香港中文大學
The Chinese University of Hong Kong

Chapter 2
Information Measures

The Chinese University of Hong Kong

In this chapter – basic tools

In this chapter − basic tools

- Probability preliminaries

In this chapter − basic tools

- Probability preliminaries
- Shannon's information measures

In this chapter – basic tools

- Probability preliminaries
- Shannon's information measures
- Other useful information measures

In this chapter − basic tools

- Probability preliminaries
- Shannon's information measures
- Other useful information measures
- Some useful identities and inequalities

2.1 Independence and Markov Chain

Notations

Notations

X discrete random variable taking values in $\mathcal{X}$

Notations

X	discrete random variable taking values in $\mathcal{X}$
$\{p_X(x)\}$	probability distribution for X

Notations

X	discrete random variable taking values in $\mathcal{X}$
$\{p_X(x)\}$	probability distribution for X
$\mathcal{S}_X$	support of X, i.e., $\{\, x \in \mathcal{X} \mid p_X(x) > 0 \,\}$

Notations

X	discrete random variable taking values in $\mathcal{X}$
$\{p_X(x)\}$	probability distribution for X
$\mathcal{S}_X$	support of X, i.e., $\{\, x \in \mathcal{X} \mid p_X(x) > 0 \,\}$

- If $\mathcal{S}_X = \mathcal{X}$, we say that p is strictly positive.

Notations

X	discrete random variable taking values in $\mathcal{X}$
$\{p_X(x)\}$	probability distribution for X
$\mathcal{S}_X$	support of X, i.e., $\{\, x \in \mathcal{X} \mid p_X(x) > 0 \,\}$

- If $\mathcal{S}_X = \mathcal{X}$, we say that p is strictly positive.
- Non-strictly positive distributions are dangerous – see Proposition 2.12 to be discussed later.

Definition 2.1 Two random variables X and Y are independent, denoted by $X \perp Y$, if

$$p(x, y) = p(x)p(y)$$

for all x and y (i.e., for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$).

Definition 2.1 Two random variables X and Y are independent, denoted by $X \perp Y$, if

$$p(x, y) = p(x)p(y)$$

for all x and y (i.e., for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$).

Definition 2.2 (Mutual Independence) For $n \geq 3$, random variables $X_1, X_2, \cdots, X_n$ are mutually independent if

$$p(x_1, x_2, \cdots, x_n) = p(x_1)p(x_2) \cdots p(x_n)$$

for all $x_1, x_2, \cdots, x_n$.

Definition 2.1 Two random variables X and Y are independent, denoted by $X \perp Y$, if

$$p(x, y) = p(x)p(y)$$

for all x and y (i.e., for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$).

Definition 2.2 (Mutual Independence) For $n \geq 3$, random variables $X_1, X_2, \cdots, X_n$ are mutually independent if

$$p(x_1, x_2, \cdots, x_n) = p(x_1)p(x_2)\cdots p(x_n)$$

for all $x_1, x_2, \cdots, x_n$.

Definition 2.3 (Pairwise Independence) For $n \geq 3$, random variables $X_1, X_2, \cdots, X_n$ are pairwise independent if X_i and X_j are independent for all $1 \leq i < j \leq n$.

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z | Y$, if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z | Y$, if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x, y, z) = \frac{p(x, y)p(y, z)}{p(y)} = p(x, y)p(z|y).$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)\underline{p(z|y)}.$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z | Y$, if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)\underline{p(z|y)}.$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

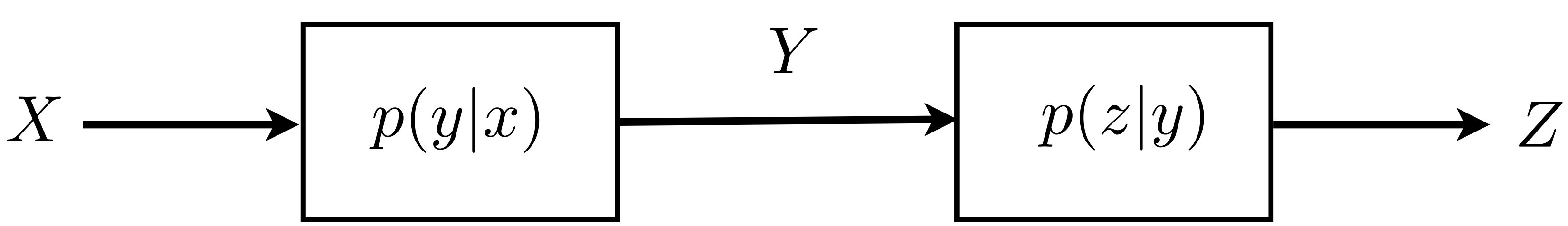
$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

