香港中文大學
The Chinese University of Hong Kong

11.9 Zero-Mean Gaussian Noise is the Worst Additive Noise

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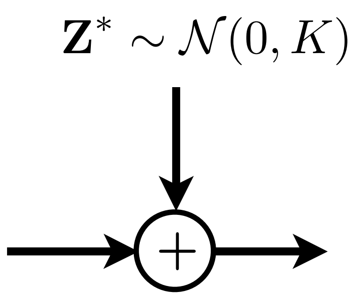
- The diagonal elements of the correlation matrix specify the power of the individual noise variables.
- The other elements in the matrix give a characterization of the correlation between the noise variables.

Main Idea

Zero-Mean Gaussian System



1. Consider a system of correlated Gaussian channels with noise vector $\mathbf{Z}^* \sim \mathcal{N}(0, K)$, and so $\tilde{K}_{\mathbf{Z}^*} = K$.

2. Let C^* be the capacity of the system.

3. Let $\mathbf{X}^*$ be the zero-mean Gaussian input vector that achieves the capacity.

4. Let $\mathbf{Y}^*$ be the output of the system with $\mathbf{X}^*$ as input, i.e.,

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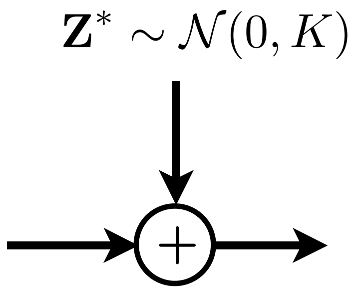

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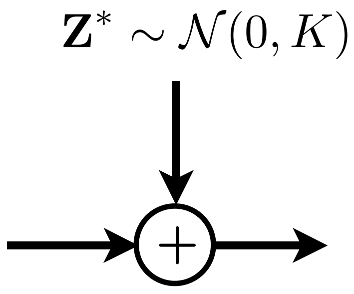

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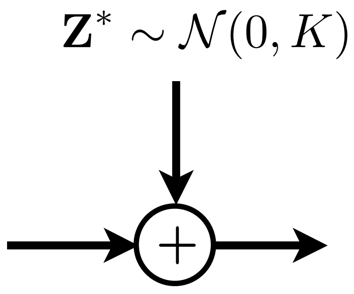

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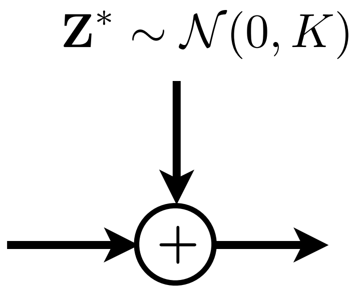

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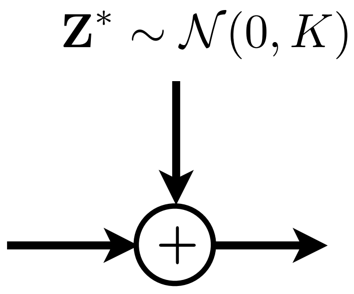

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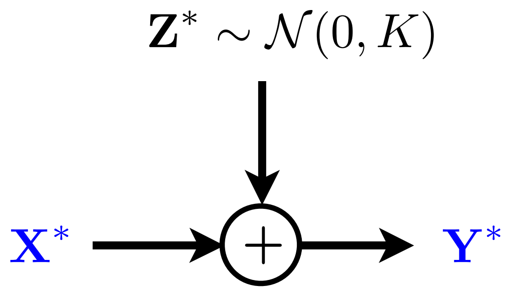

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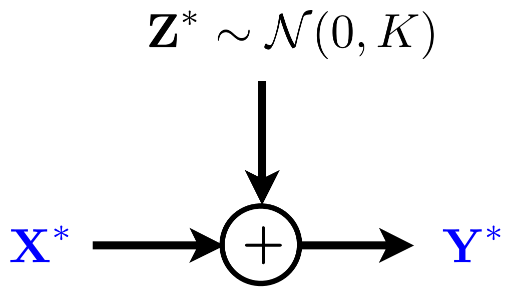

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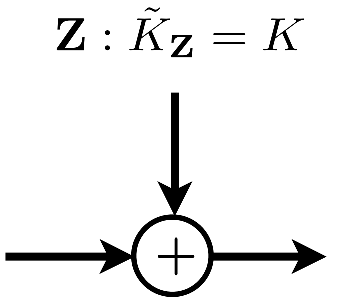

