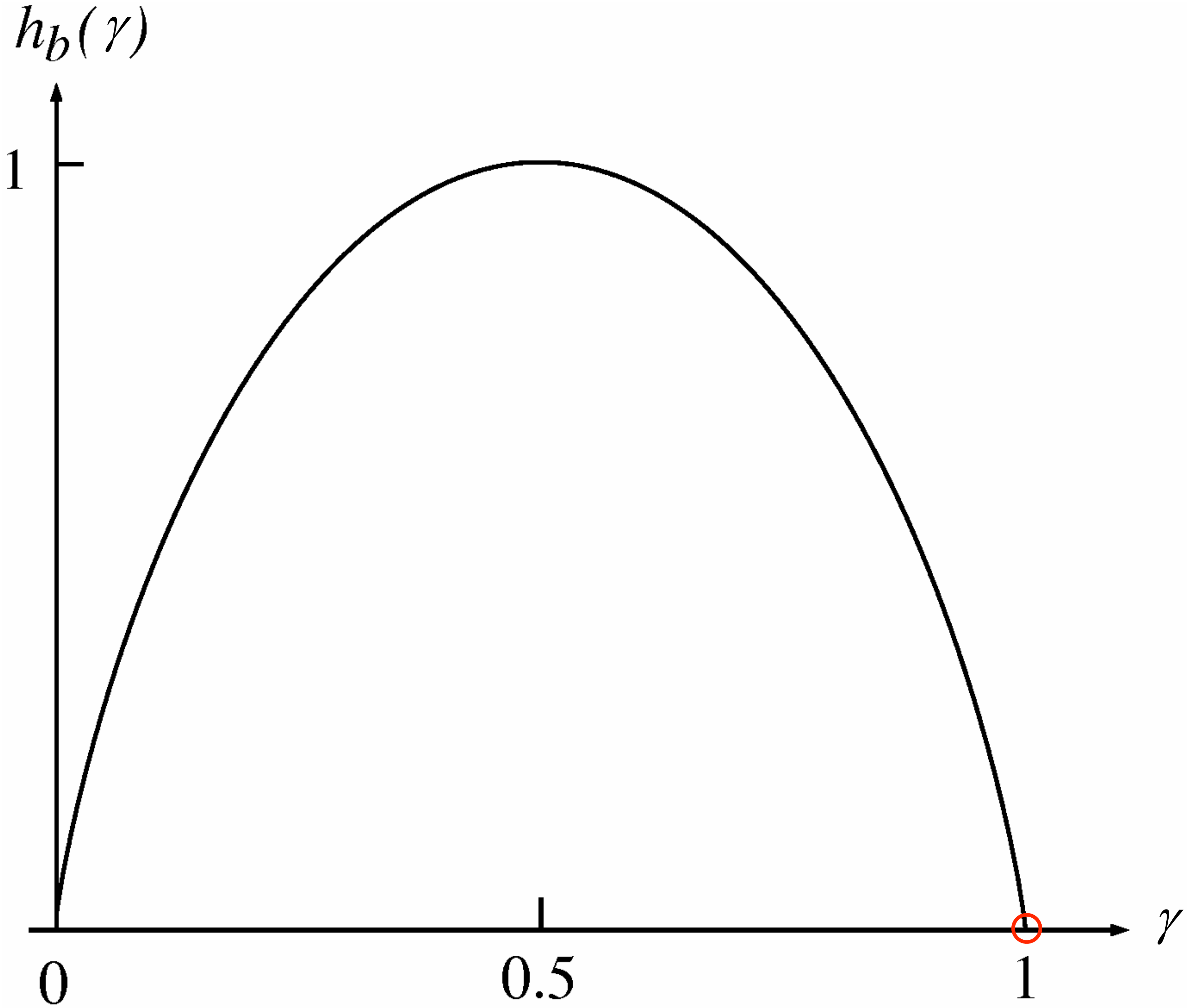
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- This interpretation will be justified in terms of the source coding theorem.

Definition 2.14 The joint entropy $H(X,Y)$ of a pair of random variables X and Y is defined as

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Proposition 2.16

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Proof

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Definition 2.17 For random variables X and Y, the mutual information between X and Y is defined as

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Proposition 2.18 The mutual information between a random variable X and itself is equal to the entropy of X, i.e., $I(X;X) = H(X)$.

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Remark The entropy of X is sometimes called the *self-information* of X.