- Conceptually, when $X \perp Z|Y$, X, Y, and Z are related as follows:



Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

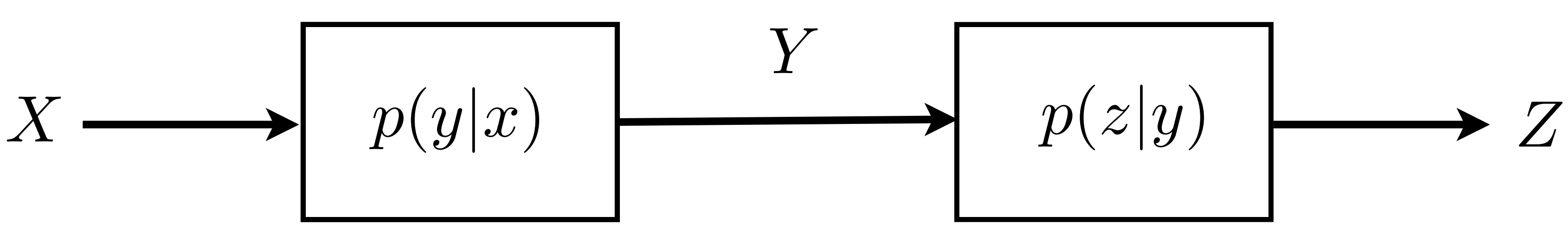
$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

- Conceptually, when $X \perp Z|Y$, X, Y, and Z are related as follows:



$$p(x,y,z) = p(x)p(y|x)p(z|y) = p(x,y)p(z|y)$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

- Conceptually, when $X \perp Z|Y$, X, Y, and Z are related as follows:

$$X \longrightarrow \boxed{p(y|x)} \xrightarrow{\;Y\;} \boxed{p(z|y)} \longrightarrow Z$$

$$p(x,y,z) = \underline{p(x)p(y|x)}p(z|y) = p(x,y)p(z|y)$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z|Y$, if

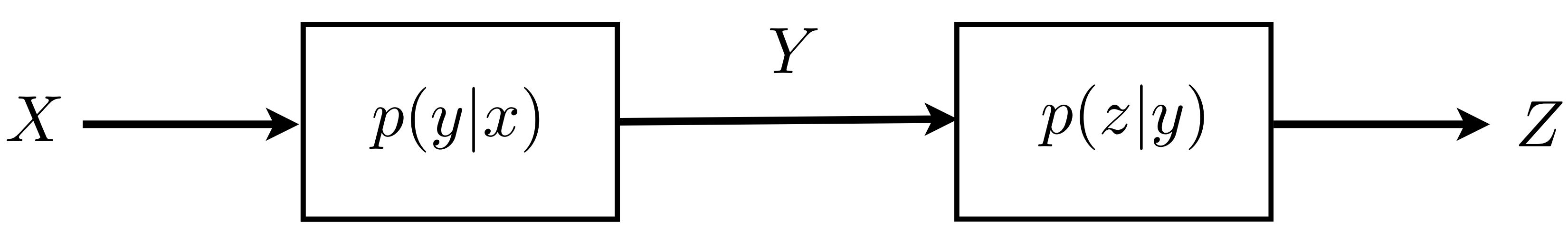
$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

- Conceptually, when $X \perp Z|Y$, X, Y, and Z are related as follows:



$$p(x,y,z) = \underline{p(x)p(y|x)}p(z|y) = \underline{p(x,y)}p(z|y)$$

Definition 2.4 (Conditional Independence) For random variables X, Y, and Z, X is independent of Z conditioning on Y, denoted by $X \perp Z | Y$, if