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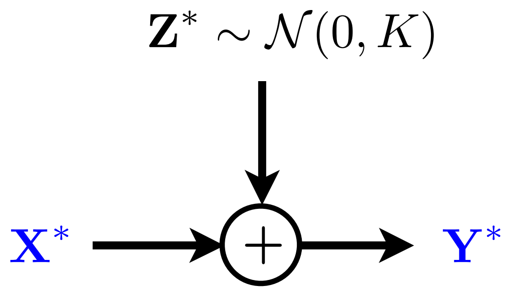

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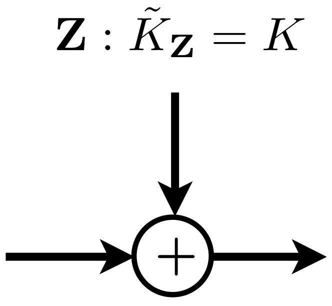

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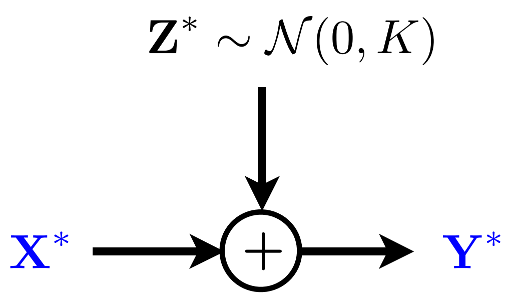

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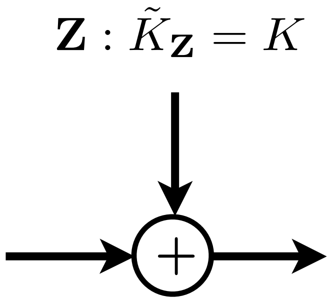

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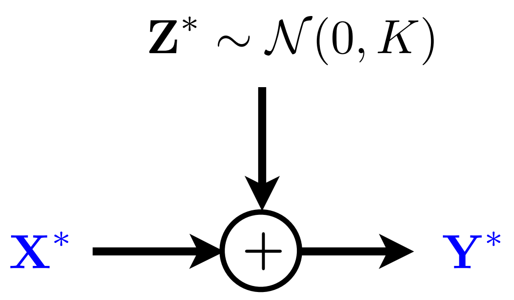

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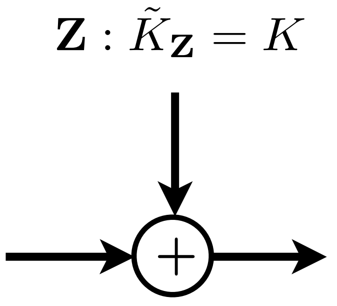

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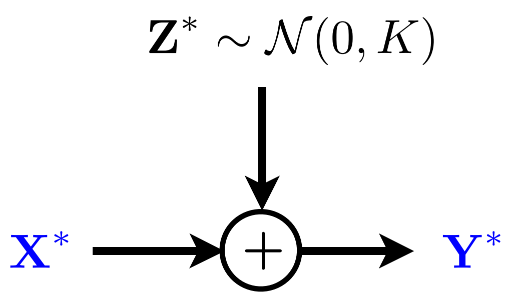

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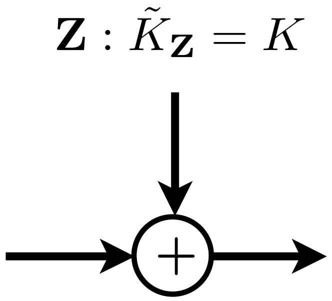

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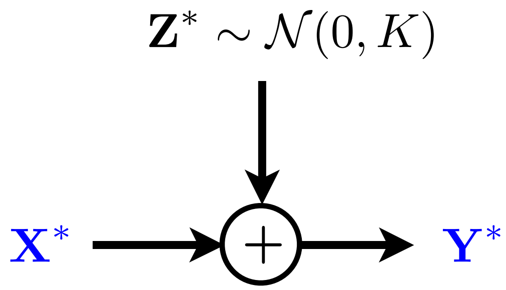

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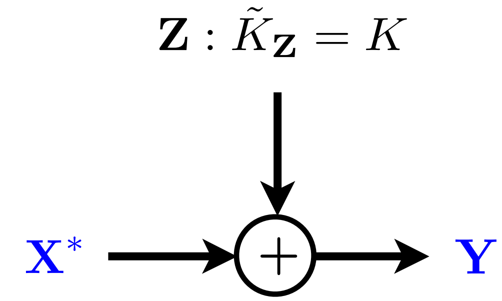

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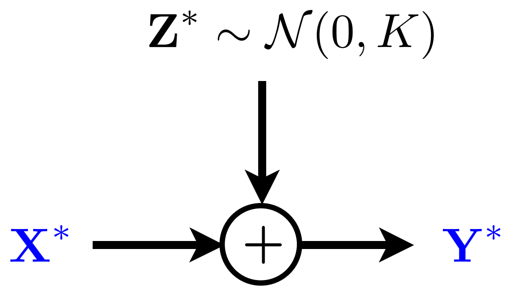