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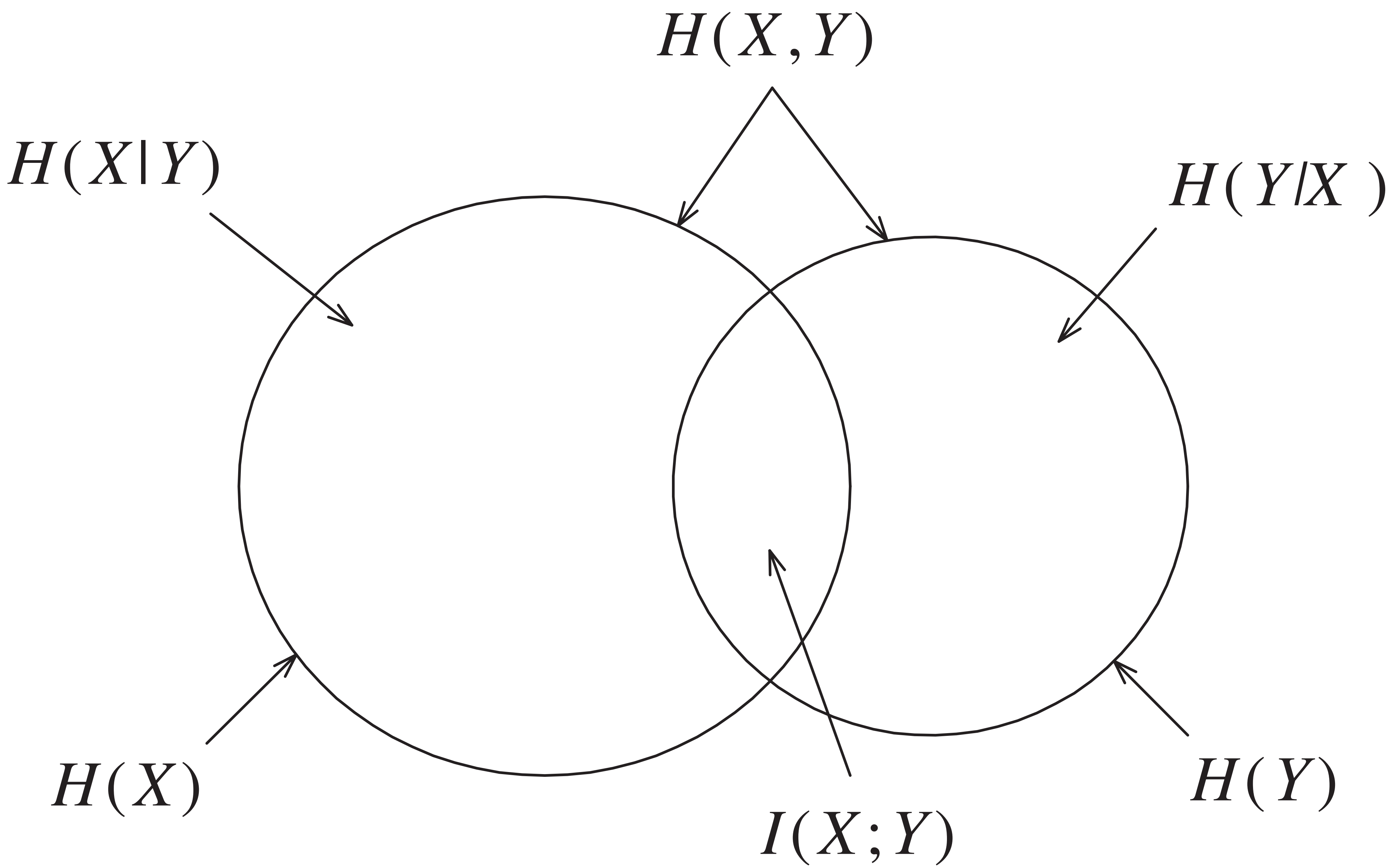