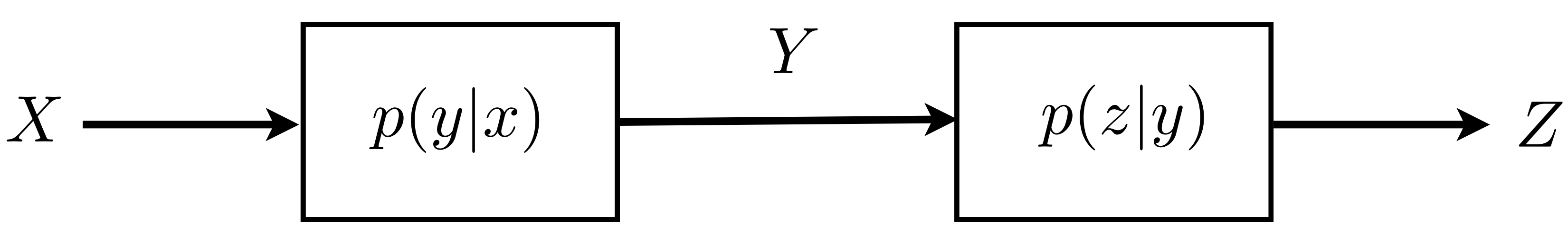
$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark

- If $p(y) > 0$, then

$$p(x,y,z) = \frac{p(x,y)p(y,z)}{p(y)} = p(x,y)p(z|y).$$

- Conceptually, when $X \perp Z|Y$, X, Y, and Z are related as follows:



$$p(x,y,z) = p(x)p(y|x)p(z|y) = p(x,y)p(z|y)$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \qquad (1)$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \qquad (1)$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.
2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \qquad (1)$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.
2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

4. Then (1) and hence the 'only if' part is proved.

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

4. Then (1) and hence the 'only if' part is proved.

5. Note that the choice of $a(x, y)$ and $b(y, z)$ is not unique. For example, one can choose

$$a(x, y) = p(x, y) \quad \text{and} \quad b(y, z) = \frac{p(y, z)}{p(y)}.$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

4. Then (1) and hence the 'only if' part is proved.

5. Note that the choice of $a(x, y)$ and $b(y, z)$ is not unique. For example, one can choose

$$a(x, y) = p(x, y) \quad \text{and} \quad b(y, z) = \frac{p(y, z)}{p(y)}.$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

4. Then (1) and hence the 'only if' part is proved.

5. Note that the choice of $a(x, y)$ and $b(y, z)$ is not unique. For example, one can choose

$$a(x, y) = p(x, y) \quad \text{and} \quad b(y, z) = \frac{p(y, z)}{p(y)}.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

4. Then (1) and hence the 'only if' part is proved.

5. Note that the choice of $a(x, y)$ and $b(y, z)$ is not unique. For example, one can choose

$$a(x, y) = p(x, y) \quad \text{and} \quad b(y, z) = \frac{p(y, z)}{p(y)}.$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z) \tag{1}$$

for all x, y, and z such that $p(y) > 0$.

Proof

A. 'Only if'

1. Assume $p(x, y, z)$ takes the form in Definition 2.4.

2. For all x and for all y such that $p(y) > 0$, let

$$a(x, y) = \frac{p(x, y)}{p(y)}.$$

3. For all y such that $p(y) > 0$ and for and all z, let

$$b(y, z) = p(y, z).$$

4. Then (1) and hence the 'only if' part is proved.

5. Note that the choice of $a(x, y)$ and $b(y, z)$ is not unique. For example, one can choose

$$a(x, y) = p(x, y) \quad \text{and} \quad b(y, z) = \frac{p(y, z)}{p(y)}.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Proposition 2.5 For random variables X, Y, and Z,
$X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$p(x, y) = \sum_z p(x, y, z)$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z \underline{a(x, y)} b(y, z) \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z \underline{a(x, y)} b(y, z) \\ &= \underline{a(x, y)} \sum_z b(y, z) \end{aligned}$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$p(y, z) = \sum_x p(x, y, z)$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \end{aligned}$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) \underline{b(y, z)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\underline{\sum_x a(x, y)} \right) \left(\sum_z b(y, z) \right) > 0.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x,y,z) = a(x,y)b(y,z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x,y,z) = a(x,y)b(y,z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x,y) &= \sum_z p(x,y,z) \\ &= \sum_z a(x,y)b(y,z) \\ &= a(x,y)\sum_z b(y,z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y,z) &= \sum_x p(x,y,z) \\ &= \sum_x a(x,y)b(y,z) \\ &= b(y,z)\sum_x a(x,y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y,z) = \left(\underline{\sum_x a(x,y)}\right)\left(\underline{\sum_z b(y,z)}\right) > 0.$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x,y,z) = a(x,y)b(y,z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x,y,z) = a(x,y)b(y,z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x,y) &= \sum_z p(x,y,z) \\ &= \sum_z a(x,y)b(y,z) \\ &= a(x,y)\sum_z b(y,z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y,z) &= \sum_x p(x,y,z) \\ &= \sum_x a(x,y)b(y,z) \\ &= b(y,z)\sum_x a(x,y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y,z) = \left(\sum_x a(x,y)\right)\left(\sum_z b(y,z)\right) > 0.$$

