香港中文大學
The Chinese University of Hong Kong

11.4 Memoryless Gaussian Channel

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1. the Gaussian channel is highly analytically tractable
2. the Gaussian noise can be regarded as the worst kind of additive noise subject to a constraint on the noise power.

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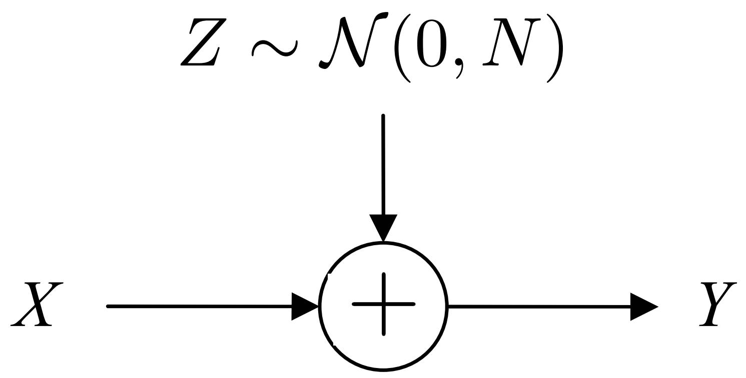

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Definition 11.20 (Memoryless Gaussian Channel) A memoryless Gaussian channel with noise power N and input power constraint P is a memoryless continuous channel with the generic continuous channel being the Gaussian channel with noise energy N. The input power constraint P refers to the input constraint (κ, P) with $\kappa(x) = x^2$.

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$$\begin{aligned} EY^2 &= E(X+Z)^2 \\ &= EX^2 + EZ^2 + 2(EXZ) \\ &= EX^2 + EZ^2 + 2(EX)(EZ) \\ &\leq P + N. \end{aligned}$$

6. By Theorem 10.43,

$$h(Y) \leq \frac{1}{2} \log[2\pi e(P + N)]$$

with equality if $Y \sim \mathcal{N}(0, P + N)$. This is achieved with $X \sim \mathcal{N}(0, P)$.

7. Hence,

$$C(P) = \sup_{F(x): EX^2 \leq P} h(Y) - h(Z)$$

Lemma 11.22 Let $Y = X + Z$. Then

$$h(Y|X) = h(Z|X)$$

provided that $f_{Z|X}(z|x)$ exists for all $x \in \mathcal{S}_X$.

Theorem 11.21 (Capacity of a Memoryless Gaussian Channel) The capacity of a memoryless Gaussian channel with noise power N and input power constraint P is

$$\frac{1}{2}\log\left(1 + \frac{P}{N}\right).$$

The capacity is achieved by the input distribution $\mathcal{N}(0, P)$.

Proof

1. Let $F(x)$ be the CDF of the input random variable X such that $EX^2 \leq P$, where X is not necessarily continuous.

2. Since $Z \sim \mathcal{N}(0, N)$, f_Z exists. Then $f_{Z|X}(z|x)$ exists and is equal to $f_Z(z)$ because Z is independent of X.

3. From the proof of Lemma 11.22,

$$f_{Y|X}(y|x) = f_{Z|X}(y - x|x) = f_Z(y - x),$$

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