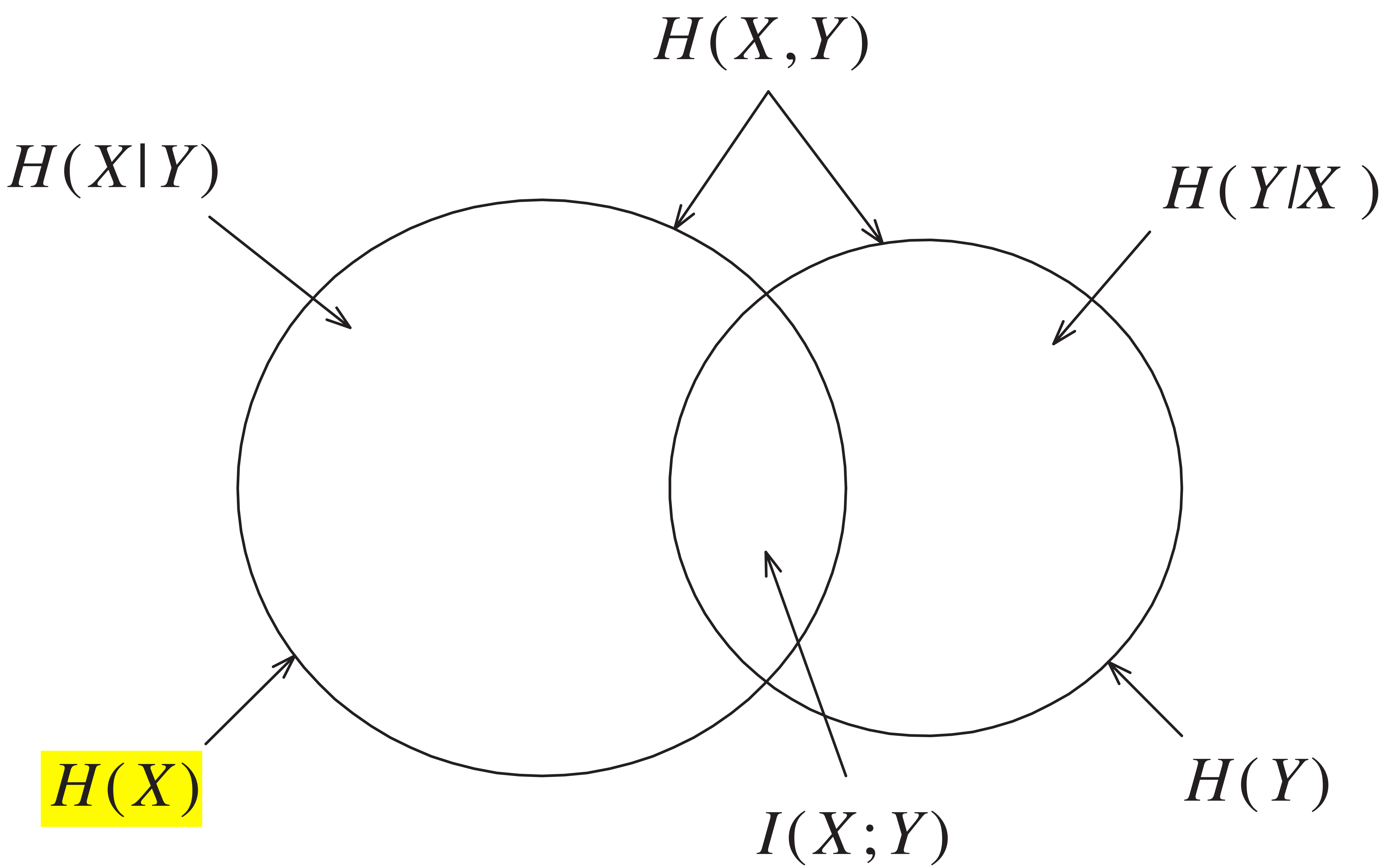
where μ is a set-additive function and A and B are sets.

