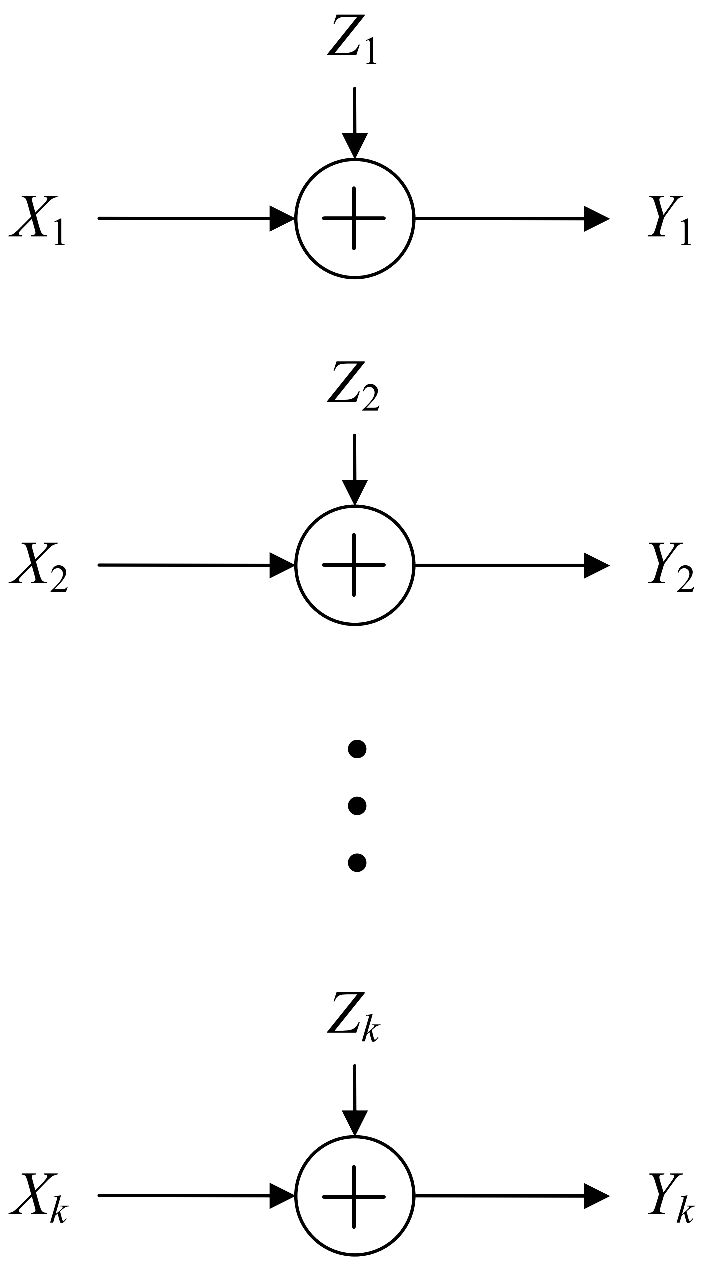
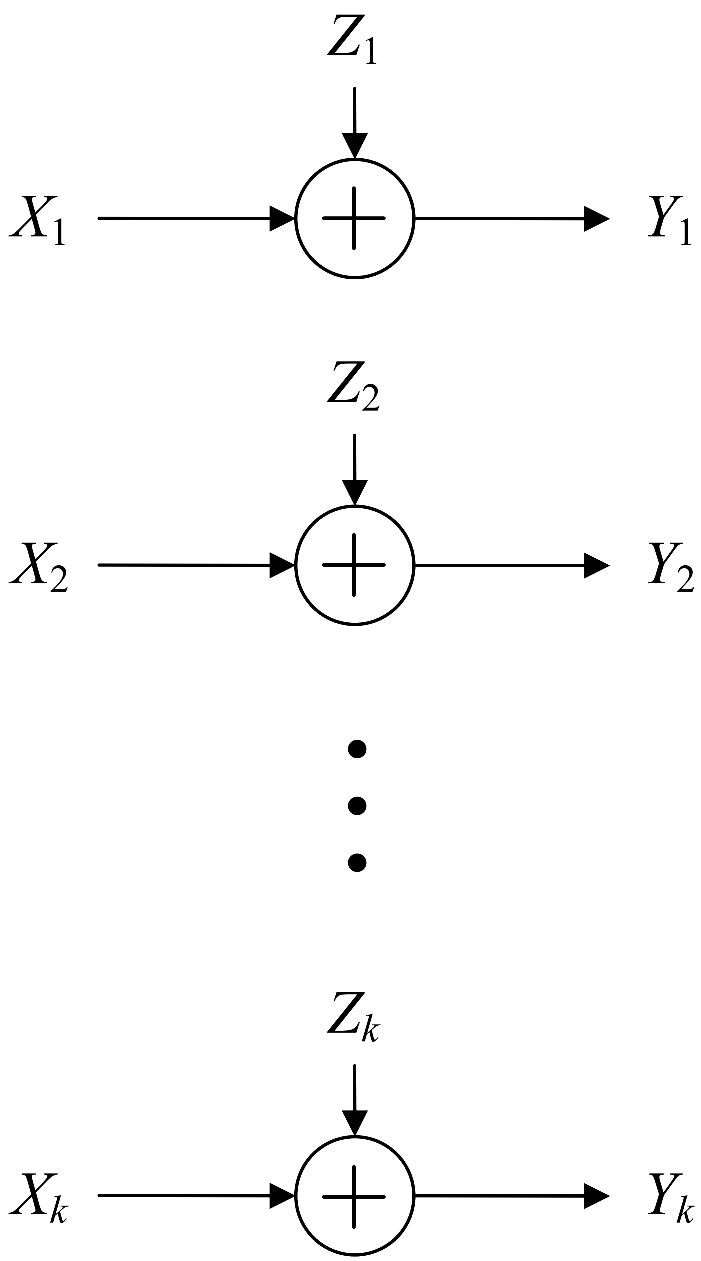
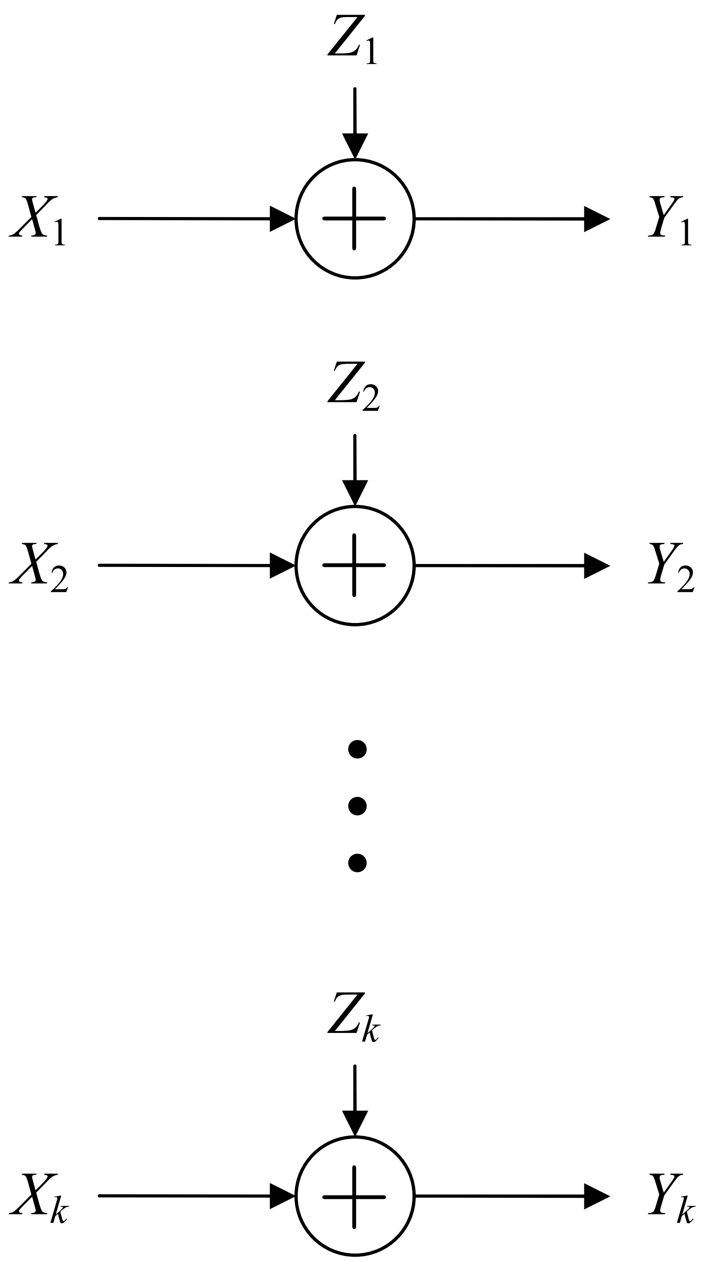
香港中文大學
The Chinese University of Hong Kong

11.5 Parallel Gaussian Channels


Z_1
X_1
Y_1
Z_2
X_2
Y_2
Z_k
X_k
Y_k



- $Z_i \sim \mathcal{N}(0, N_i)$ and Z_i, $1 \leq i \leq k$ are independent.



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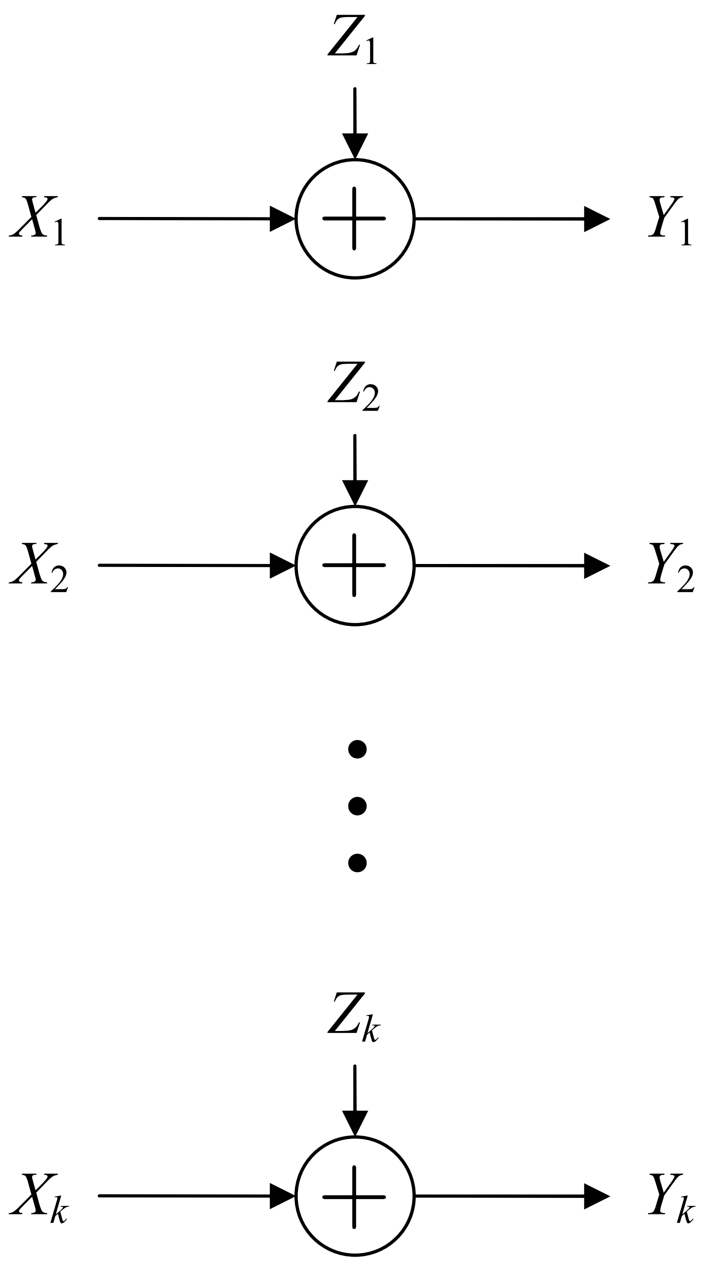
- Total input power constraint: $E \sum_{i=1}^{k} X_i^2 \leq P$.



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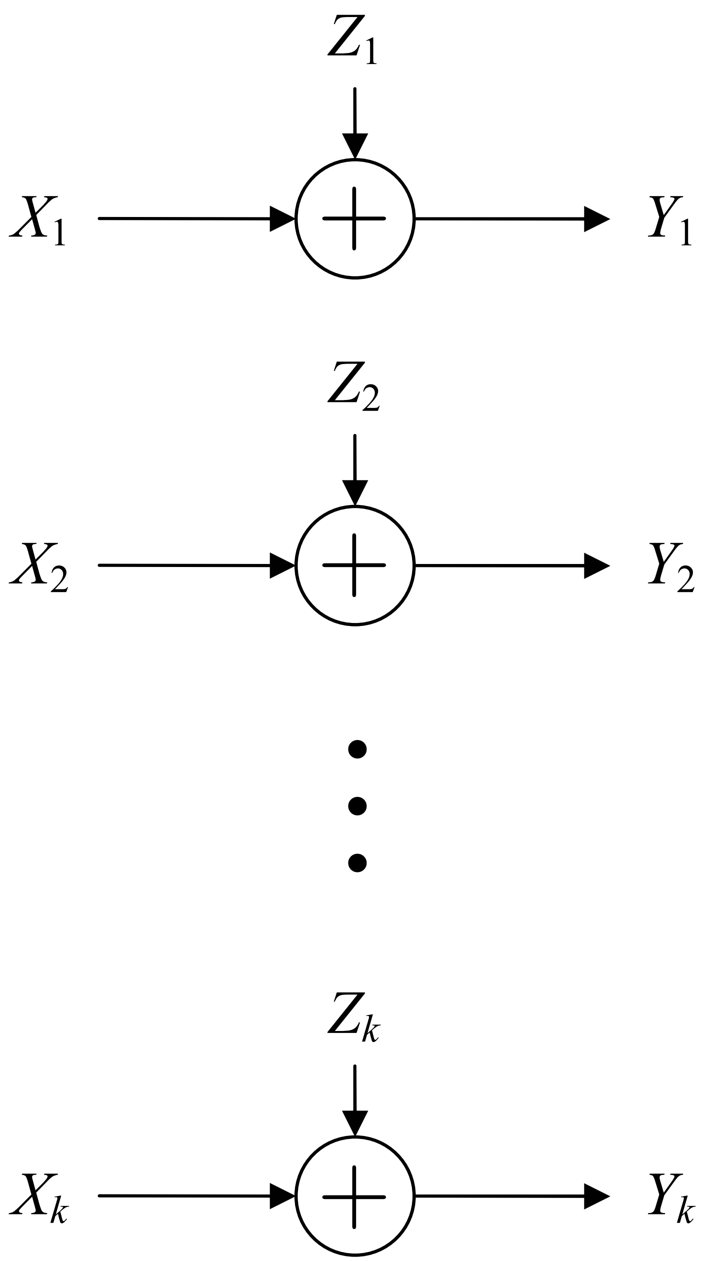
- $$C(P) = \sup_{F(\mathbf{x}): E\sum_i X_i^2 \le P} I(\mathbf{X};\mathbf{Y})$$