5. Therefore,

$$\frac{p(x,y)p(y,z)}{p(y)}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x,y,z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\frac{p(x, y)p(y, z)}{p(y)}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\frac{p(x, y)p(y, z)}{p(y)}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y)\sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z)\sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\frac{p(x, y)p(y, z)}{p(y)}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y) p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z) \right) \left(b(y, z) \sum_x a(x, y) \right)}{\left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\frac{p(x, y) p(y, z)}{p(y)} = \frac{\left(a(x, y) \sum_z b(y, z) \right) \left(b(y, z) \sum_x a(x, y) \right)}{\left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right)}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y) p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z) \right) \left(b(y, z) \sum_x a(x, y) \right)}{\left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\frac{p(x, y) p(y, z)}{p(y)} = \frac{\left(a(x, y) \sum_z b(y, z) \right) \left(b(y, z) \sum_x a(x, y) \right)}{\left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right)}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y)\sum_z b(y, z)\right)\left(b(y, z)\sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} & \frac{p(x, y)p(y, z)}{p(y)} \\ = & \frac{\left(a(x, y) \sum_z b(y, z)\right) \left(b(y, z) \cancel{\sum_x a(x, y)}\right)}{\left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z)\right) \left(b(y, z) \cancel{\sum_x a(x, y)}\right)}{\left(\cancel{\sum_x a(x, y)}\right) \left(\sum_z b(y, z)\right)} \end{aligned}$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y) p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \cancel{\sum_z b(y, z)} \right) \left(b(y, z) \cancel{\sum_x a(x, y)} \right)}{\left(\cancel{\sum_x a(x, y)} \right) \left(\sum_z b(y, z) \right)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} & \frac{p(x, y) p(y, z)}{p(y)} \\ = & \frac{\left(a(x, y) \cancel{\sum_z b(y, z)} \right) \left(b(y, z) \cancel{\sum_x a(x, y)} \right)}{\left(\cancel{\sum_x a(x, y)} \right) \left(\cancel{\sum_z b(y, z)} \right)} \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y) p(y, z)}{p(y)} \\ = &\frac{\left(a(x, y) \cancel{\sum_z b(y, z)} \right) \left(b(y, z) \cancel{\sum_x a(x, y)} \right)}{\left(\cancel{\sum_x a(x, y)} \right) \left(\cancel{\sum_z b(y, z)} \right)} \\ = &\ a(x, y) b(y, z) \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y) p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} \frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z)\right) \left(b(y, z) \sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} \frac{p(x, y)p(y, z)}{p(y)} & \\ &= \frac{\left(a(x, y) \sum_z b(y, z)\right) \left(b(y, z) \sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right) \left(\sum_z b(y, z)\right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y) p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z) \right) \left(b(y, z) \sum_x a(x, y) \right)}{\left(\sum_x a(x, y) \right) \left(\sum_z b(y, z) \right)} \\ &= a(x, y) b(y, z) \\ &= p(x, y, z). \end{aligned}$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z)\right)\left(b(y, z) \sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

6. For x, y, and z such that $p(y) = 0$, since

$$0 \leq p(x, y, z) \leq p(y) = 0,$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y) b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y) b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y) b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y) p(y, z)}{p(y)} \\ = \quad & \frac{\left(a(x, y) \sum_z b(y, z)\right)\left(b(y, z) \sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right)} \\ = \quad & a(x, y) b(y, z) \\ = \quad & p(x, y, z). \end{aligned}$$

6. For x, y, and z such that $p(y) = 0$, since

$$0 \le p(x, y, z) \le p(y) = 0,$$

we have

$$p(x, y, z) = 0.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z | Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y) \sum_z b(y, z)\right)\left(b(y, z) \sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