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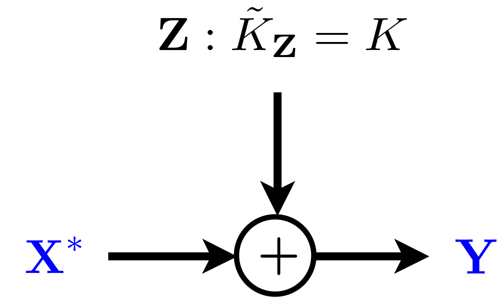

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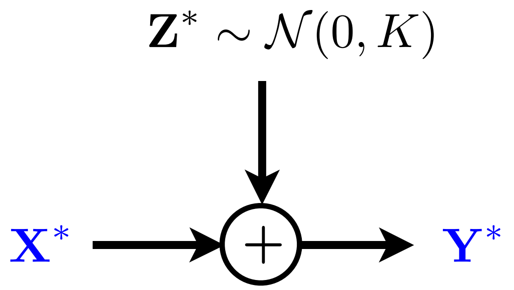

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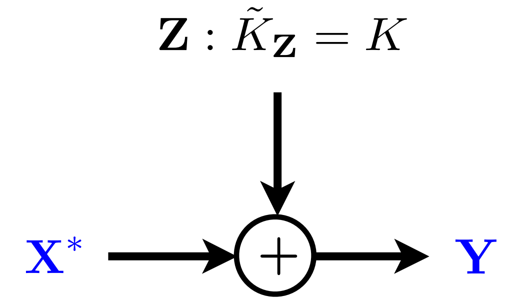

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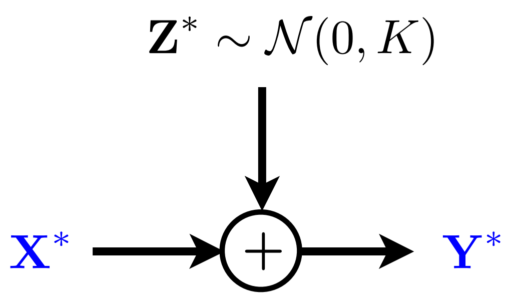

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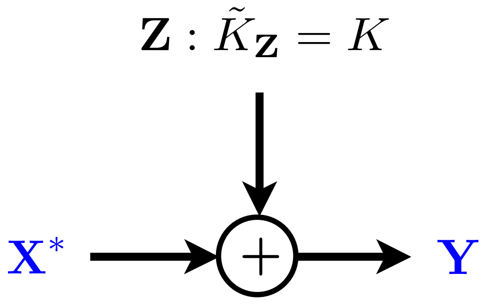

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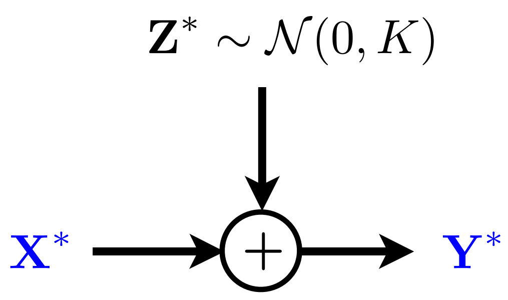

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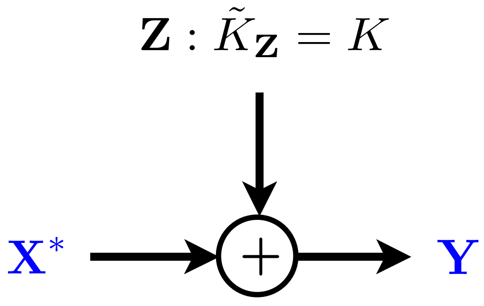

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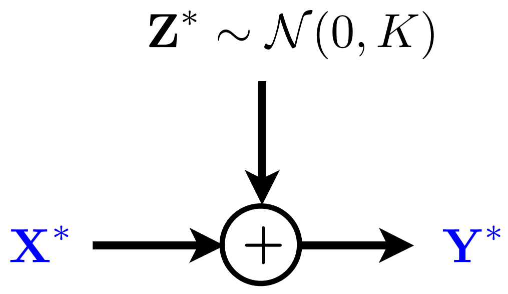

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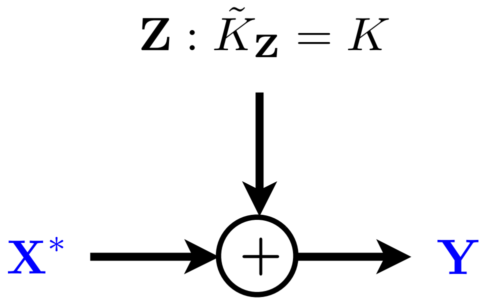