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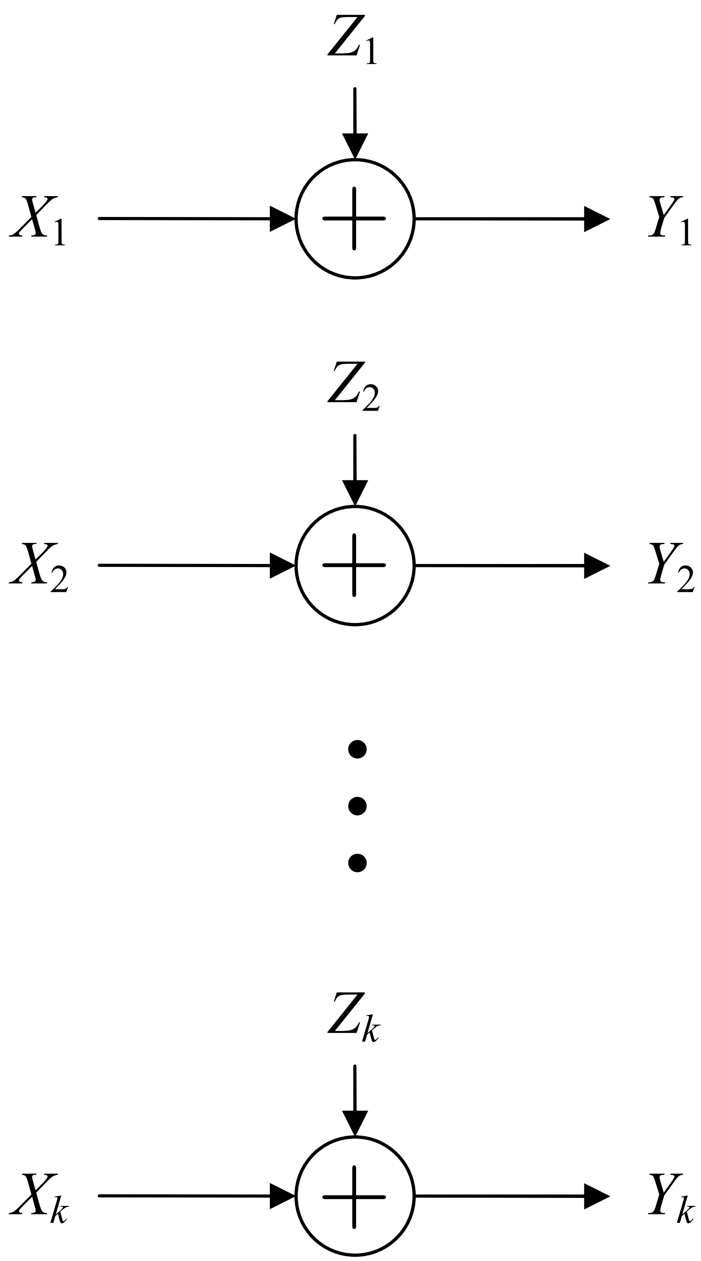
- Intuitively,

$$C(P) = \max_{P_1, P_2, \cdots, P_k : \sum_i P_i = P} \frac{1}{2}\sum_{i=1}^{k} \log\left(1 + \frac{P_i}{N_i}\right)$$



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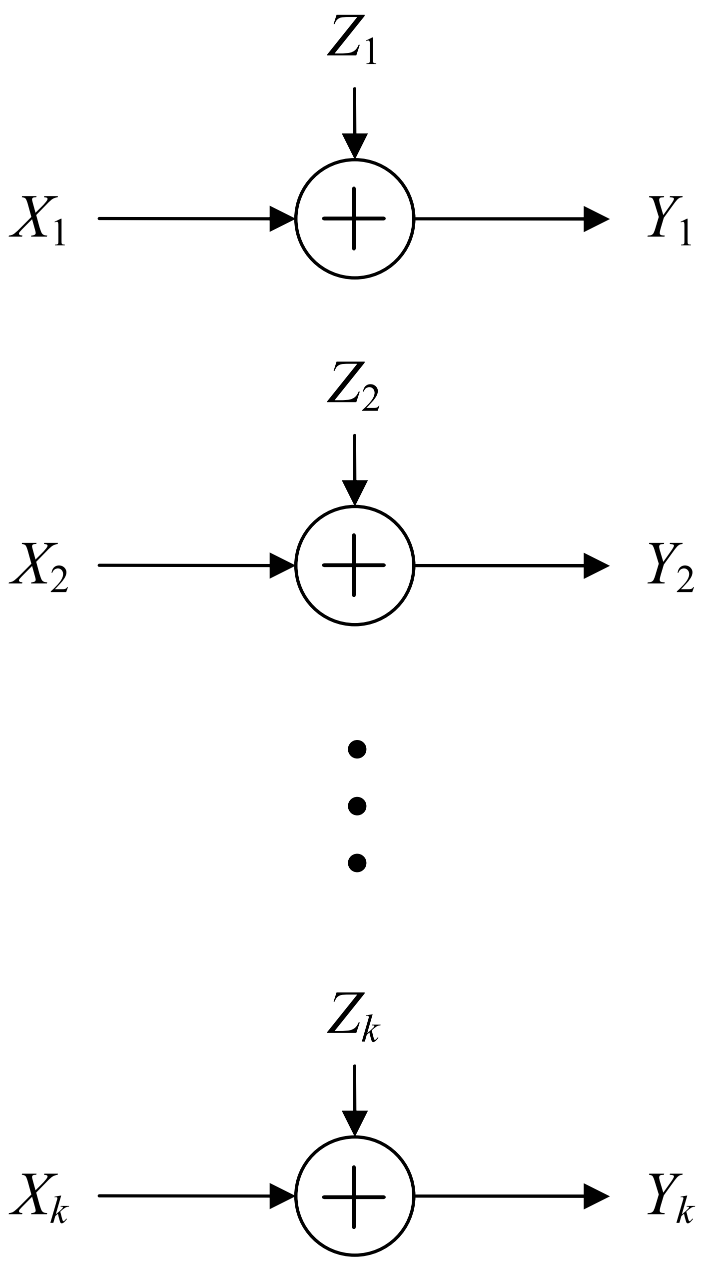
where $X_i \sim \mathcal{N}(0, P_i)$ and $X_1, X_2 \cdots, X_k$ are mutually independent.



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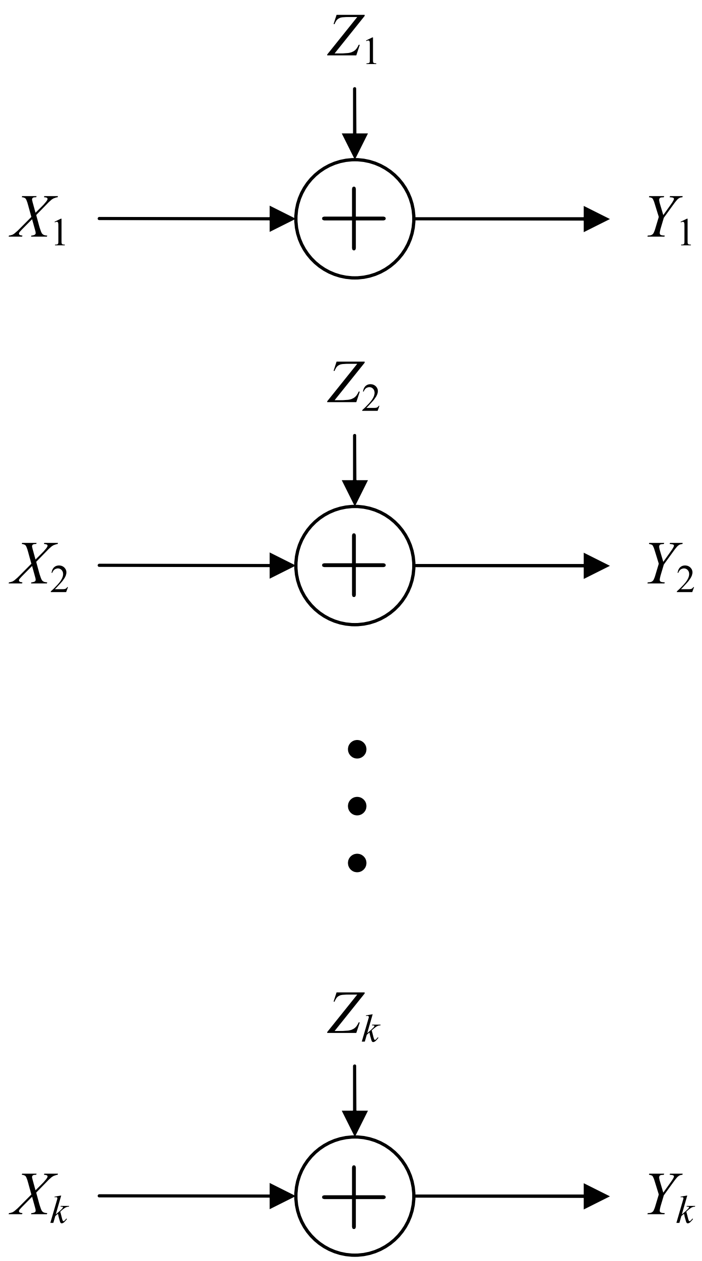
is the capacity of the ith channel when the input power is P_i.



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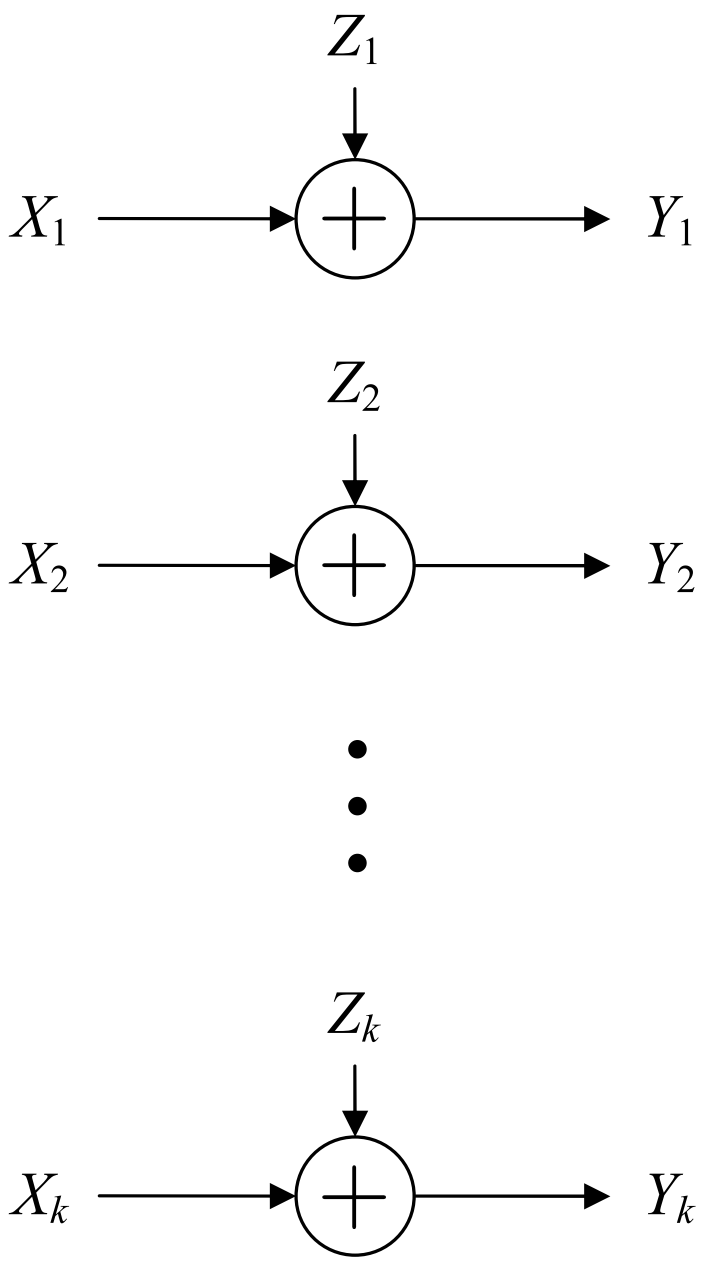
Formal Justification:



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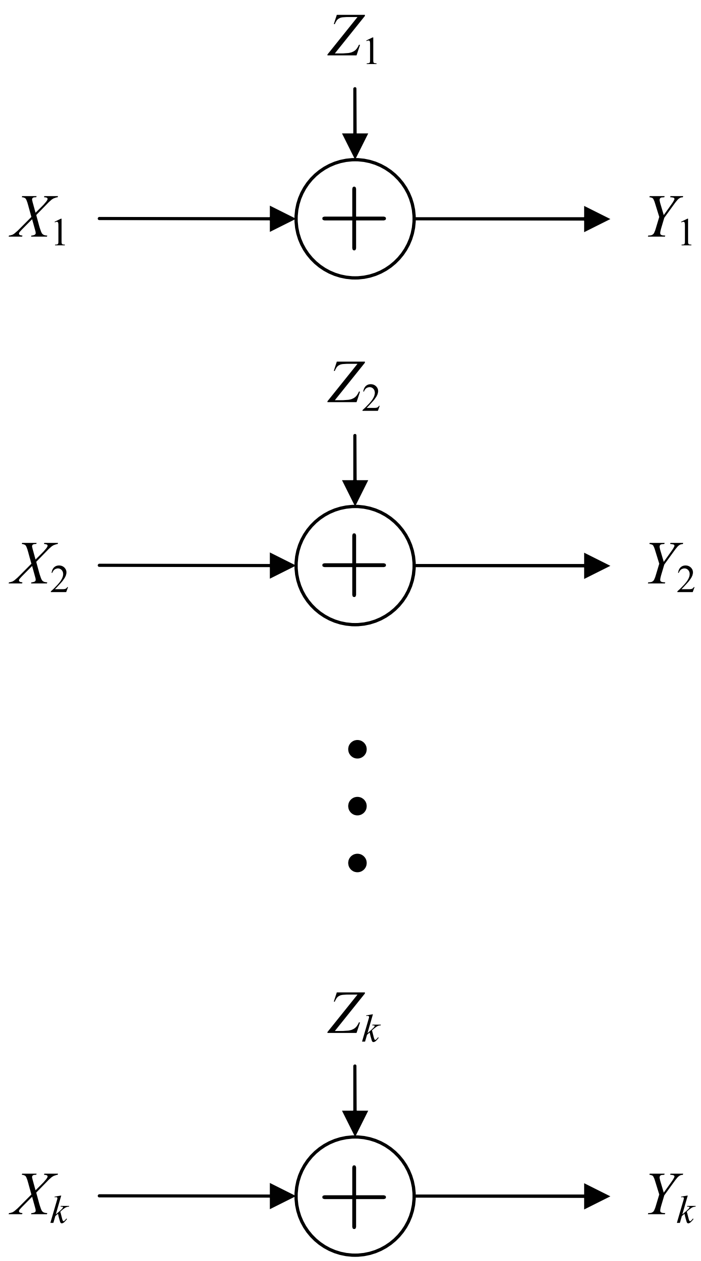
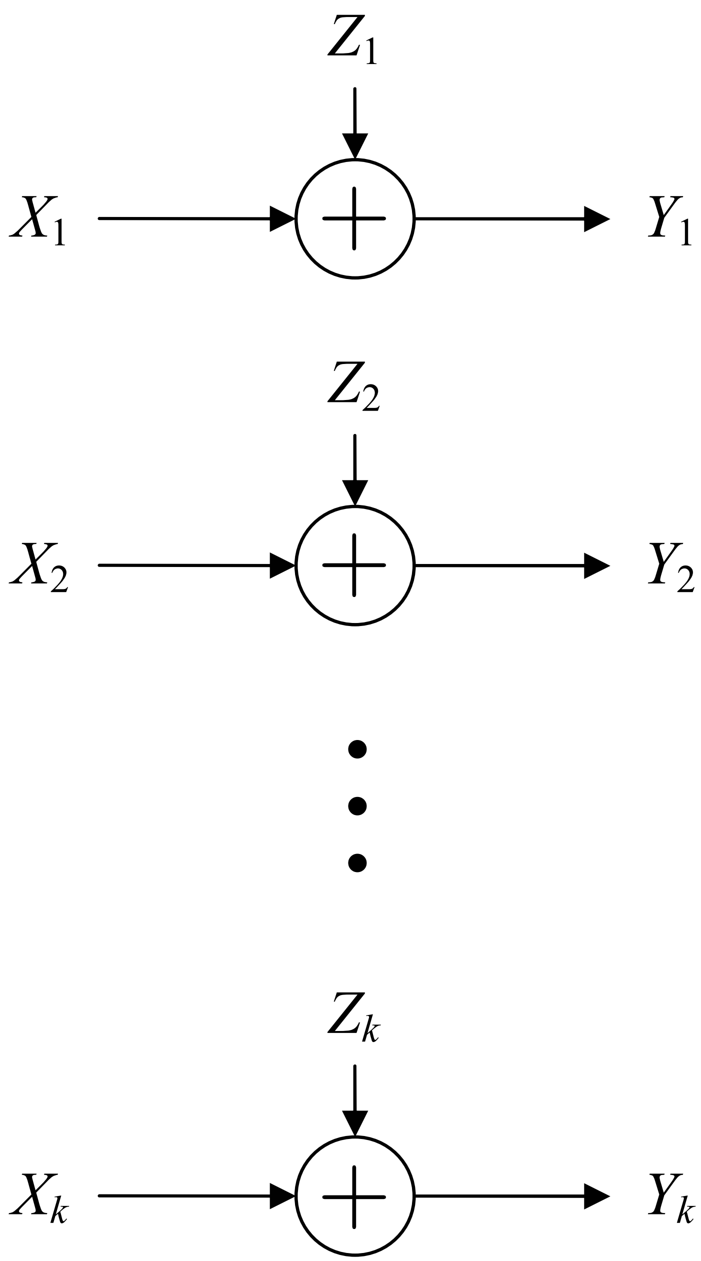