6. For x, y, and z such that $p(y) = 0$, since

$$0 \leq p(x, y, z) \leq p(y) = 0,$$

we have

$$p(x, y, z) = 0.$$

Definition 2.4 $X \perp Z | Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} \frac{p(x, y)p(y, z)}{p(y)} &= \frac{\left(a(x, y)\sum_z b(y, z)\right)\left(b(y, z)\sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

6. For x, y, and z such that $p(y) = 0$, since

$$0 \le p(x, y, z) \le p(y) = 0,$$

we have

$$p(x, y, z) = 0.$$

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5 For random variables X, Y, and Z, $X \perp Z|Y$ if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

Proof

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all x, y, and z such that $p(y) > 0$.

2. Then for such x, y, and z, we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left(a(x, y)\sum_z b(y, z)\right)\left(b(y, z)\sum_x a(x, y)\right)}{\left(\sum_x a(x, y)\right)\left(\sum_z b(y, z)\right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

6. For x, y, and z such that $p(y) = 0$, since

$$0 \le p(x, y, z) \le p(y) = 0,$$

we have

$$p(x, y, z) = 0.$$

7. Hence, $X \perp Z|Y$ according to Definition 2.4.

Definition 2.4 $X \perp Z|Y$ if

$$p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} & \text{if } p(y) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6 (Markov Chain) For random variables $X_1, X_2, \cdots, X_n$, where $n \geq 3$, $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if

$$p(x_1, x_2, \cdots, x_n) =$$

$$\begin{cases} p(x_1, x_2)p(x_3|x_2)\cdots p(x_n|x_{n-1}) & \text{if } p(x_2), p(x_3), \cdots, p(x_{n-1}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6 (Markov Chain) For random variables $X_1, X_2, \cdots, X_n$, where $n \geq 3$, $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if

$$p(x_1, x_2, \cdots, x_n) =$$

$$\begin{cases} p(x_1, x_2)p(x_3|x_2)\cdots p(x_n|x_{n-1}) & \text{if } p(x_2), p(x_3), \cdots, p(x_{n-1}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark $X_1 \to X_2 \to X_3$ is equivalent to $X_1 \perp X_3 | X_2$.

Definition 2.6 (Markov Chain) For random variables $X_1, X_2, \cdots, X_n$, where $n \geq 3$, $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if

$$p(x_1, x_2, \cdots, x_n) =$$

$$\begin{cases} p(x_1, x_2)p(x_3|x_2)\cdots p(x_n|x_{n-1}) & \text{if } p(x_2), p(x_3), \cdots, p(x_{n-1}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark $X_1 \to X_2 \to X_3$ is equivalent to $X_1 \perp X_3 | X_2$.

Proposition 2.7 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if $X_n \to X_{n-1} \to \cdots \to X_1$ forms a Markov chain. (Exercise)

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$\begin{aligned} & X_1 \to X_2 \to X_3 \\ & (X_1, X_2) \to X_3 \to X_4 \\ & \qquad\qquad \vdots \\ & (X_1, X_2, \cdots, X_{n-2}) \to X_{n-1} \to X_n \end{aligned}$$

form Markov chains. (Exercise)

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$X_1 \to X_2 \to X_3$$

form Markov chains. (Exercise)

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$X_1 \to X_2 \to X_3$$
$$(X_1, X_2) \to X_3 \to X_4$$

form Markov chains. (Exercise)

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$\begin{gathered} X_1 \to X_2 \to X_3 \\ (X_1, X_2) \to X_3 \to X_4 \\ \vdots \\ (X_1, X_2, \cdots, X_{n-2}) \to X_{n-1} \to X_n \end{gathered}$$

form Markov chains. (Exercise)

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$\begin{gathered}
X_1 \to X_2 \to X_3 \\
(X_1, X_2) \to X_3 \to X_4 \\
\vdots \\
(X_1, X_2, \cdots, X_{n-2}) \to X_{n-1} \to X_n
\end{gathered}$$

form Markov chains. (Exercise)

Proposition 2.9 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$\begin{aligned}
&X_1 \to X_2 \to X_3 \\
&(X_1, X_2) \to X_3 \to X_4 \\
&\qquad\qquad\vdots \\
&(X_1, X_2, \cdots, X_{n-2}) \to X_{n-1} \to X_n
\end{aligned}$$

form Markov chains. (Exercise)