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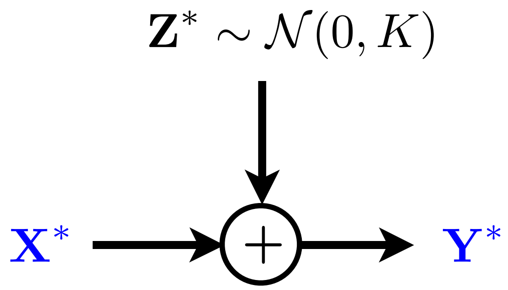

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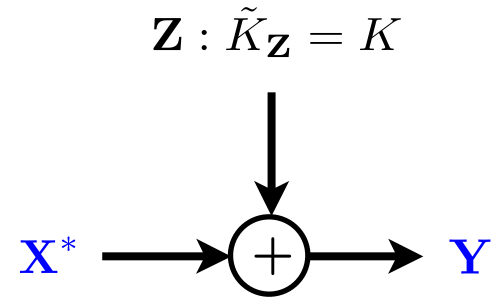

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Main Idea

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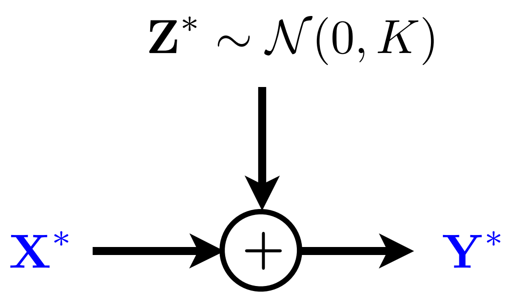


1. Consider a system of correlated Gaussian channels with noise vector $\mathbf{Z}^* \sim \mathcal{N}(0, K)$, and so $\tilde{K}_{\mathbf{Z}^*} = K$.

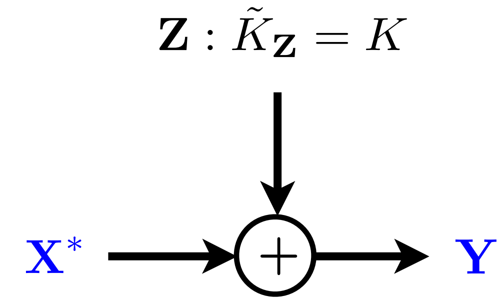
2. Let C^* be the capacity of the system.

3. Let $\mathbf{X}^*$ be the zero-mean Gaussian input vector that achieves the capacity.

4. Let $\mathbf{Y}^*$ be the output of the system with $\mathbf{X}^*$ as input, i.e.,

$$\mathbf{Y}^* = \mathbf{X}^* + \mathbf{Z}^*.$$

Alternative System



1. Consider a system exactly the same as the Zero-Mean Gaussian System except that the noise vector $\mathbf{Z}$, which has the same correlation matrix as $\mathbf{Z}^*$, may neither be zero-mean nor Gaussian.

2. Assume that the joint pdf of $\mathbf{Z}$ exists.

3. Let C be the capacity of the system.

4. Let $\mathbf{Y}$ be the output of the system with $\mathbf{X}^*$ as input, i.e.,

$$\mathbf{Y} = \mathbf{X}^* + \mathbf{Z}.$$

- For the Zero-Mean Gaussian System, $C^* = I(\mathbf{X}^*; \mathbf{Y}^*)$.
- For the alternative system, $\underline{I(\mathbf{X}^*; \mathbf{Y}) \leq C}$.
- We will show that $I(\mathbf{X}^*; \mathbf{Y}^*) \leq I(\mathbf{X}^*; \mathbf{Y})$.
- Hence,

$$C^* = I(\mathbf{X}^*; \mathbf{Y}^*) \leq \underline{I(\mathbf{X}^*; \mathbf{Y}) \leq C}.$$

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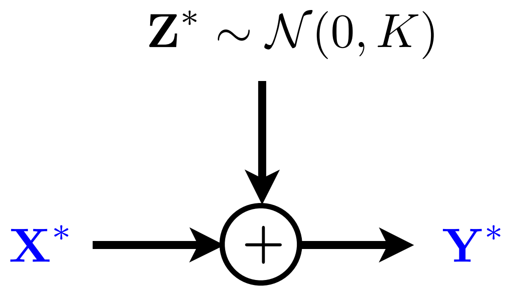