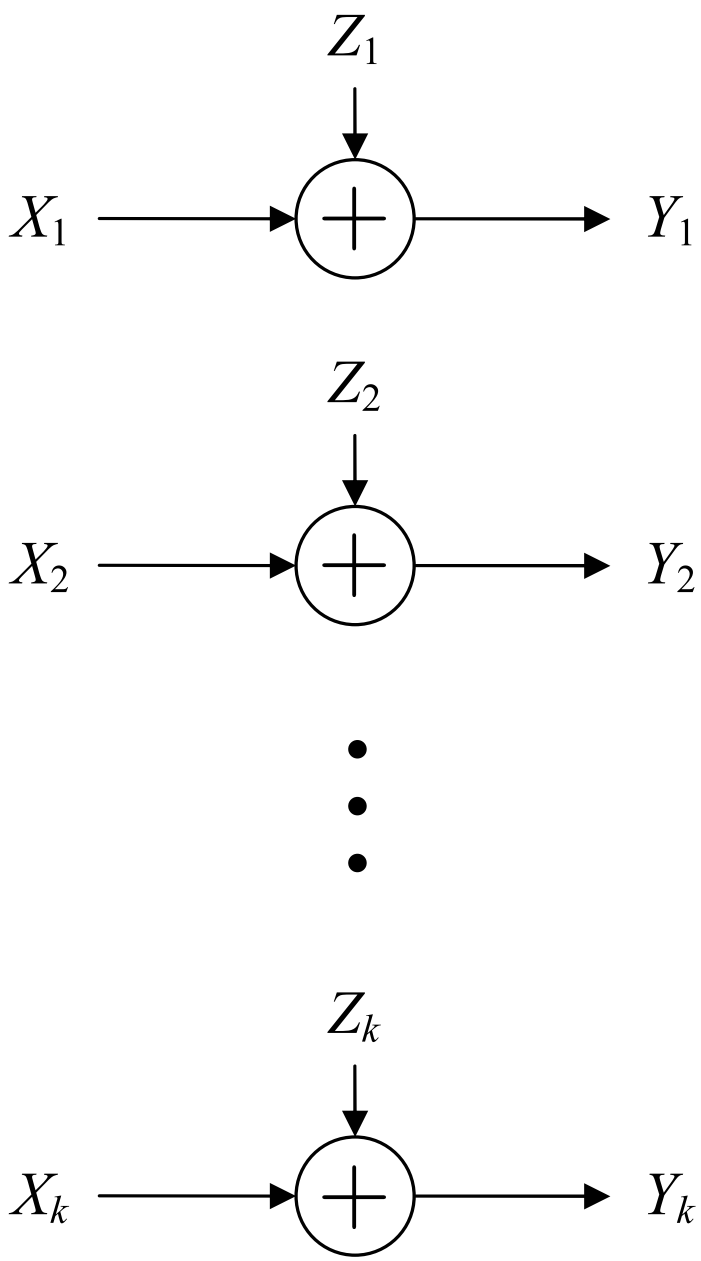
1. Let $P_i = EX_i^2$ be the input power of the ith channel. Consider



Formal Justification:

1. Let $P_i = EX_i^2$ be the input power of the ith channel. Consider

$$I(\mathbf{X};\mathbf{Y}) \quad = \quad h(\mathbf{Y}) - h(\mathbf{Z})$$



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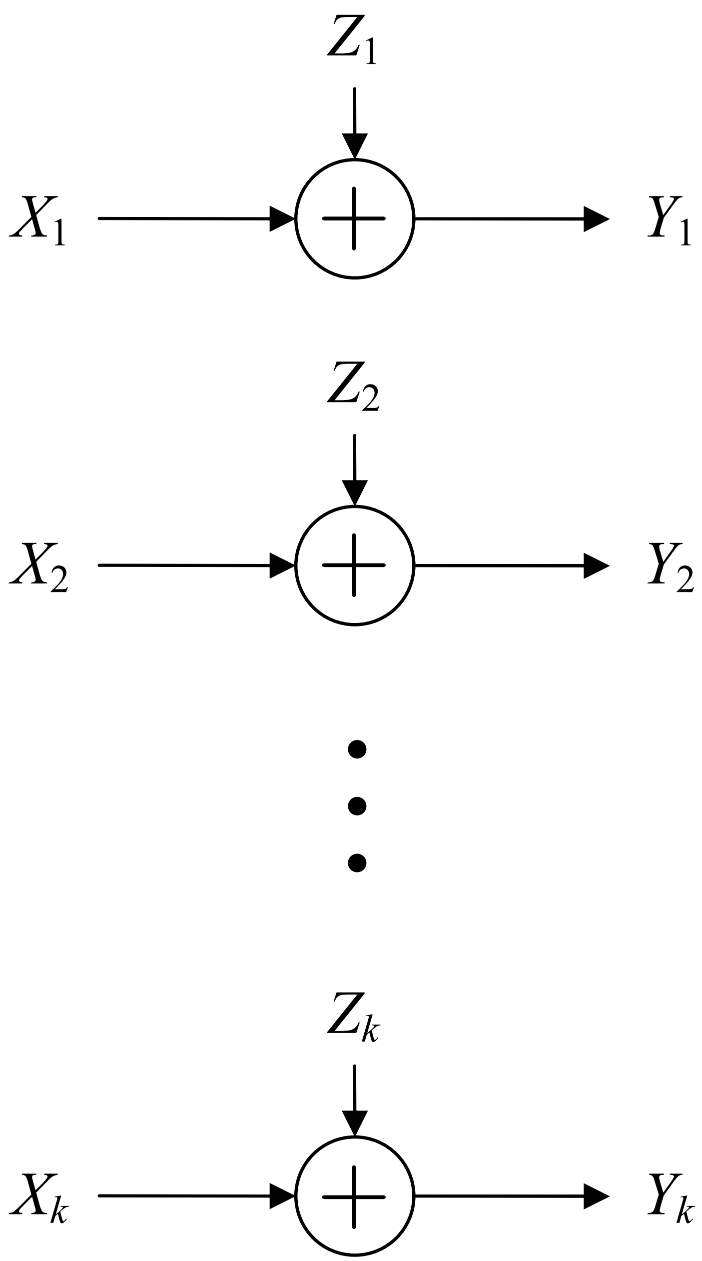

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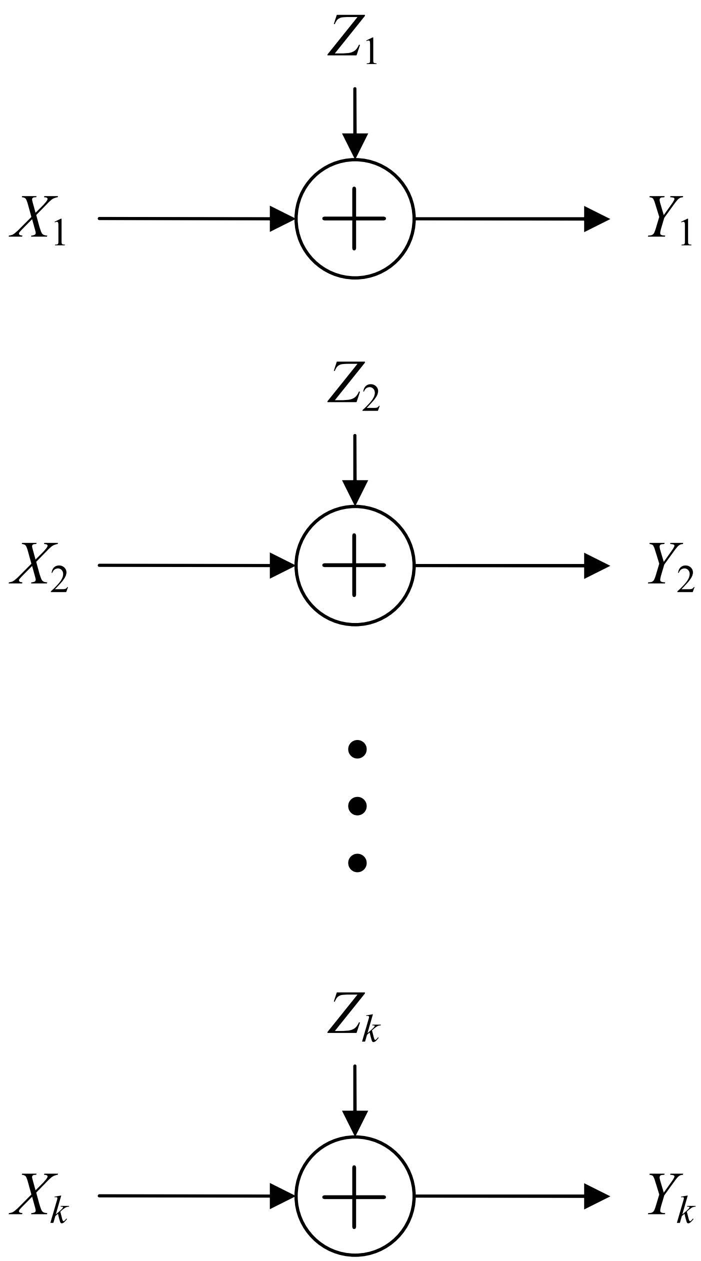
$$\begin{aligned} I(\mathbf{X};\mathbf{Y}) &= h(\mathbf{Y}) - h(\mathbf{Z}) \\ &\le \sum_{i=1}^{k} h(Y_i) - \sum_{i=1}^{k} h(Z_i) \qquad (1) \end{aligned}$$



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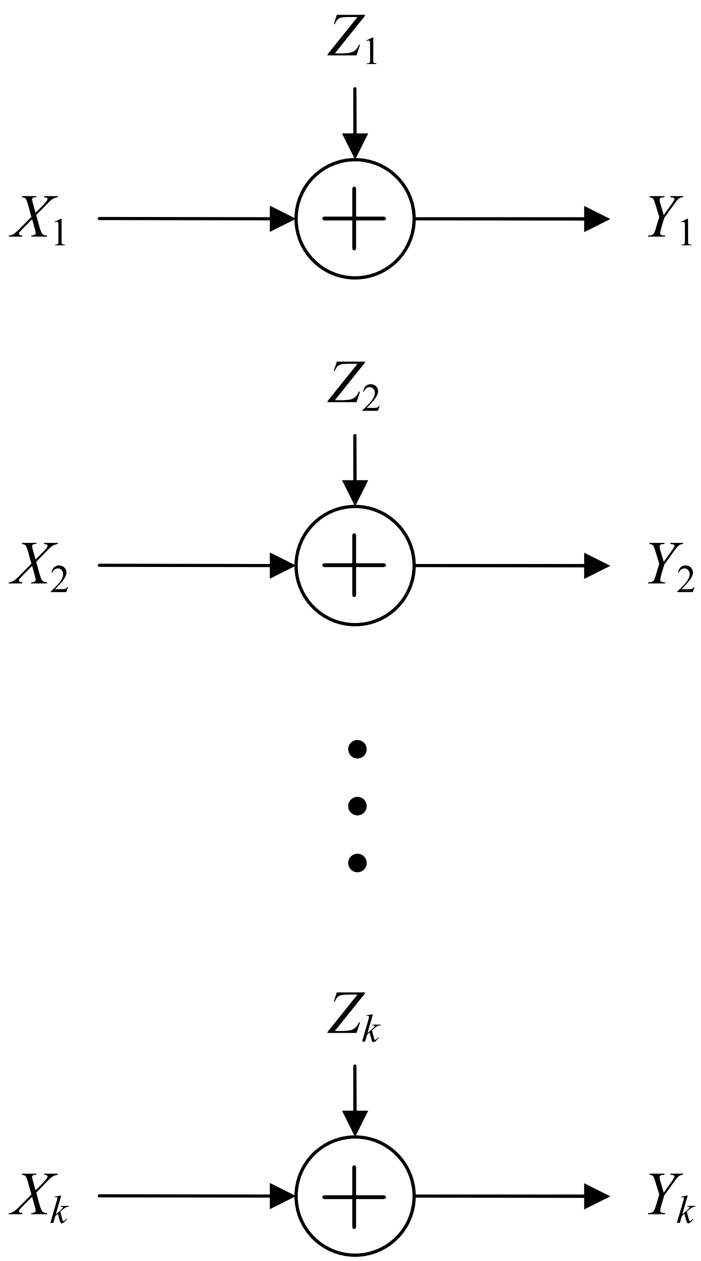
$$\begin{aligned} I(\mathbf{X};\mathbf{Y}) &= h(\mathbf{Y}) - \underline{h(\mathbf{Z})} \\ &\leq \sum_{i=1}^{k} h(Y_i) - \sum_{i=1}^{k} h(Z_i) \qquad (1) \end{aligned}$$



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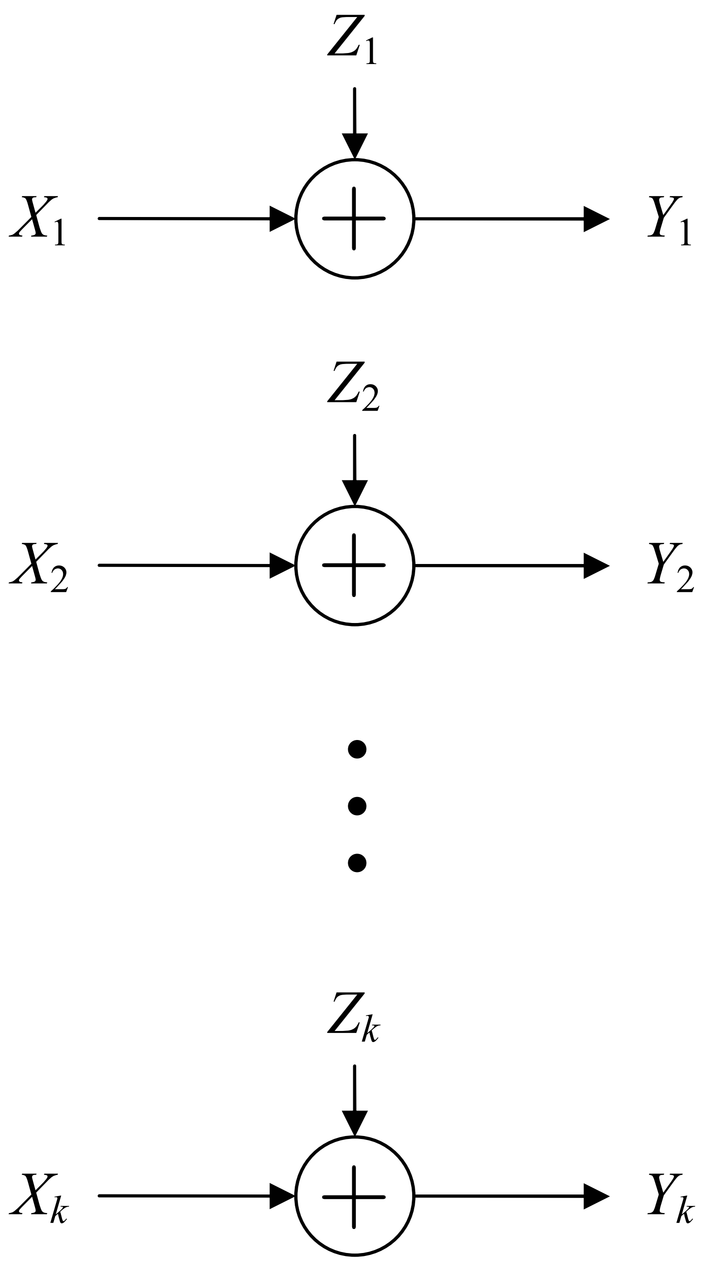