Proposition 2.9 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$p(x_1, x_2, \cdots, x_n) = f_1(x_1, x_2) f_2(x_2, x_3) \cdots f_{n-1}(x_{n-1}, x_n)$$

Proposition 2.8 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$\begin{aligned} & X_1 \to X_2 \to X_3 \\ & (X_1, X_2) \to X_3 \to X_4 \\ & \qquad\qquad \vdots \\ & (X_1, X_2, \cdots, X_{n-2}) \to X_{n-1} \to X_n \end{aligned}$$

form Markov chains. (Exercise)

Proposition 2.9 $X_1 \to X_2 \to \cdots \to X_n$ forms a Markov chain if and only if

$$p(x_1, x_2, \cdots, x_n) = f_1(x_1, x_2) f_2(x_2, x_3) \cdots f_{n-1}(x_{n-1}, x_n)$$

for all $x_1, x_2, \cdots, x_n$ such that $p(x_2), p(x_3), \cdots, p(x_{n-1}) > 0$.

Proposition 2.10 (Markov subchains) Let $\mathcal{N}_n = \{1, 2, \cdots, n\}$ and let $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$ form a Markov chain. For any subset α of $\mathcal{N}_n$, denote $(X_i, i \in \alpha)$ by X_α. Then for any disjoint subsets $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $\mathcal{N}_n$ such that

$$k_1 < k_2 < \cdots < k_m$$

for all $k_j \in \alpha_j$, $j = 1, 2, \cdots, m$,

$$X_{\alpha_1} \rightarrow X_{\alpha_2} \rightarrow \cdots \rightarrow X_{\alpha_m}$$

forms a Markov chain. That is, a subchain of $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n$ is also a Markov chain. (Exercise)

Proposition 2.10 (Markov subchains) Let $\mathcal{N}_n = \{1, 2, \cdots, n\}$ and let $X_1 \to X_2 \to \cdots \to X_n$ form a Markov chain. For any subset α of $\mathcal{N}_n$, denote $(X_i, i \in \alpha)$ by X_α. Then for any disjoint subsets $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $\mathcal{N}_n$ such that

$$k_1 < k_2 < \cdots < k_m$$

for all $k_j \in \alpha_j$, $j = 1, 2, \cdots, m$,

$$X_{\alpha_1} \to X_{\alpha_2} \to \cdots \to X_{\alpha_m}$$

forms a Markov chain. That is, a subchain of $X_1 \to X_2 \to \cdots \to X_n$ is also a Markov chain. (Exercise)

Proposition 2.10 (Markov subchains) Let $\mathcal{N}_n = \{1, 2, \cdots, n\}$ and let $X_1 \to X_2 \to \cdots \to X_n$ form a Markov chain. For any subset α of $\mathcal{N}_n$, denote $(X_i, i \in \alpha)$ by X_α. Then for any disjoint subsets $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $\mathcal{N}_n$ such that

$$k_1 < k_2 < \cdots < k_m$$

for all $k_j \in \alpha_j$, $j = 1, 2, \cdots, m$,

$$X_{\alpha_1} \to X_{\alpha_2} \to \cdots \to X_{\alpha_m}$$

forms a Markov chain. That is, a subchain of $X_1 \to X_2 \to \cdots \to X_n$ is also a Markov chain. (Exercise)

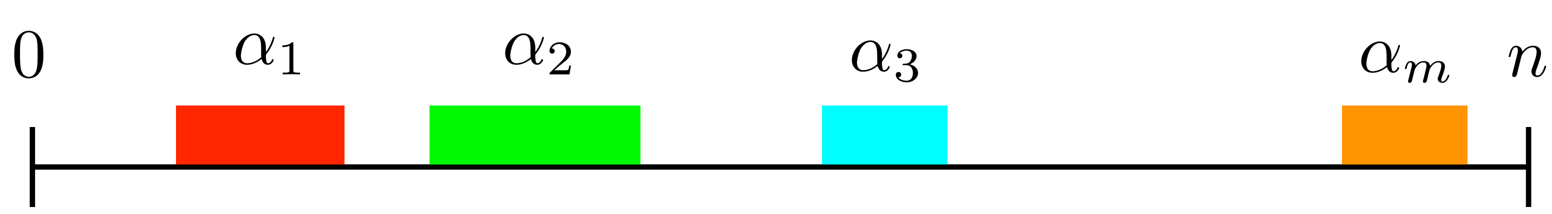
Proposition 2.10 (Markov subchains) Let $\mathcal{N}_n = \{1, 2, \cdots, n\}$ and let $X_1 \to X_2 \to \cdots \to X_n$ form a Markov chain. For any subset α of $\mathcal{N}_n$, denote $(X_i, i \in \alpha)$ by X_α. Then for any disjoint subsets $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $\mathcal{N}_n$ such that