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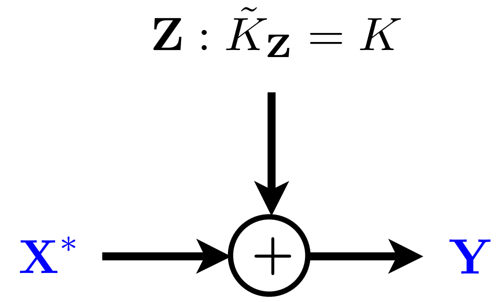

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- Hence,
$$C^* = I(\mathbf{X}^*; \mathbf{Y}^*) \leq I(\mathbf{X}^*; \mathbf{Y}) \leq C.$$

Theorem 11.32 For a fixed zero-mean Gaussian random vector $\mathbf{X}^*$, let

$$\mathbf{Y} = \mathbf{X}^* + \mathbf{Z},$$

where the joint pdf of $\mathbf{Z}$ exists and $\mathbf{Z}$ is independent of $\mathbf{X}^*$. Under the constraint that the correlation matrix of $\mathbf{Z}$ is equal to K, where K is any symmetric positive definite matrix, $I(\mathbf{X}^*; \mathbf{Y})$ is minimized if and only if $\mathbf{Z} = \mathbf{Z}^* \sim \mathcal{N}(0, K)$.

Lemma 11.33 Let $\mathbf{X}$ be a zero-mean random vector and

$$\mathbf{Y} = \mathbf{X} + \mathbf{Z}$$

where $\mathbf{Z}$ is independent of $\mathbf{X}$. Then

$$\tilde{K}_{\mathbf{Y}} = \tilde{K}_{\mathbf{X}} + \tilde{K}_{\mathbf{Z}}.$$

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Remark The scalar case has been proved in the proof of Theorem 11.21.

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Lemma 11.34 Let $\mathbf{Y}^* \sim \mathcal{N}(0, K)$ and $\mathbf{Y}$ be any random vector with correlation matrix K. Then

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Remark A similar technique has been used in proving Theorems 2.50 and 10.41 (maximum entropy distributions).

Theorem 2.50 Let

$$p^*(x) = e^{-\lambda_0 - \sum_{i=1}^{m} \lambda_i r_i(x)}$$

for all $x \in \mathcal{S}$, where $\lambda_0, \lambda_1, \cdots, \lambda_m$ are chosen such that

$$\sum_{x \in \mathcal{S}_p} p(x) r_i(x) = a_i \quad \text{for } 1 \leq i \leq m. \qquad (1)$$

Then p^* maximizes $H(p)$ over all probability distribution p on $\mathcal{S}$ subject to (1).

Sketch of Proof

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$$H(p^*) - H(p)$$

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$$\begin{aligned} & H(p^*) - H(p) \\ = & \quad - \sum_{x \in \mathcal{S}} p^*(x) \ln p^*(x) + \sum_{x \in \mathcal{S}_p} p(x) \ln p(x) \end{aligned}$$

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Theorem 10.45 Let $\mathbf{X}$ be a vector of n continuous random variables with correlation matrix $\tilde{K}$. Then

$$h(\mathbf{X}) \le \frac{1}{2} \log \left[(2\pi e)^n |\tilde{K}| \right]$$

with equality if and only if $\mathbf{X} \sim \mathcal{N}(0, \tilde{K})$.

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Data Processing Inequality for Informational Divergence

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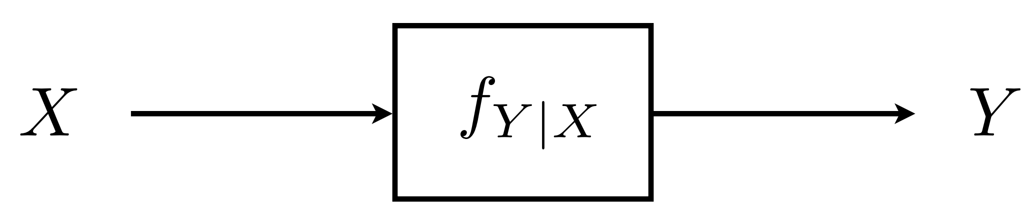