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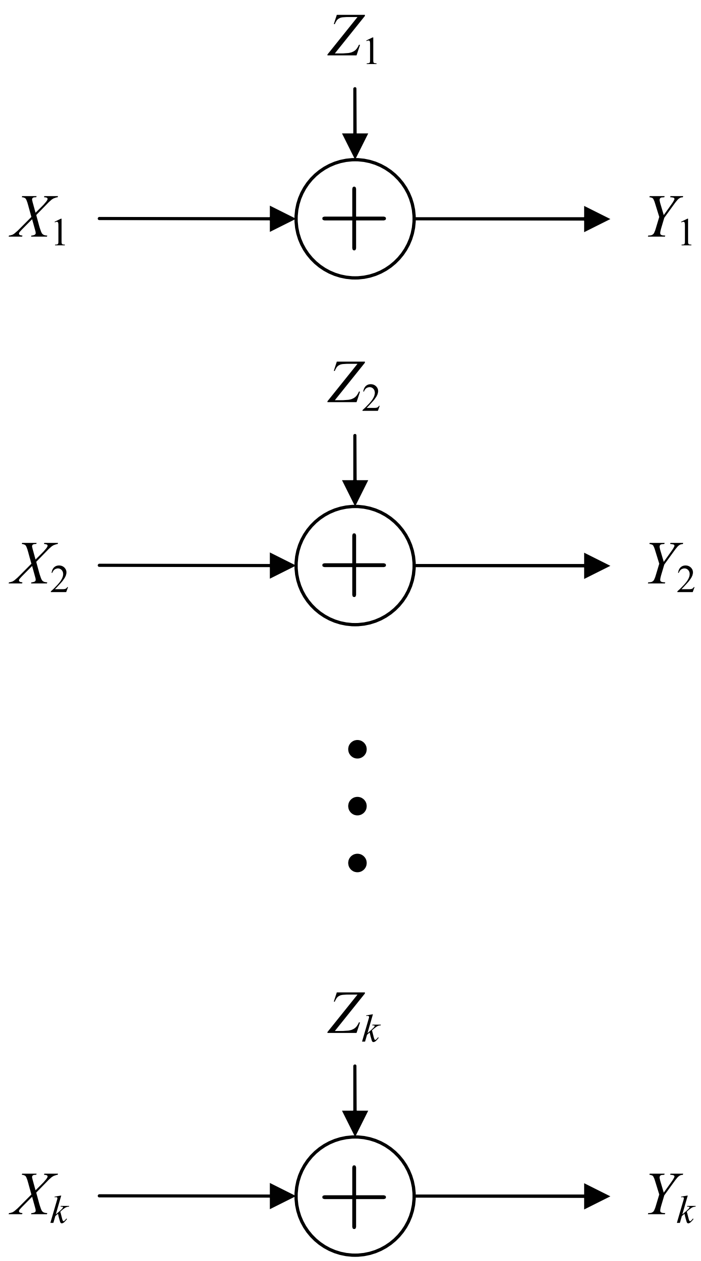

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I(\mathbf{X};\mathbf{Y}) &= h(\mathbf{Y}) - h(\mathbf{Z}) \\
&\le \sum_{i=1}^{k} h(Y_i) - \sum_{i=1}^{k} h(Z_i) && (1) \\
&\le \frac{1}{2}\sum_{i=1}^{k} \log[2\pi e(EY_i^2)] - \frac{1}{2}\sum_{i=1}^{k} \log(2\pi e N_i) && (2)
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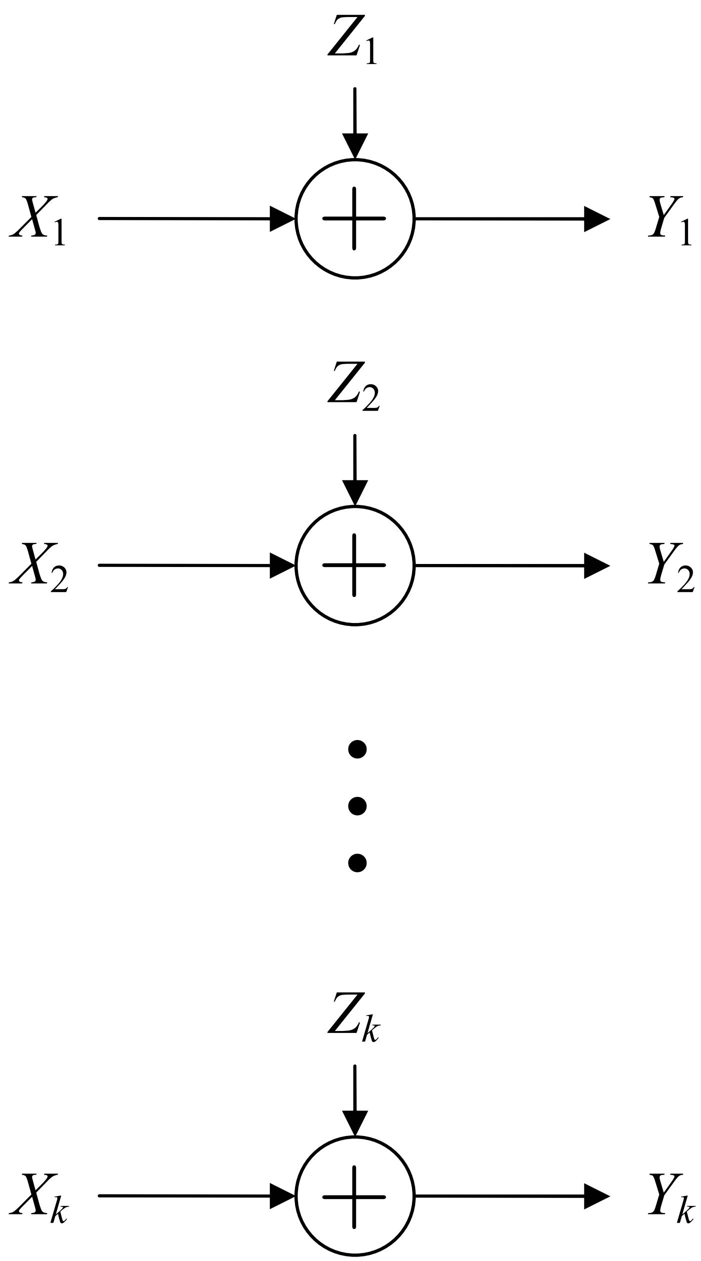

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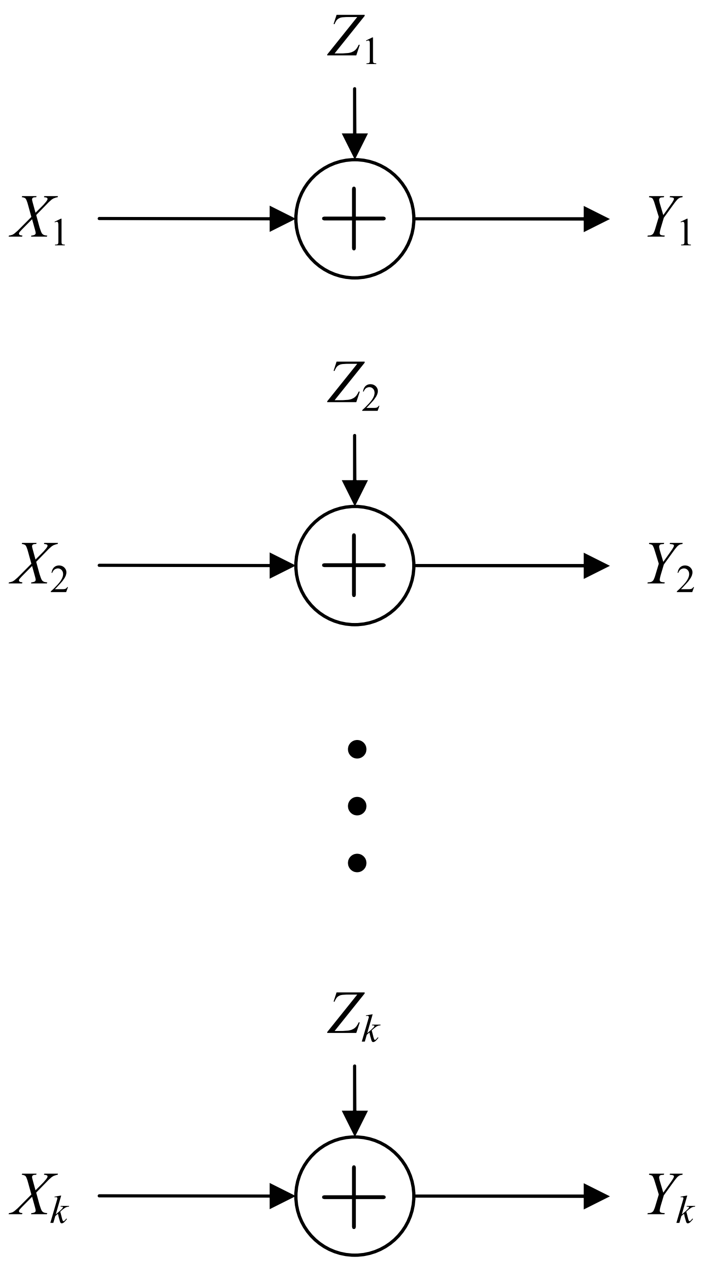

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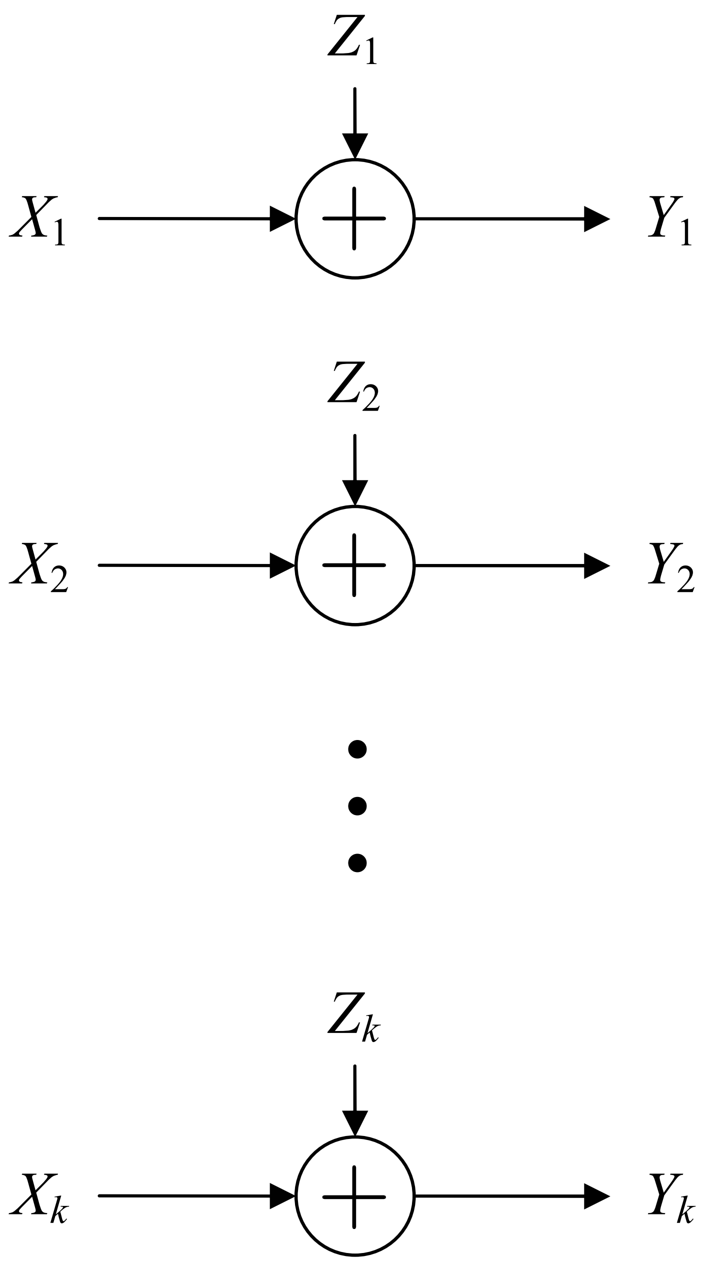

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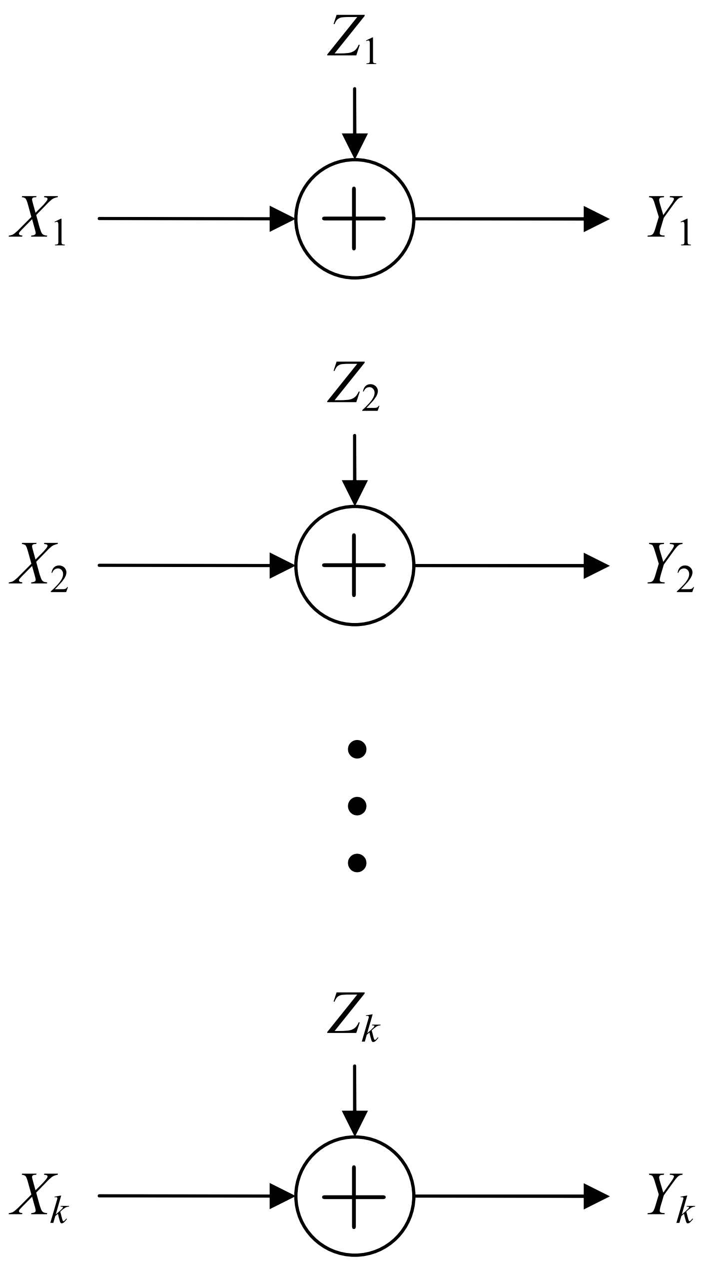