$$k_1 < k_2 < \cdots < k_m$$

for all $k_j \in \alpha_j$, $j = 1, 2, \cdots, m$,

$$X_{\alpha_1} \to X_{\alpha_2} \to \cdots \to X_{\alpha_m}$$

forms a Markov chain. That is, a subchain of $X_1 \to X_2 \to \cdots \to X_n$ is also a Markov chain. (Exercise)



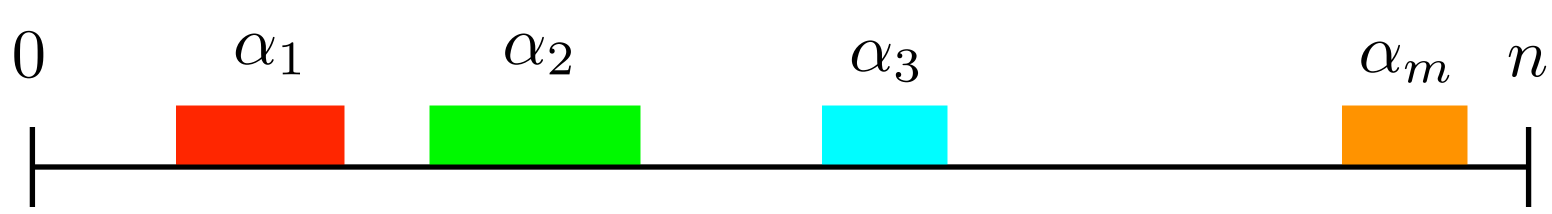
Proposition 2.10 (Markov subchains) Let $\mathcal{N}_n = \{1, 2, \cdots, n\}$ and let $X_1 \to X_2 \to \cdots \to X_n$ form a Markov chain. For any subset α of $\mathcal{N}_n$, denote $(X_i, i \in \alpha)$ by X_α. Then for any disjoint subsets $\alpha_1, \alpha_2, \cdots, \alpha_m$ of $\mathcal{N}_n$ such that

$$k_1 < k_2 < \cdots < k_m$$

for all $k_j \in \alpha_j$, $j = 1, 2, \cdots, m$,

$$\boxed{X_{\alpha_1} \to X_{\alpha_2} \to \cdots \to X_{\alpha_m}}$$

forms a Markov chain. That is, a subchain of $X_1 \to X_2 \to \cdots \to X_n$ is also a Markov chain. (Exercise)



Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $\underline{X_1 = Y}$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $\underline{X_2 = Z}$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $\underline{X_2 = Z}$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = \underline{Y}$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $\underline{X_1} \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp \underline{X_4} | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = \underline{(Y, Z)}$, where $Y \perp Z$
- Then $X_1 \perp \underline{X_4} | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (\underline{X_2,} X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = \underline{Z,}$ and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (\underline{X_2,} X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, \underline{X_3})$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.
  $Y \perp (Y, Z) | (Y, Z)$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

  $Y \perp (Y, Z) | (\underline{Y}, Z)$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

  $\cancel{Y} \perp (Y, Z) | (\underline{Y}, Z)$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

  $\cancel{Y} \perp (\cancel{Y}, Z) | (\underline{Y}, Z)$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

  $\cancel{Y} \perp (\cancel{Y}, Z) | (\underline{Y}, \underline{Z})$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$\cancel{Y} \perp (\cancel{Y}, \cancel{Z}) | (\underline{Y}, \underline{Z})$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $\underline{X_1 \perp X_3 | (X_2, X_4)}$, but $X_1 \not\perp (X_3, X_4) | X_2$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$$Y \perp (Y, Z) | (Y, Z)$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$$Y \perp (Y, Z) | (\underline{Y}, Z)$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$$\cancel{Y} \perp (Y, Z) | (\underline{Y}, Z)$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4|(X_2, X_3) \\ X_1 \perp X_3|(X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4)|X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4|(X_2, X_3)$, $X_1 \perp X_3|(X_2, X_4)$, but $X_1 \not\perp (X_3, X_4)|X_2$.