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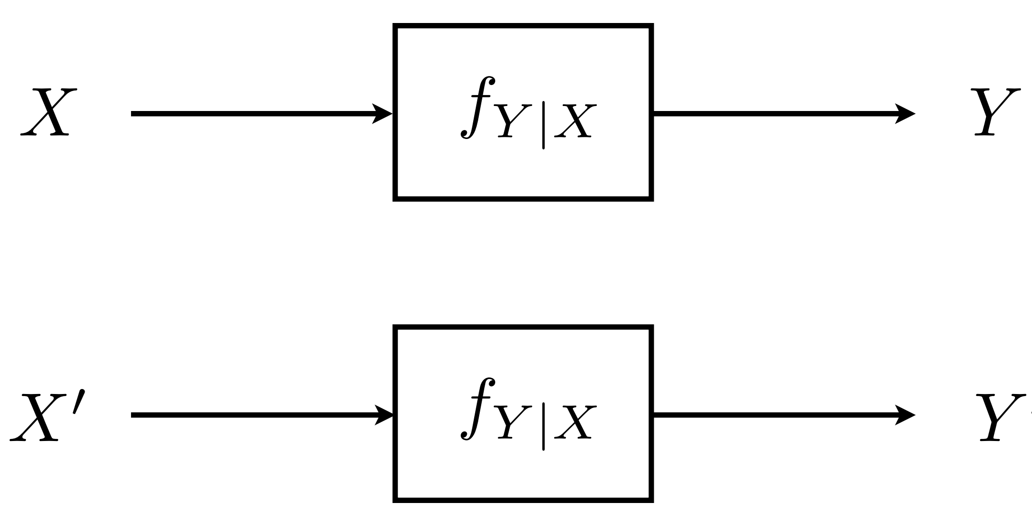

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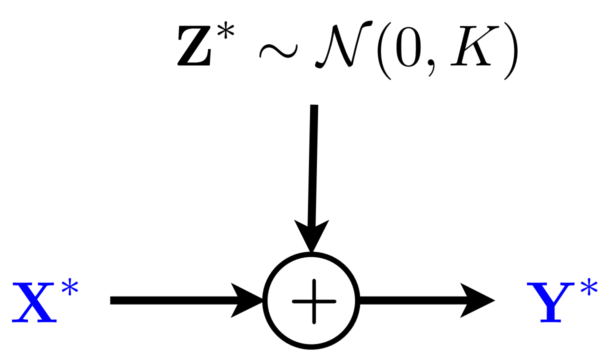

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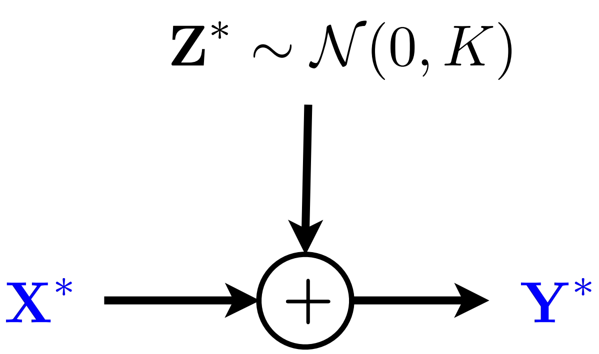

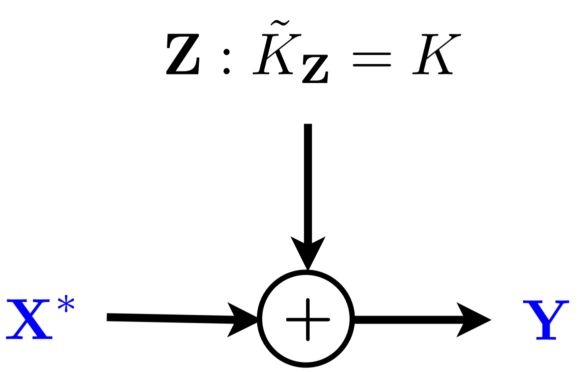

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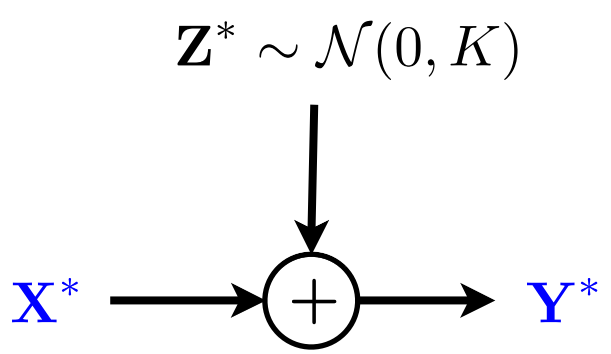

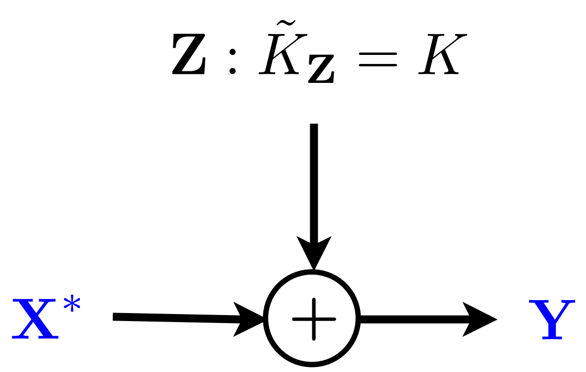

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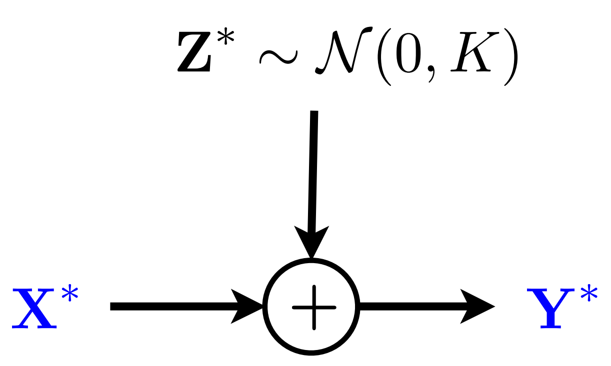