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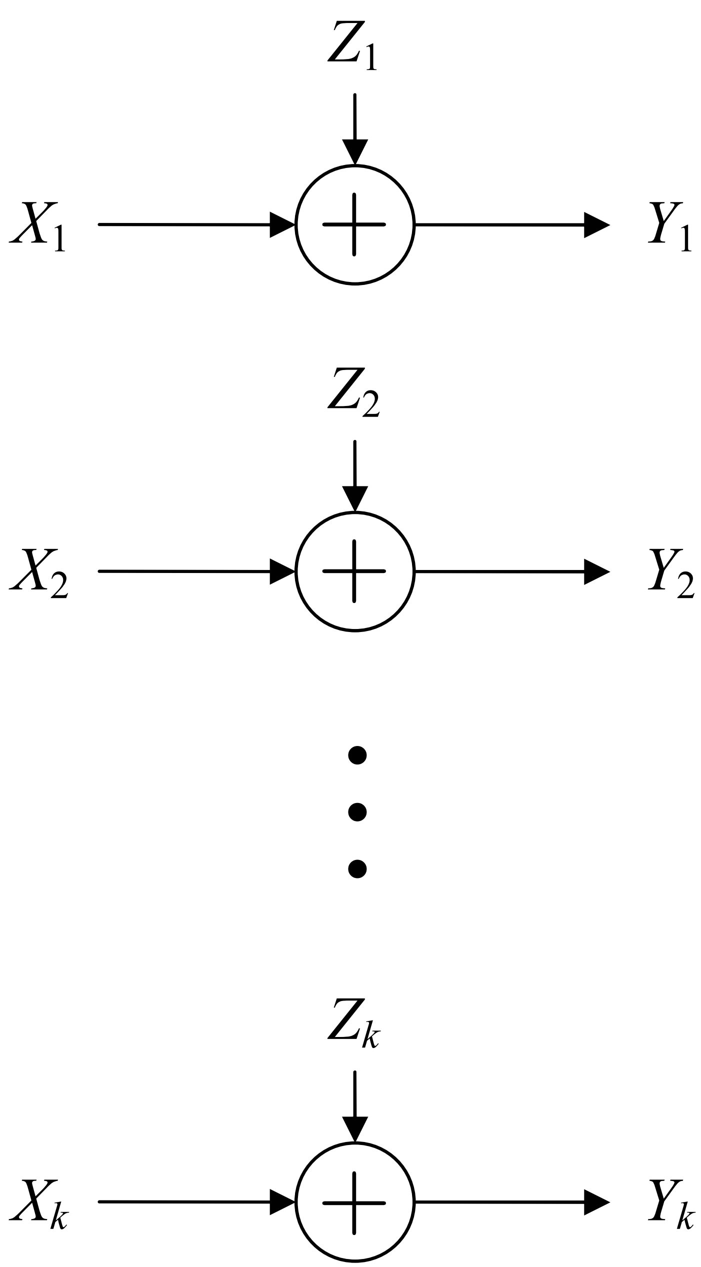

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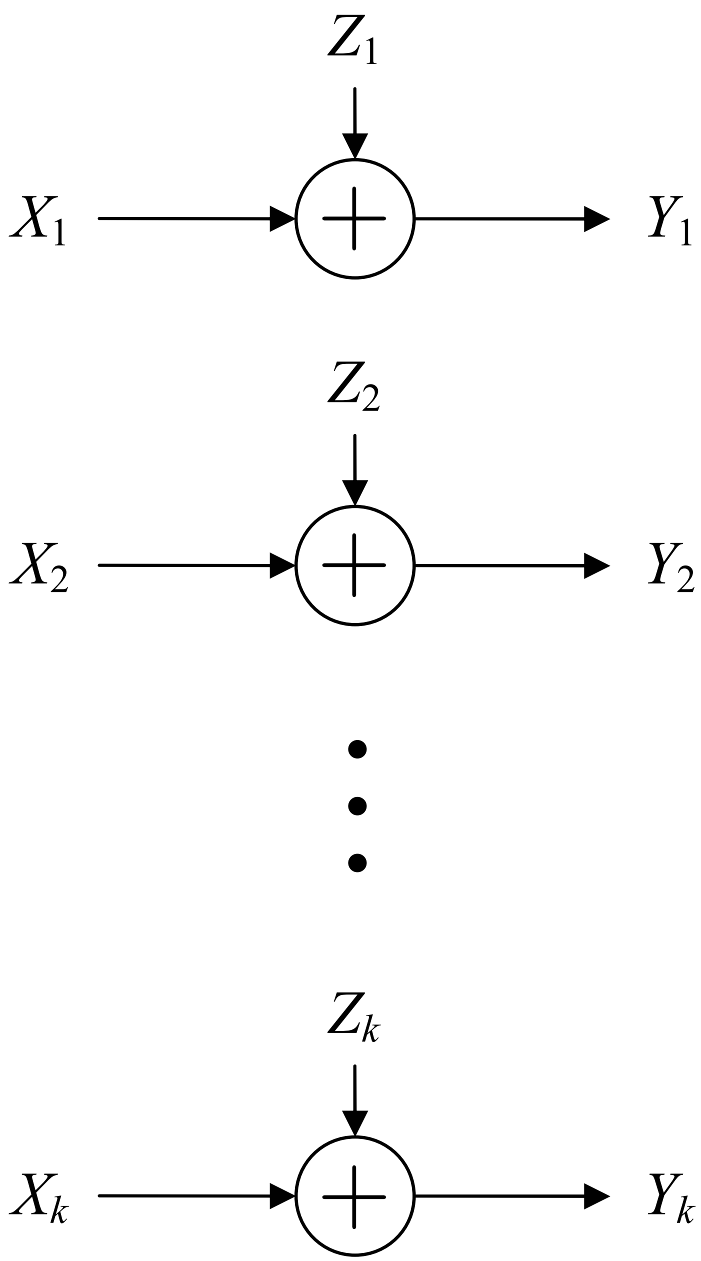

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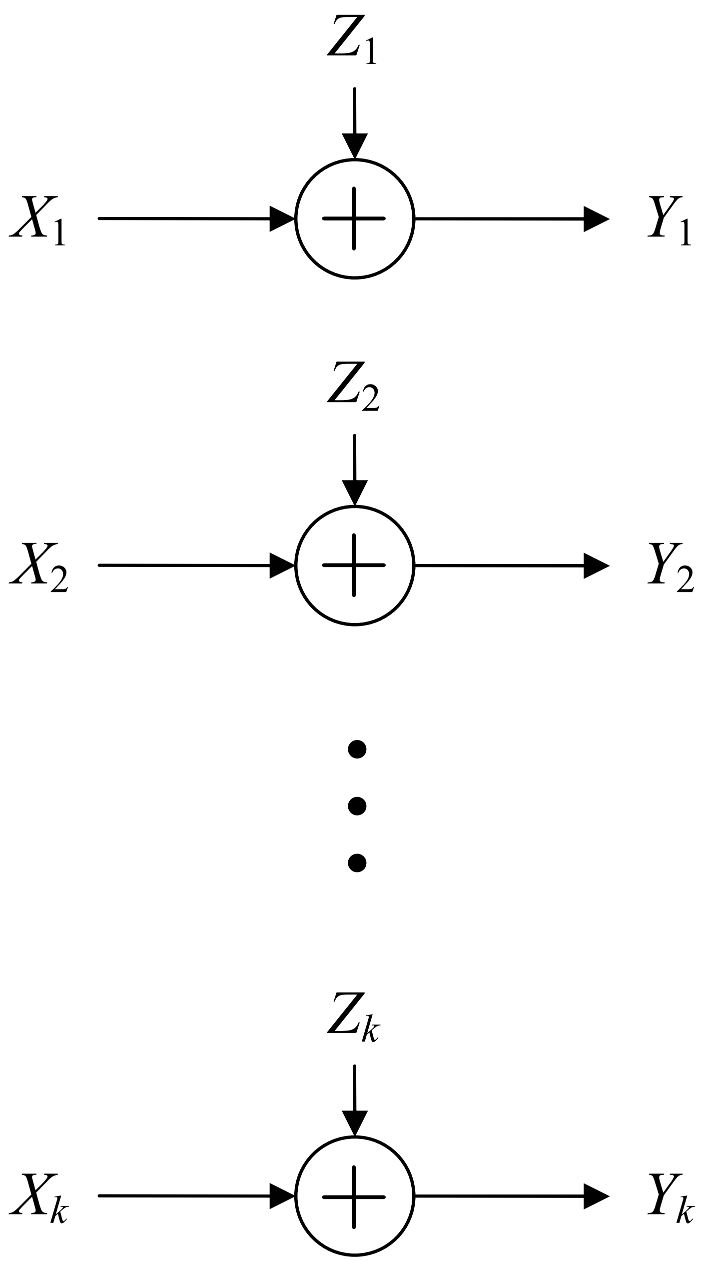

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&= \frac{1}{2}\sum_{i=1}^{k} \log(\underline{EX_i^2 + EZ_i^2}) - \frac{1}{2}\sum_{i=1}^{k} \log N_i
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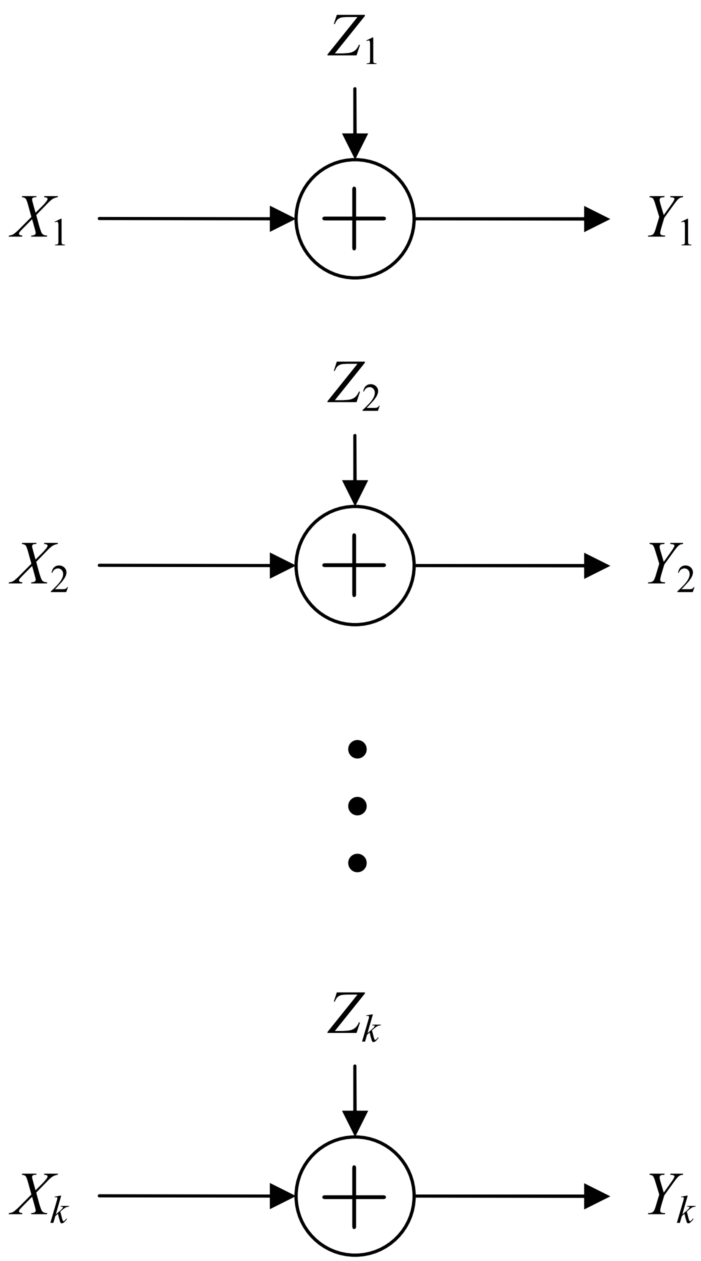

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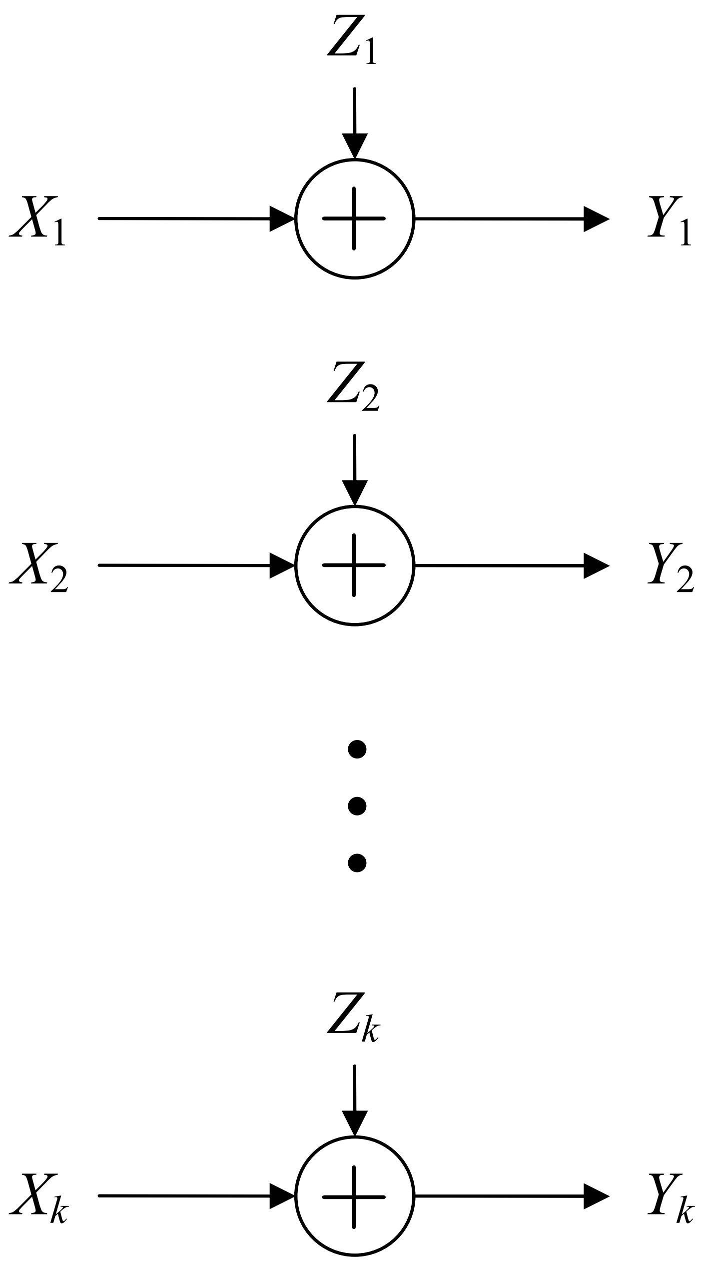

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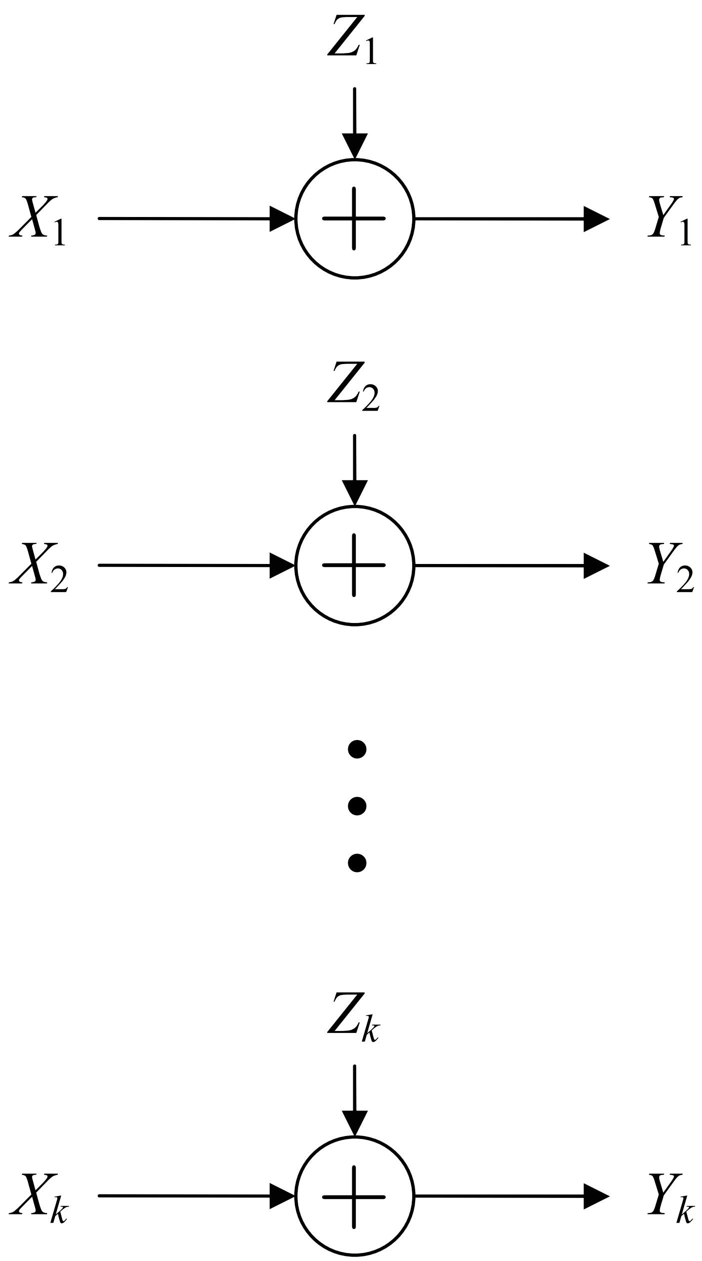
&= \frac{1}{2}\sum_{i=1}^{k} \log(EX_i^2 + EZ_i^2) - \frac{1}{2}\sum_{i=1}^{k} \log N_i \\
&= \frac{1}{2}\sum_{i=1}^{k} \log(P_i + N_i) - \frac{1}{2}\sum_{i=1}^{k} \log N_i \qquad (3)
\end{aligned}$$