Information Diagram



Information Diagram



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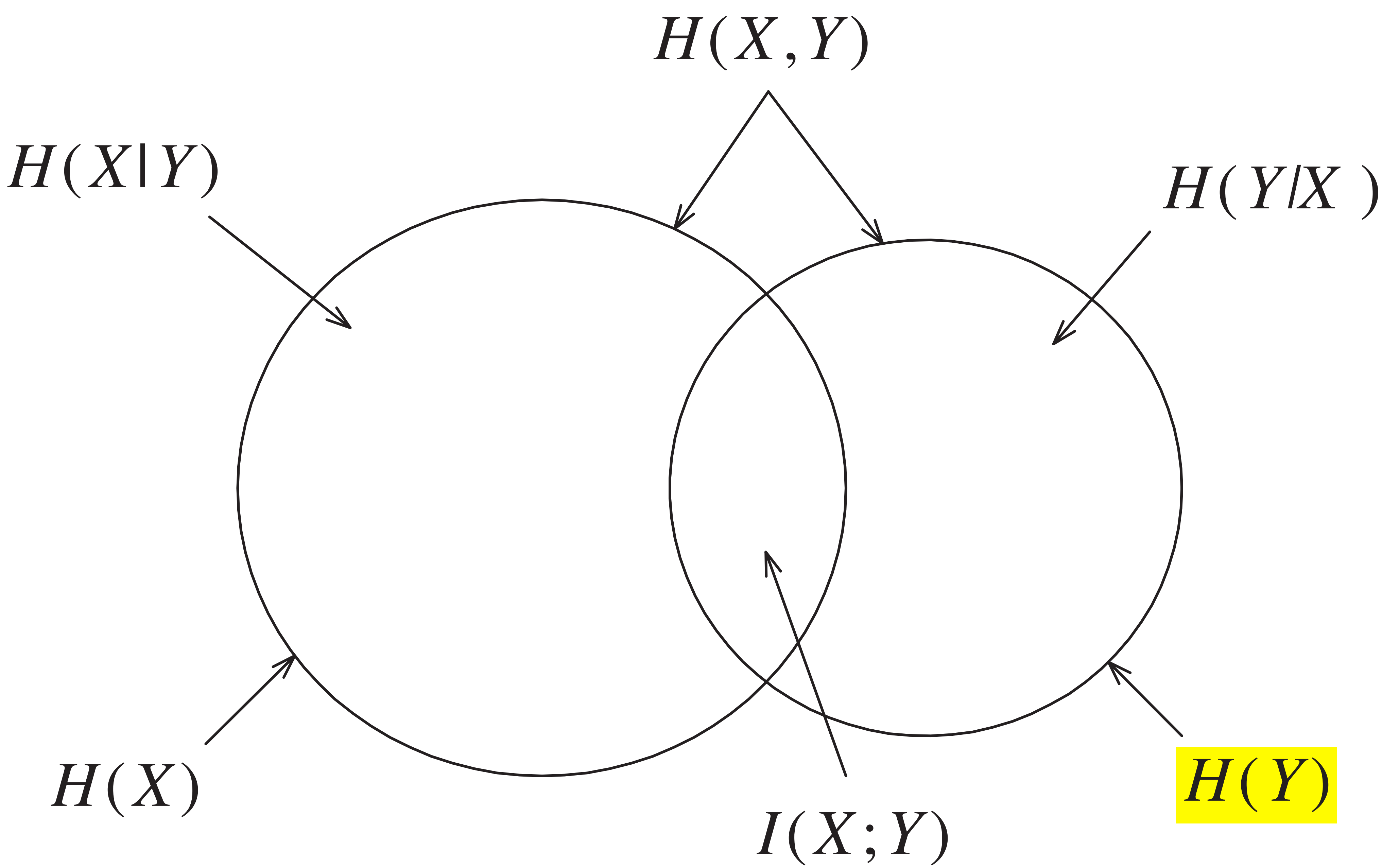

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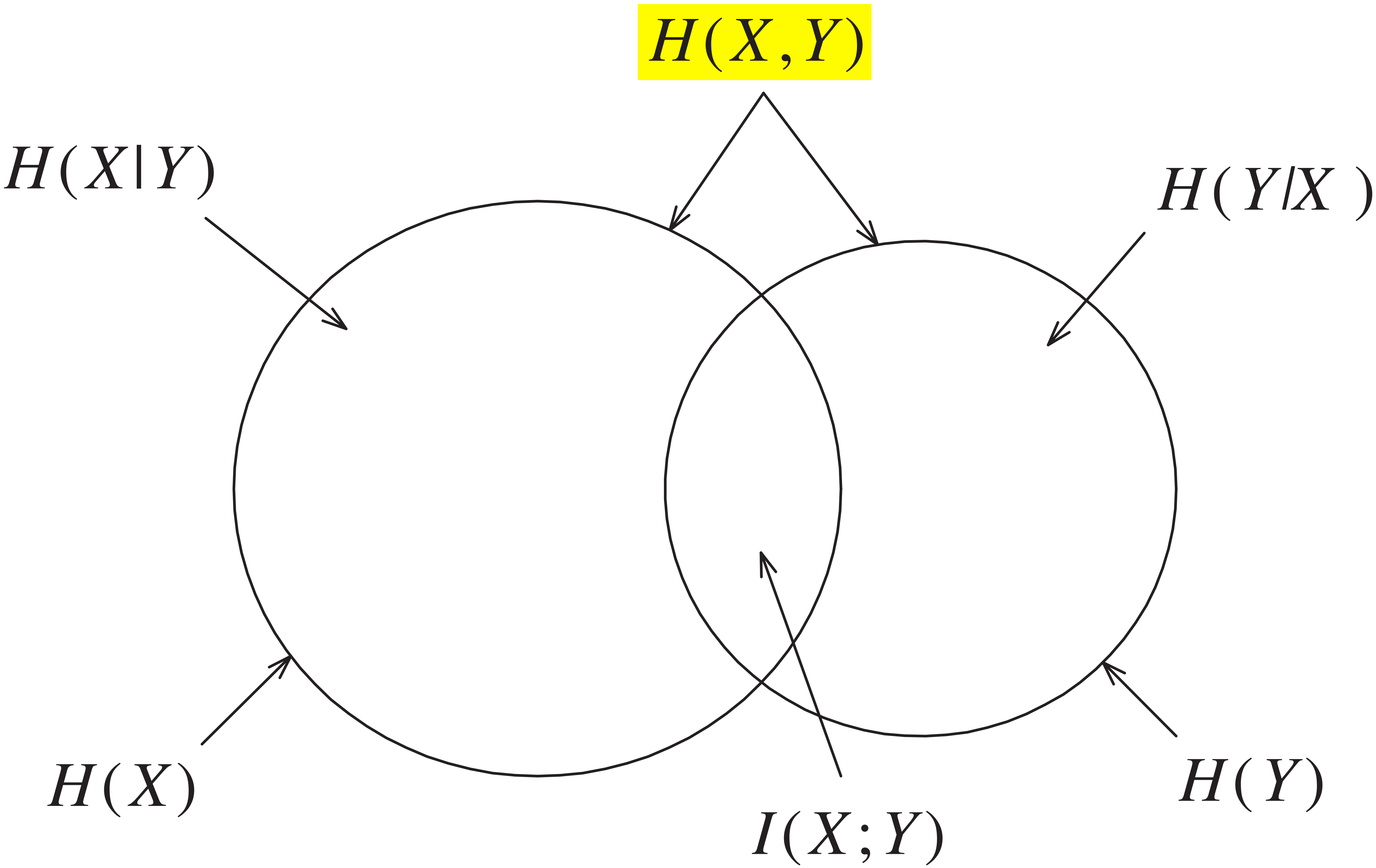