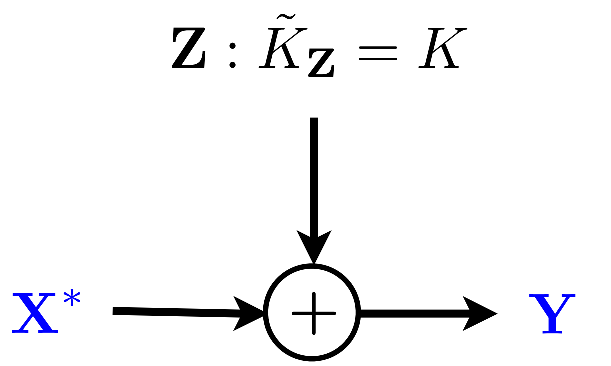

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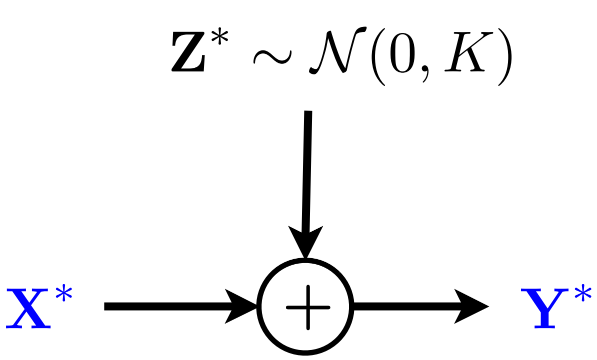

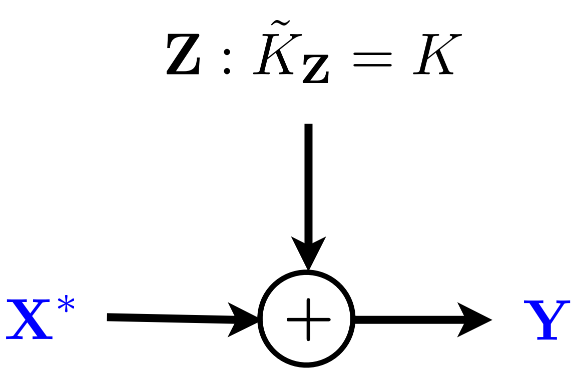

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Lemma 11.34 Let $\mathbf{Y}^* \sim \mathcal{N}(0, K)$ and $\mathbf{Y}$ be any random vector with correlation matrix K. Then

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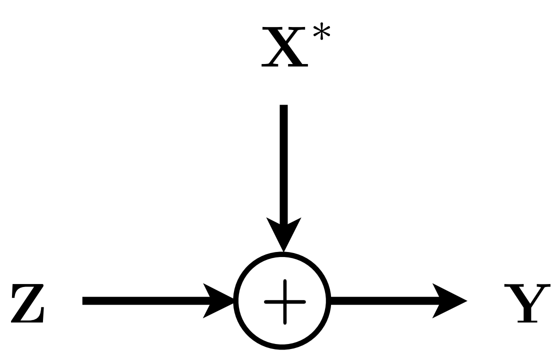

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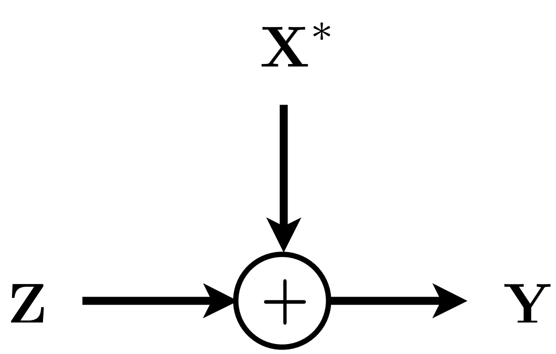