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&= \frac{1}{2}\sum_{i=1}^{k} \log(EX_i^2 + \underline{EZ_i^2}) - \frac{1}{2}\sum_{i=1}^{k} \log N_i \\
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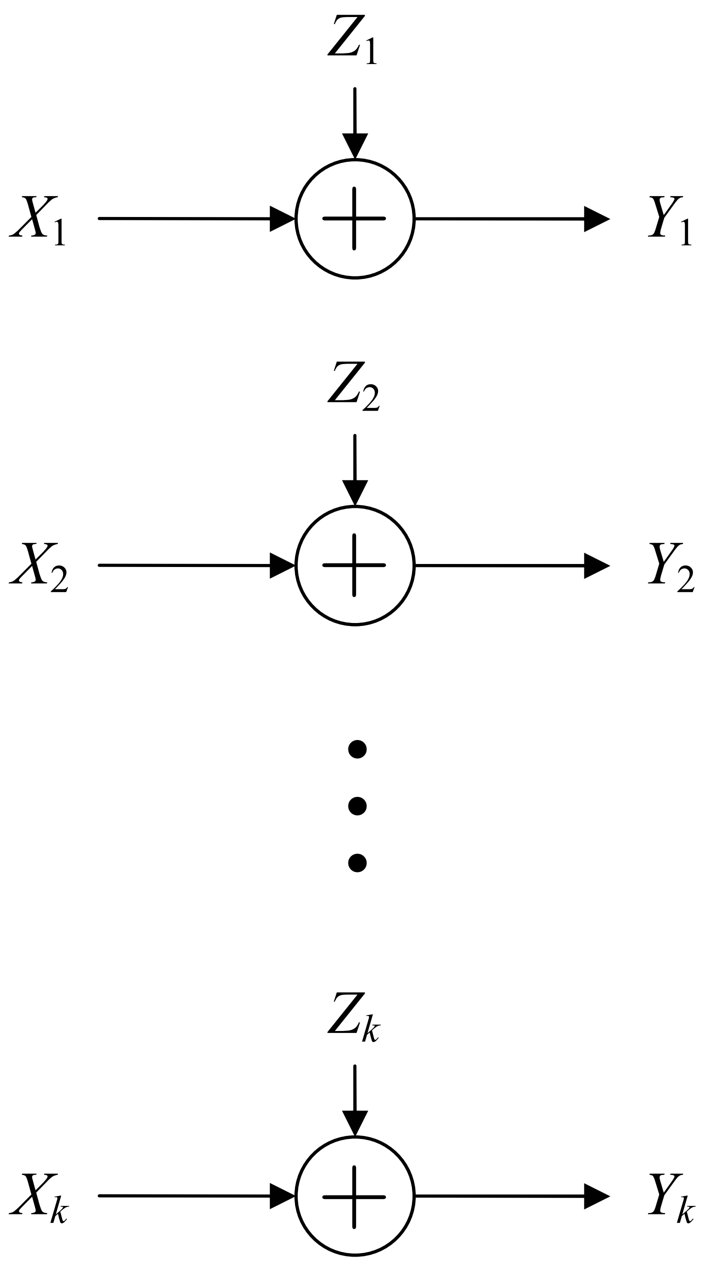

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&\leq \frac{1}{2}\sum_{i=1}^{k} \log[2\pi e(EY_i^2)] - \frac{1}{2}\sum_{i=1}^{k} \log(2\pi e N_i) && (2) \\
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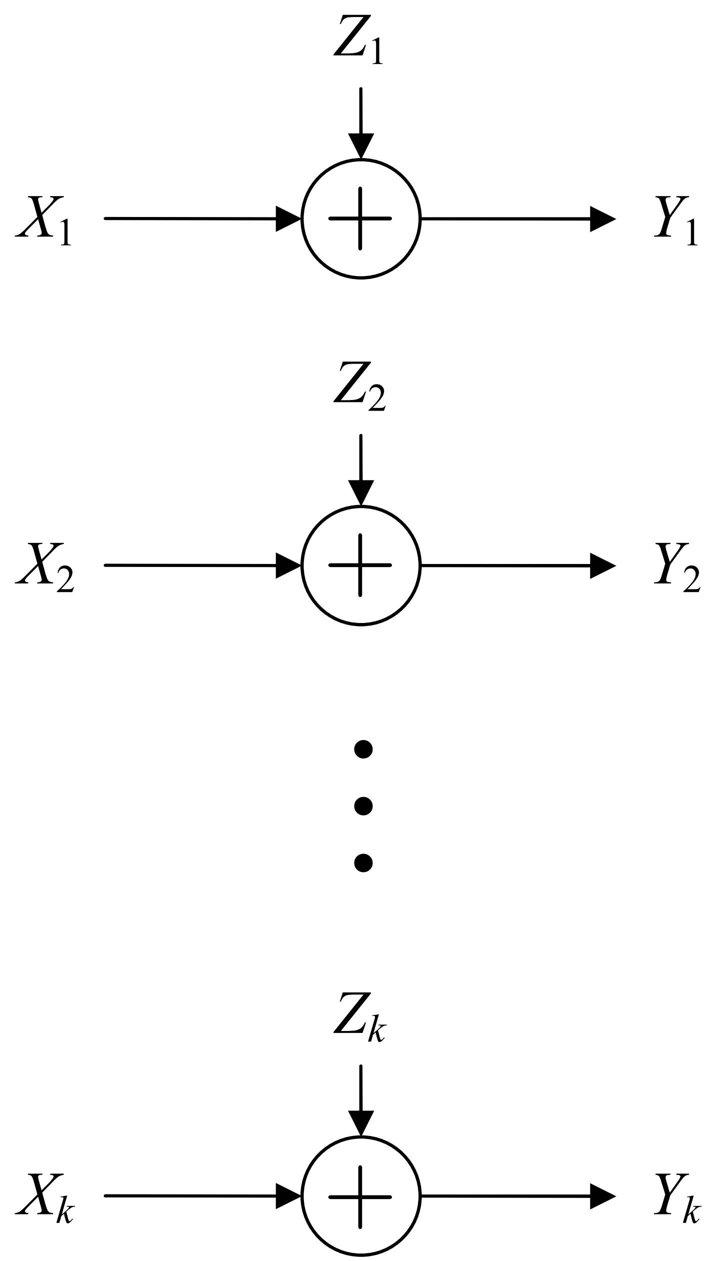

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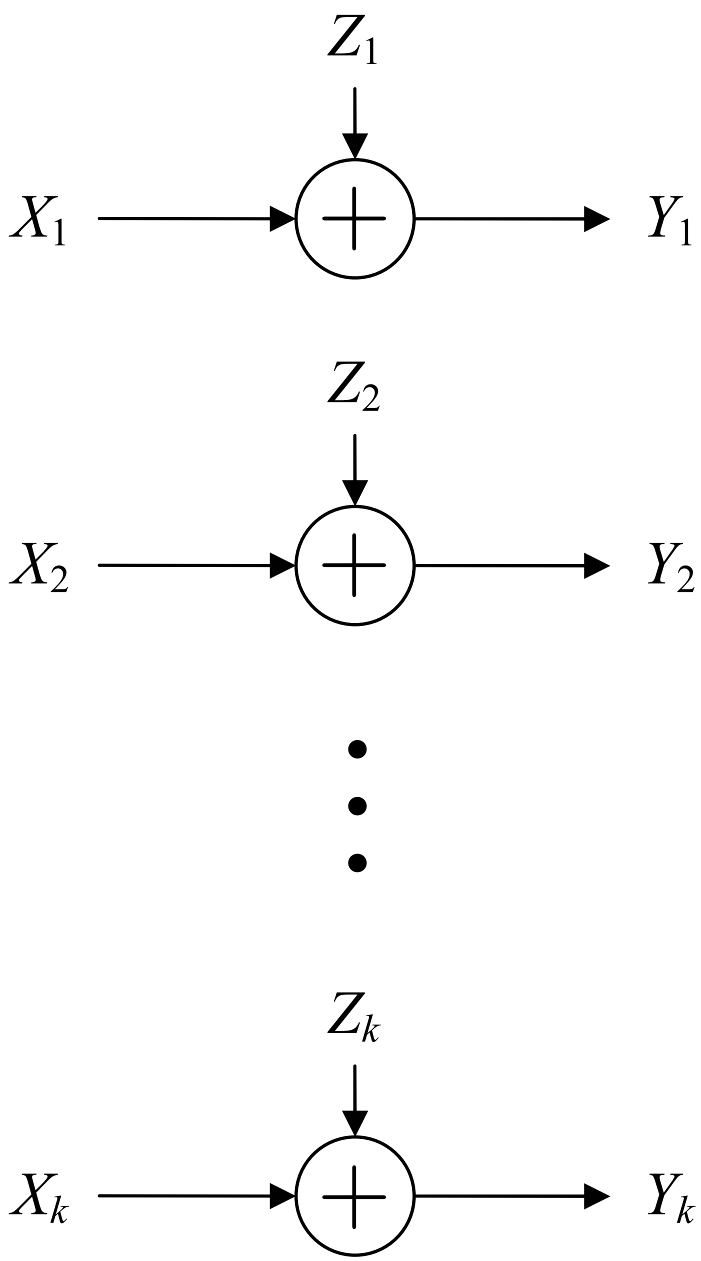
\end{aligned}
$$



Formal Justification:

1. Let $P_i = EX_i^2$ be the input power of the ith channel. Consider

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I(\mathbf{X};\mathbf{Y}) &= h(\mathbf{Y}) - h(\mathbf{Z}) \\
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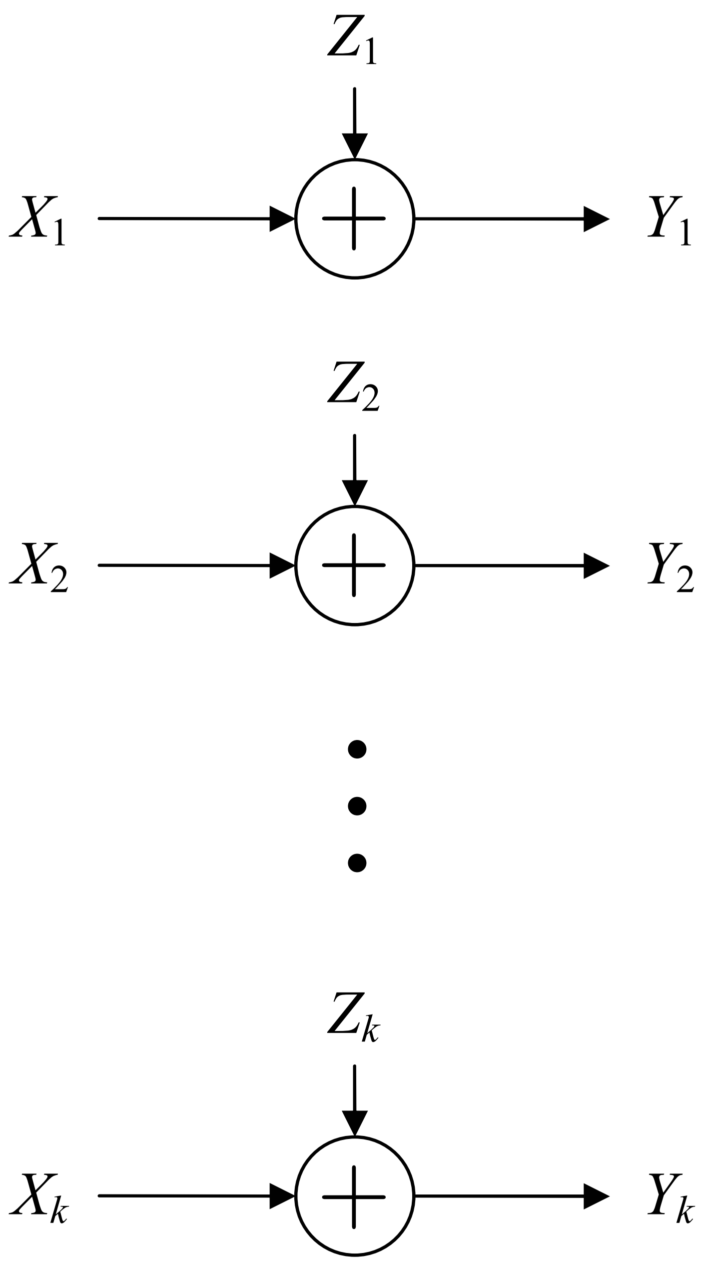

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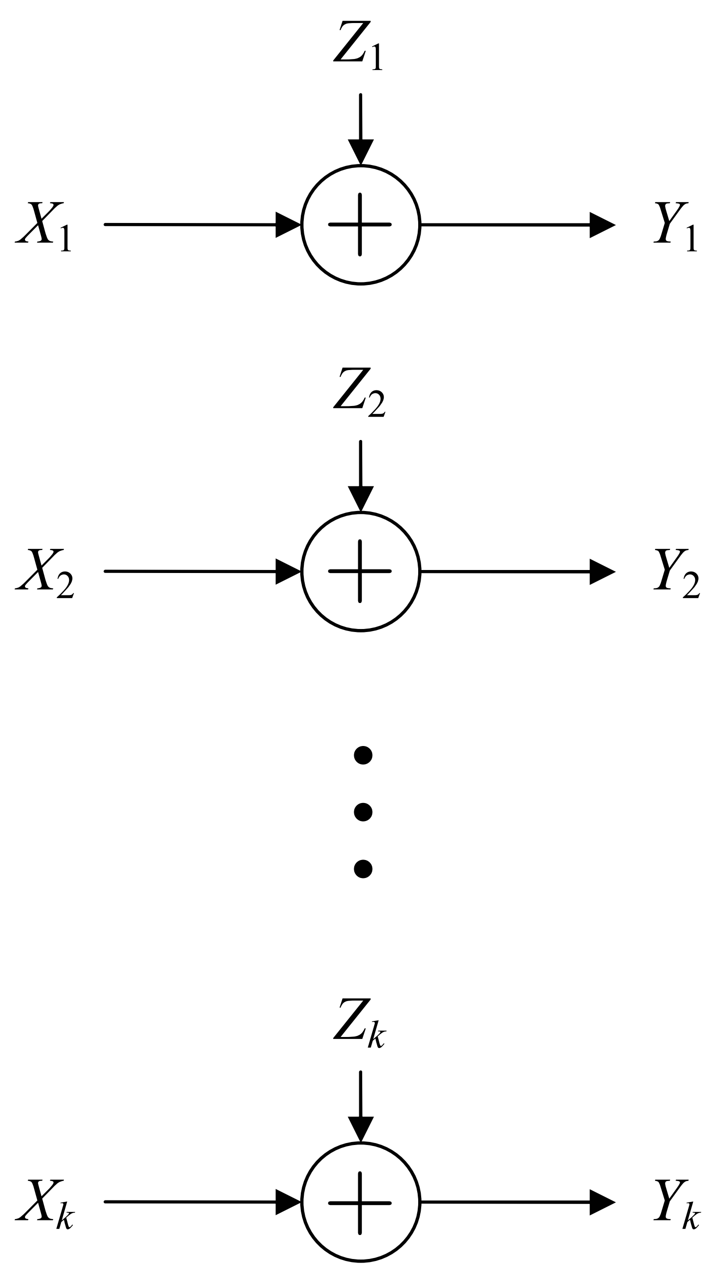