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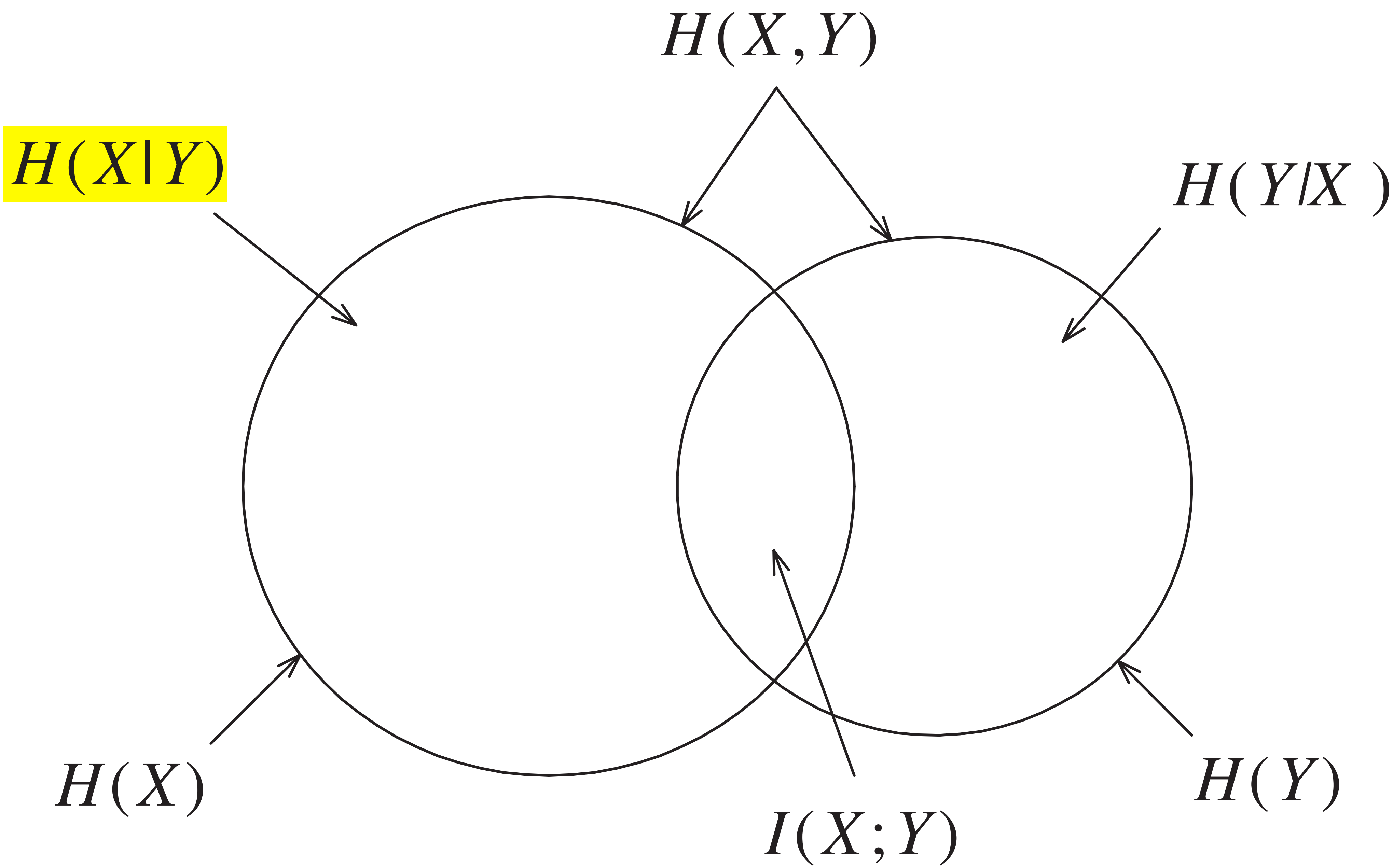