$$\cancel{Y} \perp (\cancel{Y}, Z)|(\underline{Y}, Z)$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$$\cancel{Y} \perp (\cancel{Y}, Z) | (\underline{Y}, \underline{Z})$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left. \begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array} \right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$$\cancel{Y} \perp (\cancel{Y}, \cancel{Z}) | (\underline{Y}, \underline{Z})$$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $\underline{X_1 \not\perp (X_3, X_4) | X_2}$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$Y \not\perp (Y, Z) | Z$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$Y \not\perp (Y, Z) | \underline{Z}$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4|(X_2, X_3) \\ X_1 \perp X_3|(X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4)|X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4|(X_2, X_3)$, $X_1 \perp X_3|(X_2, X_4)$, but $X_1 \not\perp (X_3, X_4)|X_2$.

$Y \not\perp (Y, \cancel{Z})|\underline{Z}$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4|(X_2, X_3) \\ X_1 \perp X_3|(X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4)|X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4|(X_2, X_3)$, $X_1 \perp X_3|(X_2, X_4)$, but $X_1 \not\perp (X_3, X_4)|X_2$.

$Y \not\perp (Y, \cancel{Z})|\underline{Z}$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

$Y \not\perp (Y, \cancel{Z}) | \cancel{Z}$

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $\underline{X_1 \not\perp (X_3, X_4) | X_2}$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4|(X_2, X_3) \\ X_1 \perp X_3|(X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4)|X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4|(X_2, X_3)$, $X_1 \perp X_3|(X_2, X_4)$, but $X_1 \not\perp (X_3, X_4)|X_2$.

- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $\underline{X_3 = X_4 = (Y, Z)}$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4|(X_2, X_3) \\ X_1 \perp X_3|(X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4)|X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $\underline{X_1 = Y}$, $X_2 = Z$, and $\underline{X_3 = X_4 = (Y, Z)}$, where $Y \perp Z$
- Then $X_1 \perp X_4|(X_2, X_3)$, $X_1 \perp X_3|(X_2, X_4)$, but $X_1 \not\perp (X_3, X_4)|X_2$.

- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $\underline{X_1 = Y}$, $\underline{X_2 = Z}$, and $\underline{X_3 = X_4 = (Y, Z)}$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $\underline{X_1 = Y}$, $\underline{X_2 = Z}$, and $\underline{X_3 = X_4 = \overset{= (X_1, X_2)}{(Y, Z)}}$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!

$= (X_1, X_2)$

- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.
- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!
- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$

  $= (X_1, X_2)$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

- p is not strictly positive because $p(x_1, x_2, \underline{x_3}, x_4) = 0$ if $\underline{x_3 \neq (x_1, x_2)}$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.

- Not true if p is not strictly positive!

- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = \overset{= (X_1, X_2)}{(Y, Z)}$, where $Y \perp Z$

- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

- p is not strictly positive because $p(x_1, x_2, x_3, x_4) = 0$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.

Proposition 2.12 Let X_1, X_2, X_3, and X_4 be random variables such that $p(x_1, x_2, x_3, x_4)$ is strictly positive. Then

$$\left.\begin{array}{l} X_1 \perp X_4 | (X_2, X_3) \\ X_1 \perp X_3 | (X_2, X_4) \end{array}\right\} \Rightarrow X_1 \perp (X_3, X_4) | X_2.$$

- See textbook for a proof of the proposition.
- Not true if p is not strictly positive!

$= (X_1, X_2)$

- Let $X_1 = Y$, $X_2 = Z$, and $X_3 = X_4 = (Y, Z)$, where $Y \perp Z$
- Then $X_1 \perp X_4 | (X_2, X_3)$, $X_1 \perp X_3 | (X_2, X_4)$, but $X_1 \not\perp (X_3, X_4) | X_2$.

- p is not strictly positive because $\underline{p(x_1, x_2, x_3, x_4) = 0}$ if $x_3 \neq (x_1, x_2)$ or $x_4 \neq (x_1, x_2)$.