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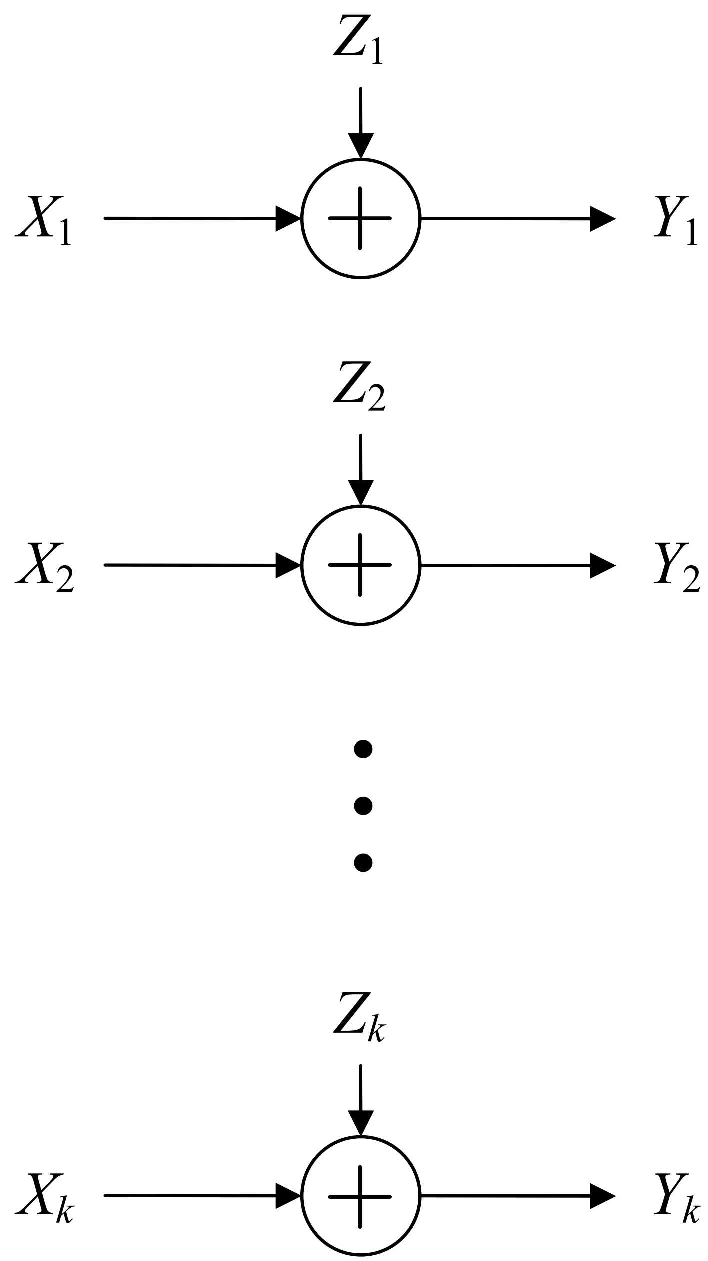
2. The inequalities in (1) and (2) are tight when X_i's are independent and $X_i \sim \mathcal{N}(0, P_i)$.



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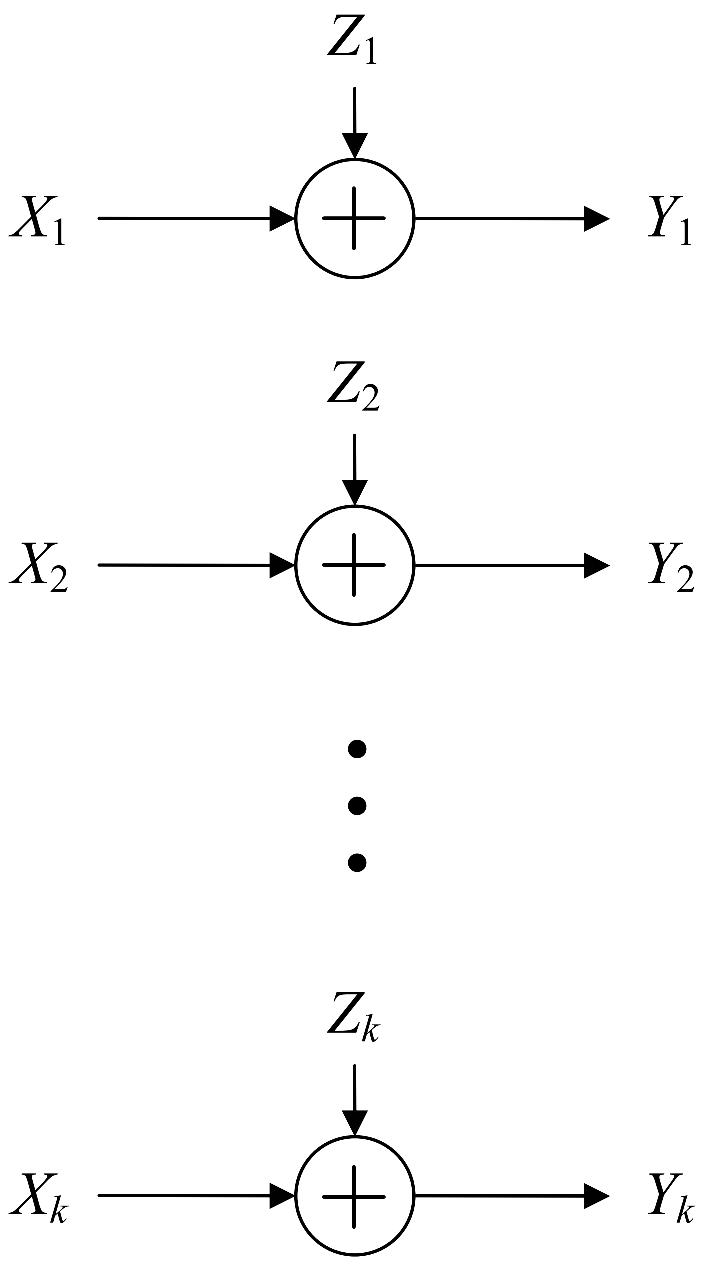

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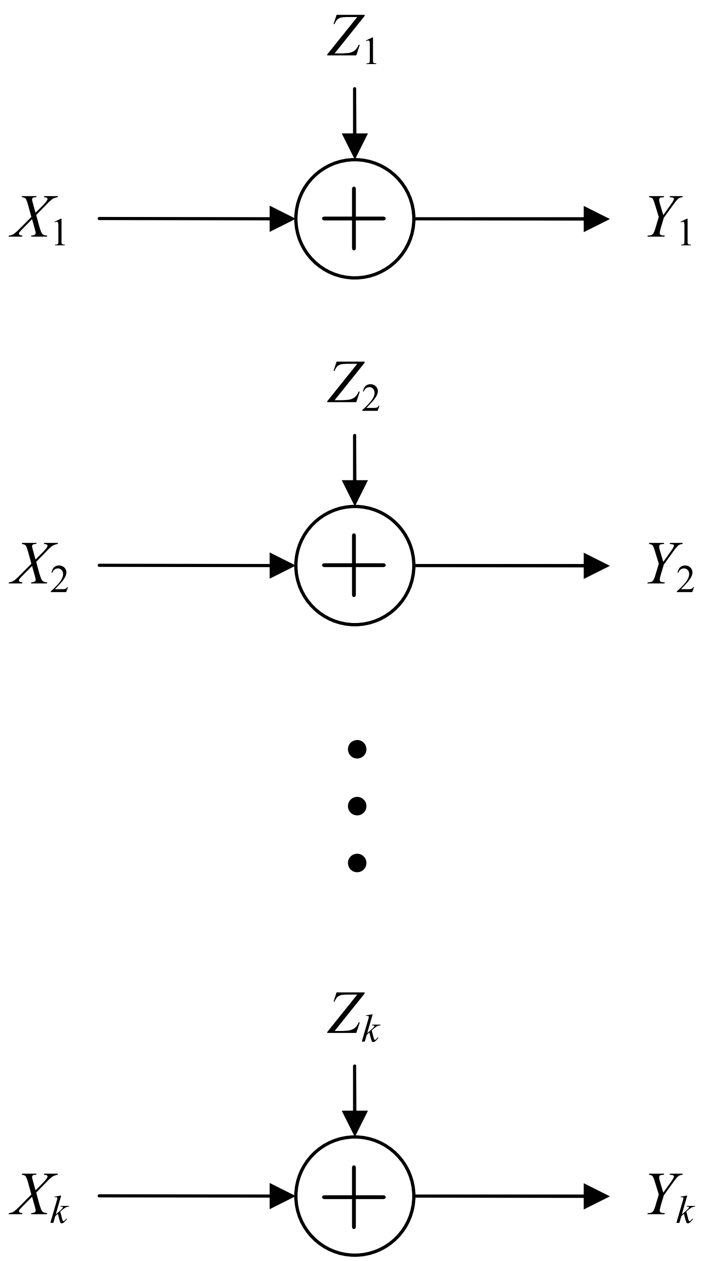

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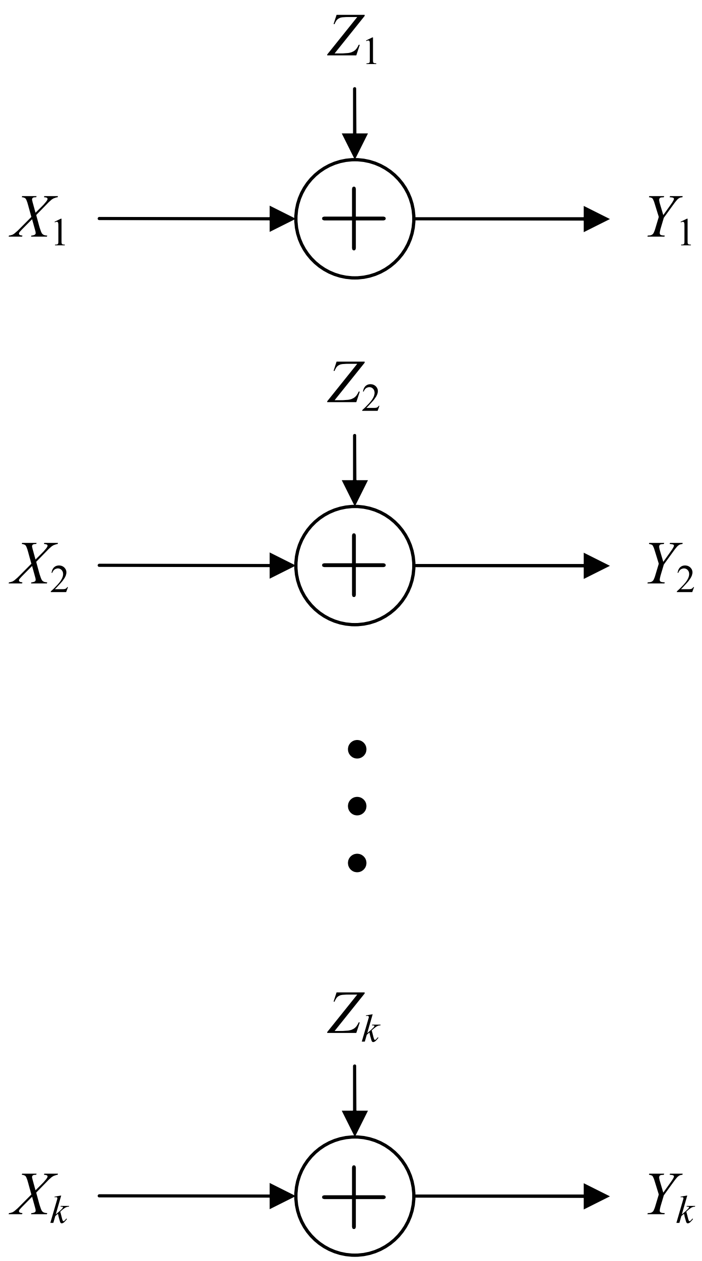
3. Therefore, maximizing $I(\mathbf{X};\mathbf{Y})$ becomes maximizing $\sum_i \log(P_i + N_i)$ in (3).



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4. The capacity of the system of parallel Gaussian channels is equal to the sum of the capacities of the individual Gaussian channels with the input power optimally allocated.

Maximize $\sum_i \log(P_i + N_i)$ subject to $\sum_i P_i \leq P$ and $P_i \geq 0$.

Lagrange Multipliers:

Maximize $\sum_i \log(P_i + N_i)$ subject to $\sum_i P_i \leq P$ and $P_i \geq 0$.

Lagrange Multipliers:

1. Apply the method of Lagrange multipliers by temporarily ignoring the nonnegativity constraints on P_i.

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Lagrange Multipliers:

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2. Observe that in order for $\sum_i \log(P_i + N_i)$ to be maximized, $\sum_i P_i = P$ must hold because $\log\left(P_i + N_i\right)$ is increasing in P_i.

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4. Let

$$J = \sum_{i=1}^{k} \log\left(P_i + N_i\right) - \mu \sum_{i=1}^{k} P_i .$$

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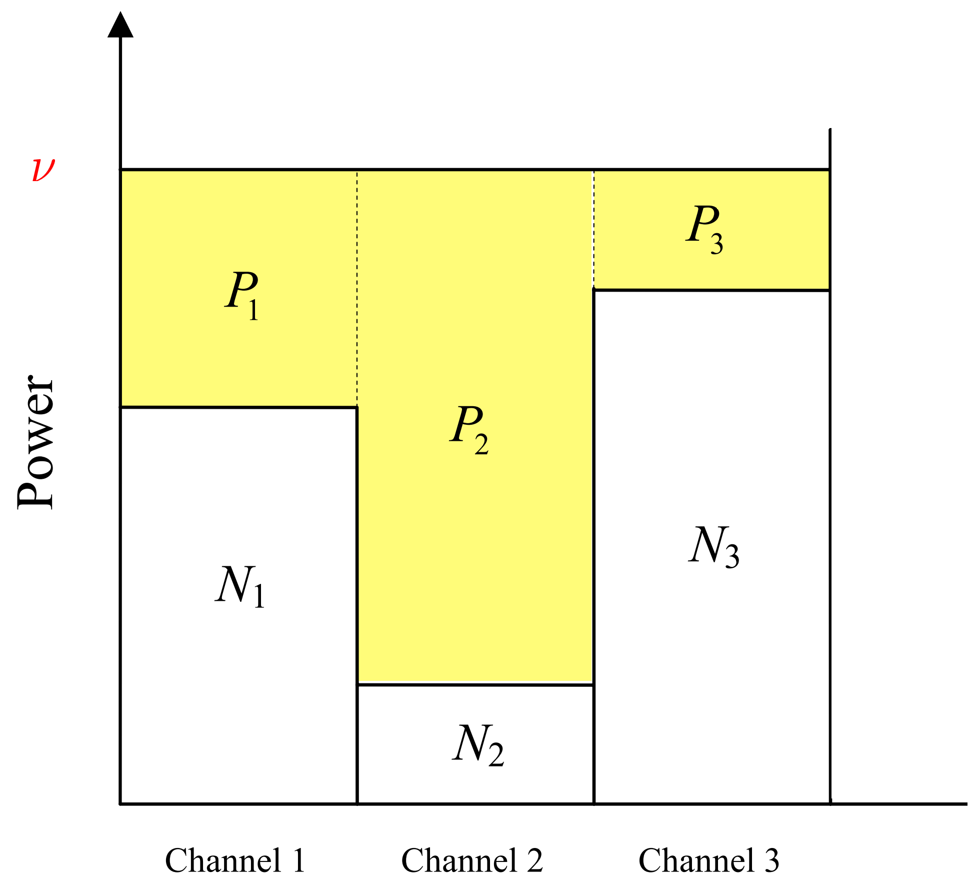

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where ν is chosen to satisfy the power constraint