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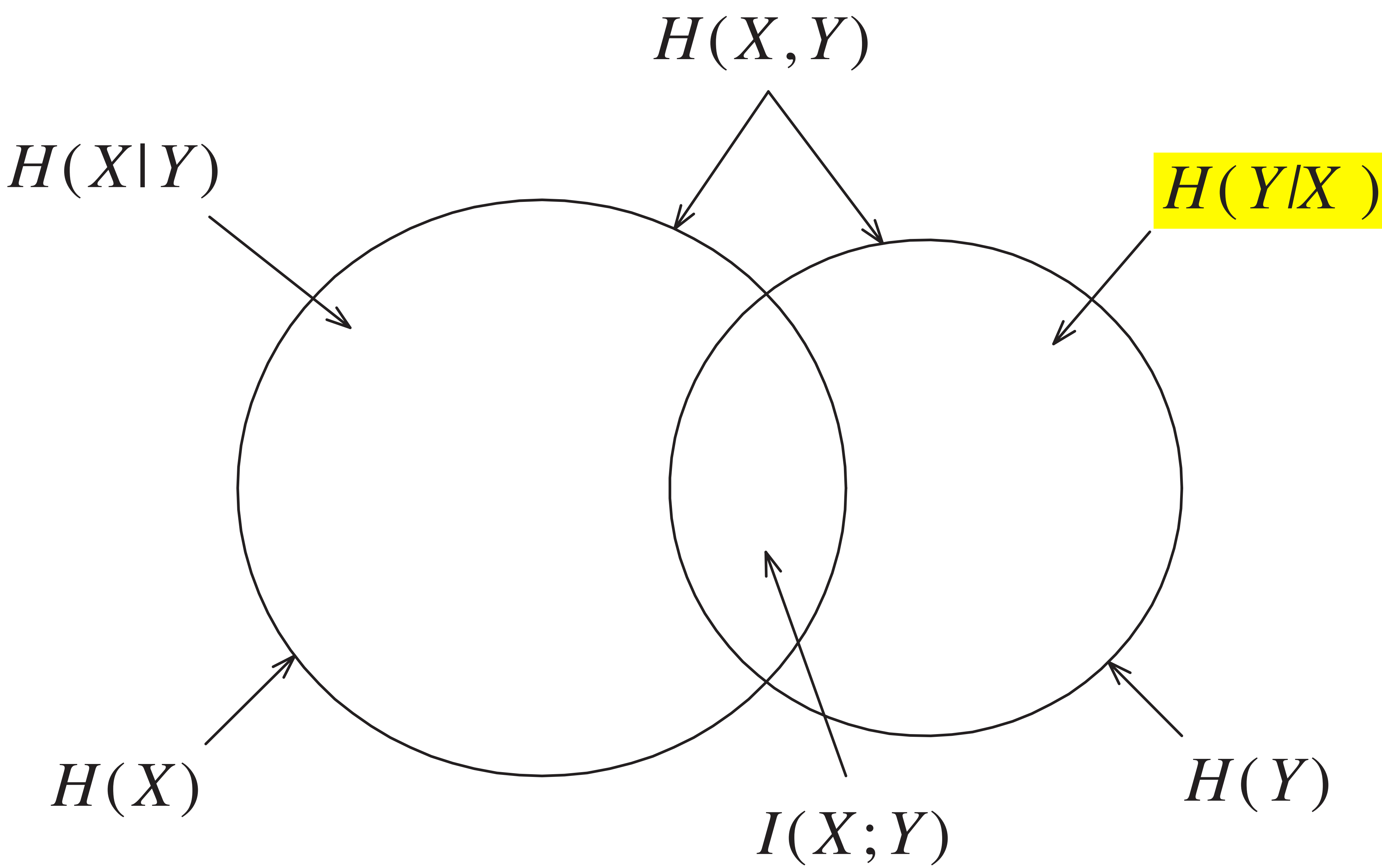