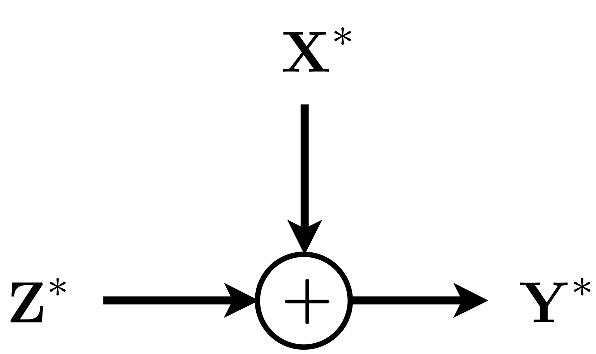

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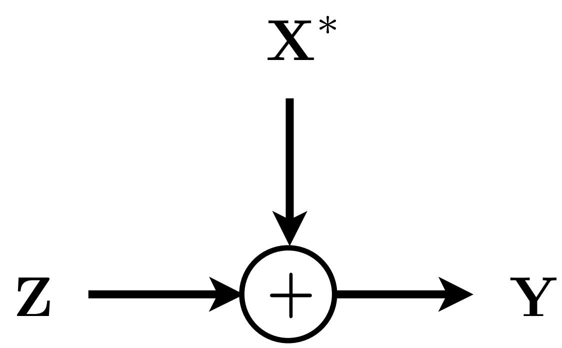

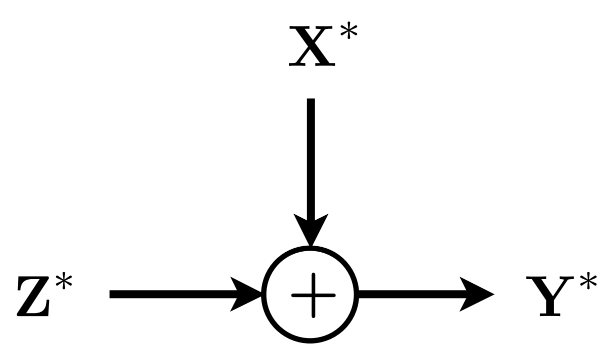

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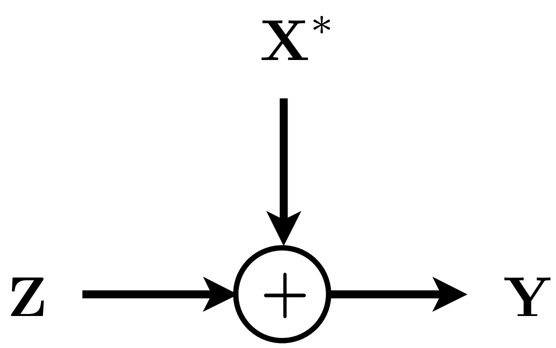

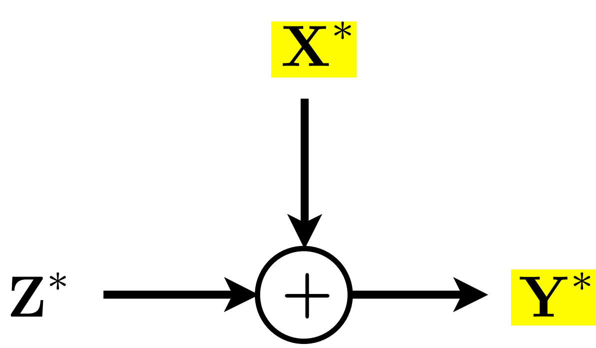

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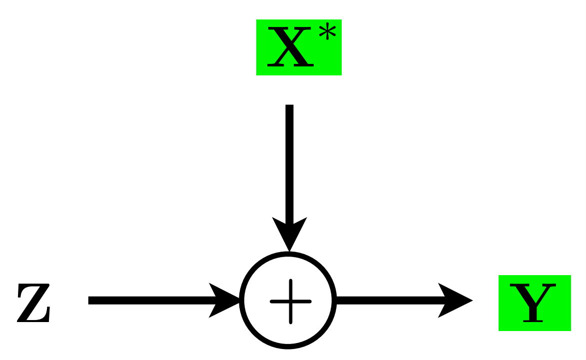

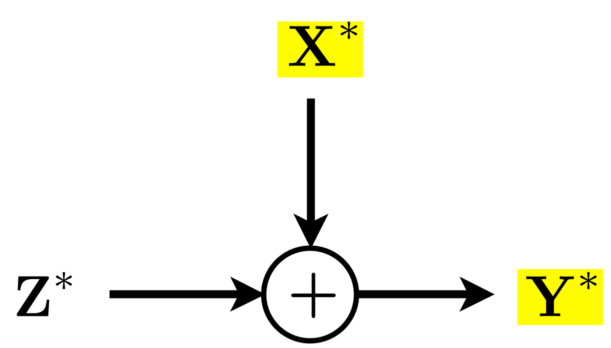

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