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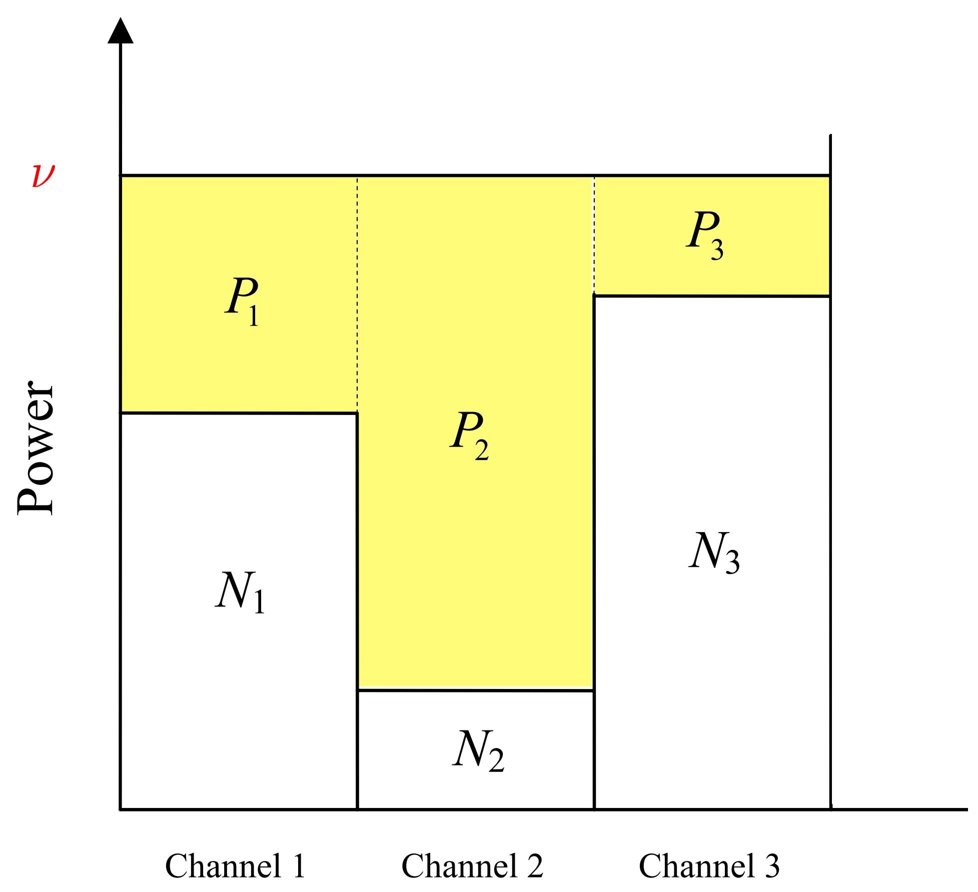

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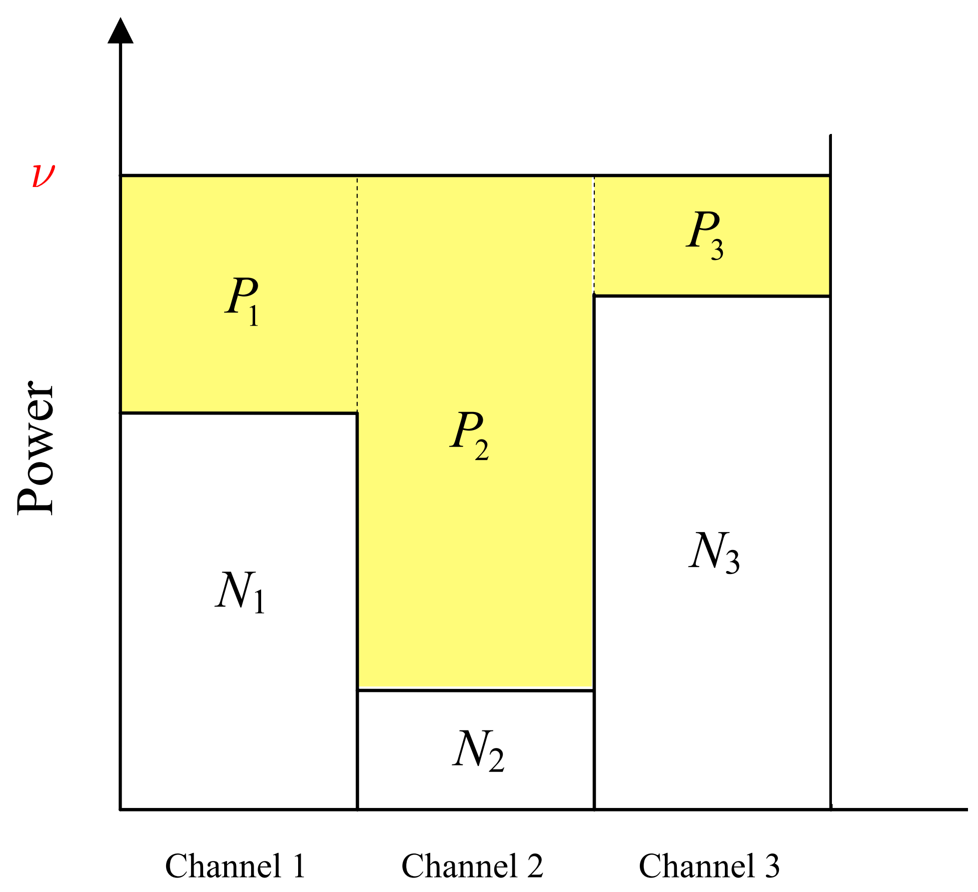

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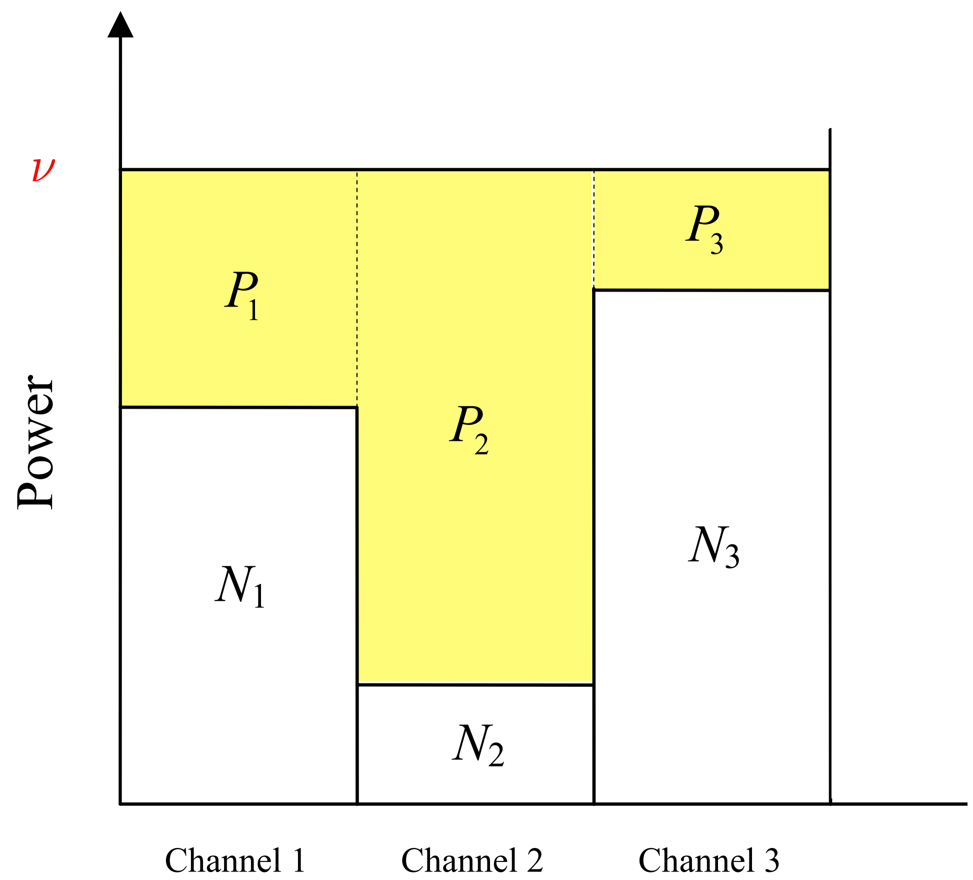

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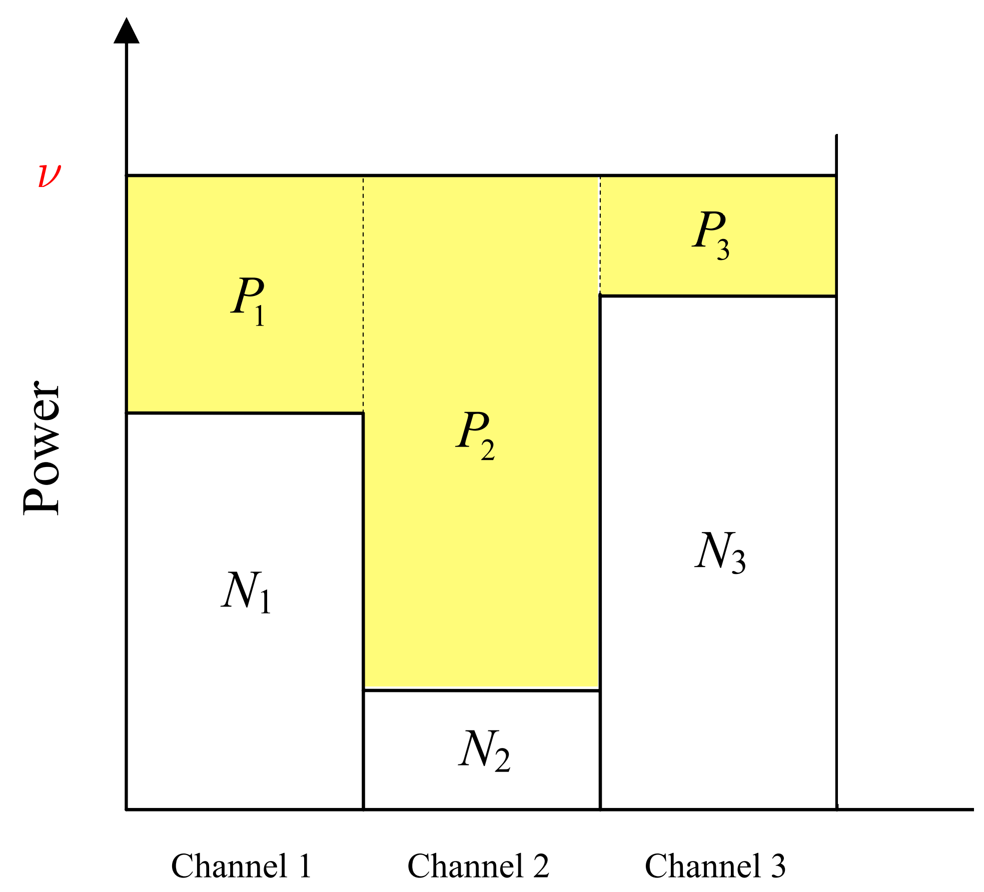

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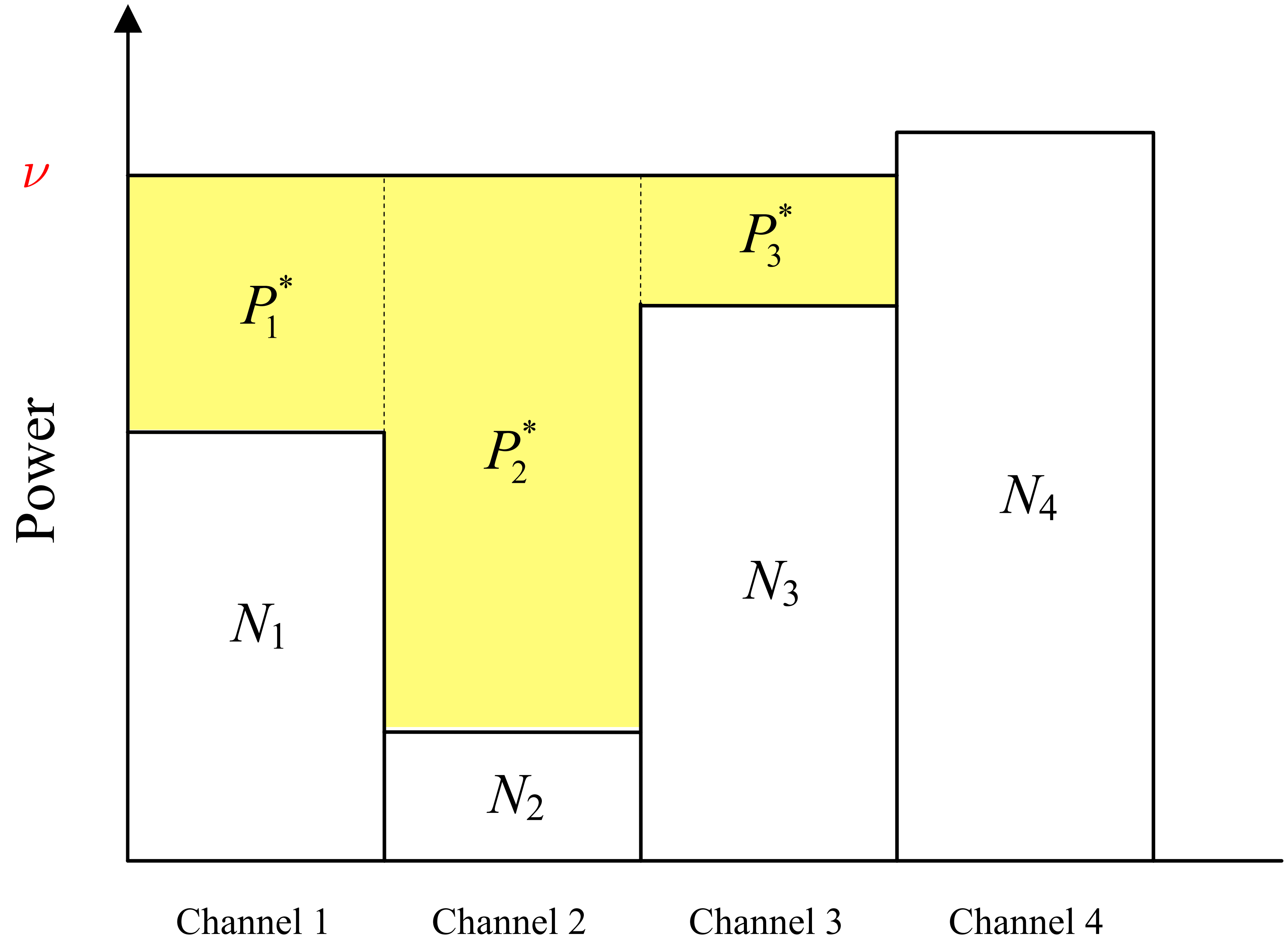

ν
Power
P*1
P*2
P*3
N1
N2
N3
N4
Channel 1
Channel 2
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Channel 4

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where μ and μ_i are the multipliers associated with the constraints in (1) and (2), respectively.

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For given $\lambda_i \geq 0$, maximize $\sum_{i=1}^{k} \log(a_i + \lambda_i)$ subject to

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which implies

$$\mu_i = (\log e)\left(\frac{1}{\nu} - \frac{1}{\lambda_i}\right) \geq 0.$$

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4. For i such that $a_i^* = 0$,

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which implies

$$\mu_i = (\log e)\left(\frac{1}{\nu} - \frac{1}{\lambda_i} \right) \geq 0.$$

Proposition 11.23 The problem

For given $\lambda_i \geq 0$, maximize $\sum_{i=1}^{k} \log(a_i + \lambda_i)$ subject to

$$\begin{aligned} \sum_i a_i &\leq P \quad (1) \\ -a_i &\leq 0. \quad (2) \end{aligned}$$

has the solution

$$a_i^* = (\nu - \lambda_i)^+, \quad 1 \leq i \leq k,$$

where ν satisfies

$$\sum_{i=1}^{k} (\nu - \lambda_i)^+ = P.$$

Proof

1. We will prove the proposition by verifying that the proposed solution satisfies the KKT condition. This is done by finding nonnegative μ and μ_i satisfying the equations

$$\begin{aligned} \frac{\log e}{(a_i^* + \lambda_i)} - \mu + \mu_i &= 0 \quad (3) \\ \mu \left(P - \sum_{i=1}^{k} a_i^* \right) &= 0 \quad (4) \\ \mu_i a_i^* &= 0, \quad 1 \leq i \leq k, \quad (5) \end{aligned}$$

where μ and μ_i are the multipliers associated with the constraints in (1) and (2), respectively.

2. To avoid triviality, assume $P > 0$ so that $\nu > 0$, and observe that there exists at least one i such that $a_i^* > 0$.

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5. Thus we have obtained nonnegative μ and μ_i satisfying the KKT condition.