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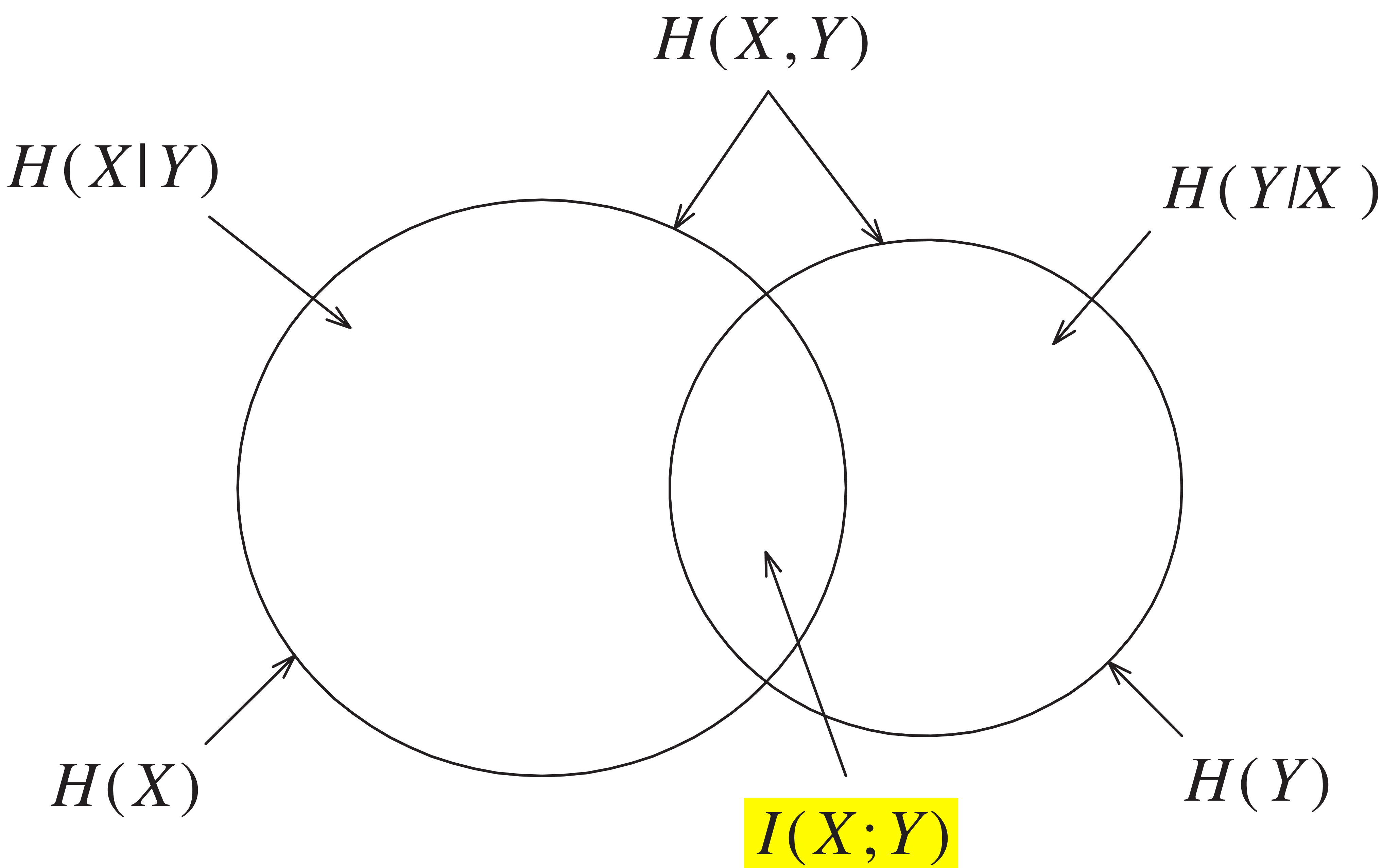

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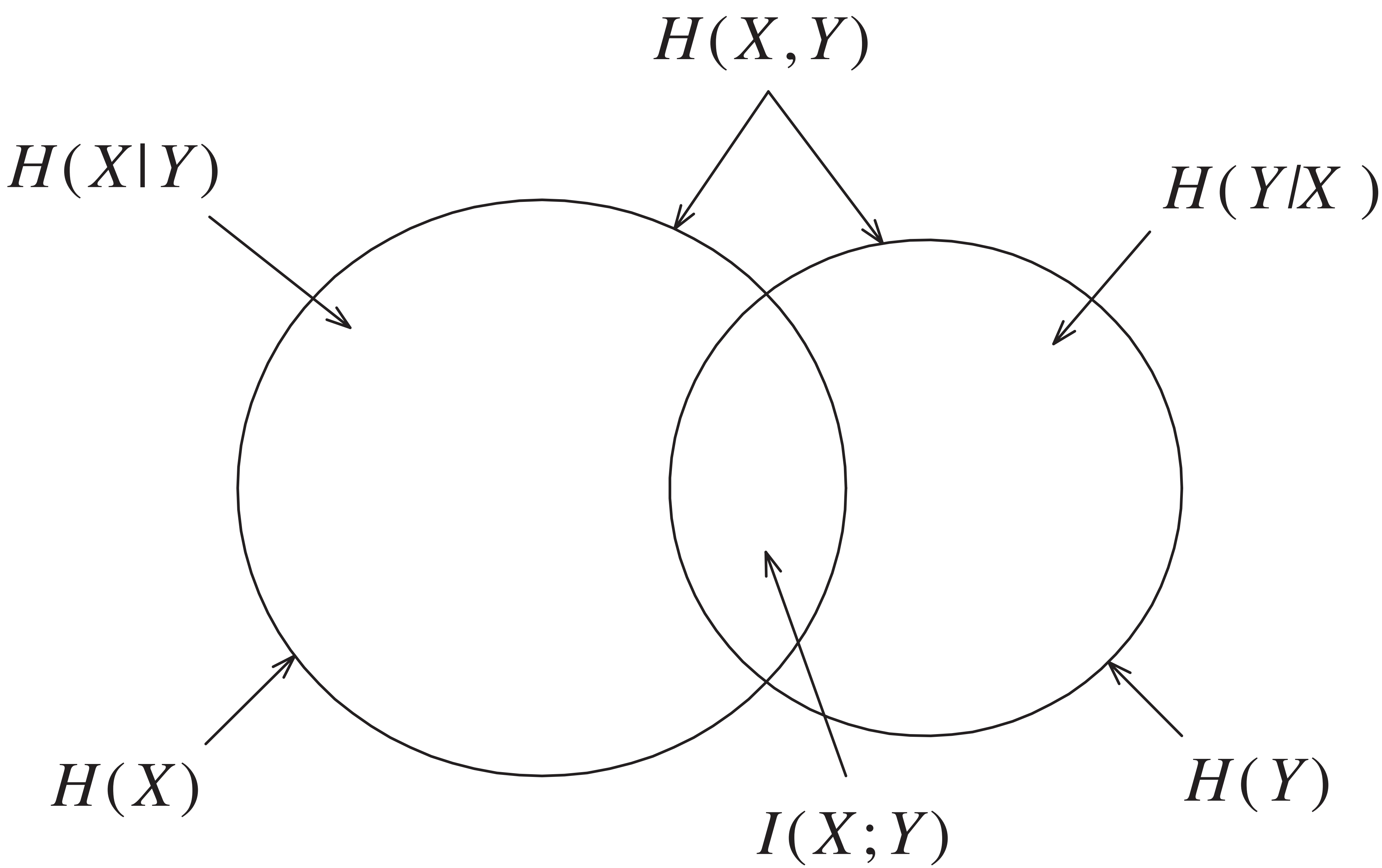

Definition 2.20 For random variables X, Y and Z, the mutual information between X and Y conditioning on Z is defined as

$$I(X;Y|Z) = \sum_{x,y,z} p(x,y,z) \log \frac{p(x,y|z)}{p(x|z)p(y|z)} = E \log \frac{p(X,Y|Z)}{p(X|Z)p(Y|Z)}.$$

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$$\begin{aligned} I(X;Y|Z) &= H(X|Z) - H(X|Y,Z), \\ I(X;Y|Z) &= H(Y|Z) - H(Y|X,Z), \end{aligned}$$

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