香港中文大學
The Chinese University of Hong Kong

Chapter 11
Continuous-Valued Channels

The Chinese University of Hong Kong

Preamble

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- A waveform channel is one such that transmission is in continuous time.
- At the physical layer, we need to deal with channels such that the values taken are continuous and transmission is in continuous time.

11.1 Discrete-Time Channel

Definition 11.1 (Continuous Channel I) Let $f(y|x)$ be a conditional pdf defined for all x, where

$$-\int_{\mathcal{S}_Y(x)} f(y|x) \log f(y|x) dy < \infty$$

for all x. A (discrete-time) continuous channel $f(y|x)$ is a system with input random variable X and output random variable Y such that Y is related to X through $f(y|x)$.

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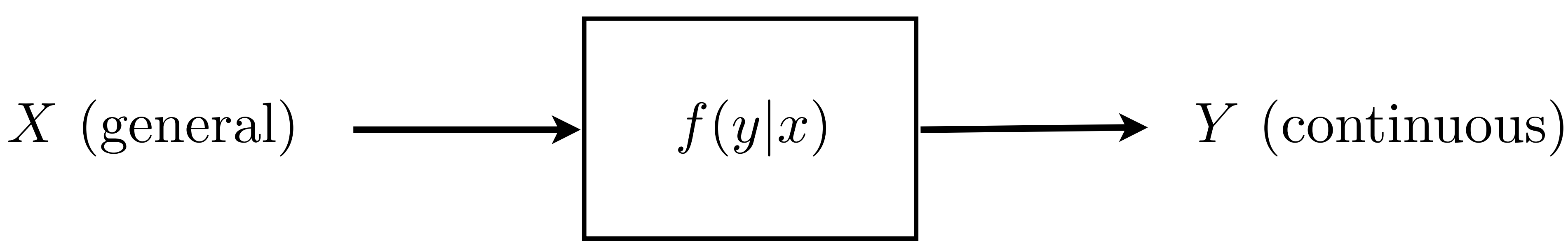

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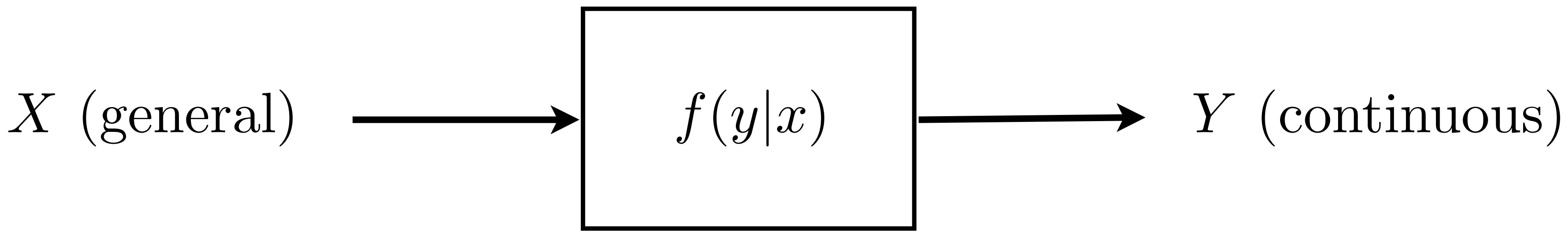
Remark The integral in Definition 11.1 is precisely the conditional differential entropy $h(Y|X = x)$, which is required to be finite.



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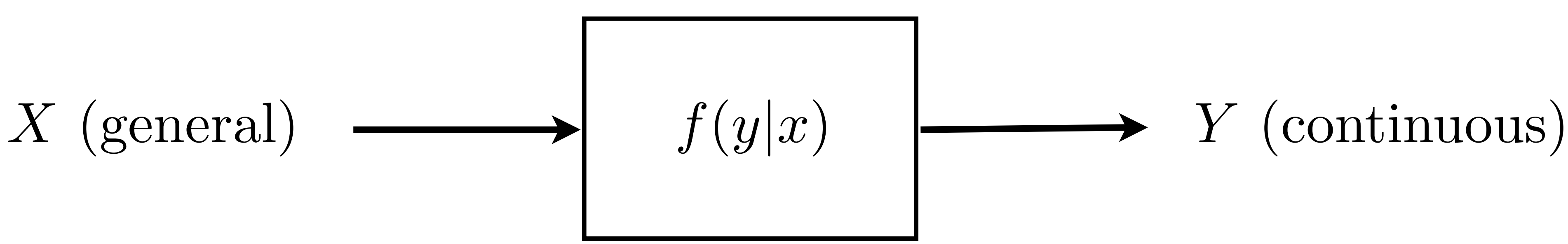

Definition 11.2 (Continuous Channel II) Let $\alpha : \Re \times \Re \to \Re$, and Z be a real random variable, called the noise variable. A (discrete-time) continuous channel (α, Z) is a system with a real input and a real output. For any input random variable X, the noise random variable Z is independent of X, and the output random variable Y is given by

$$Y = \alpha(X, Z).$$

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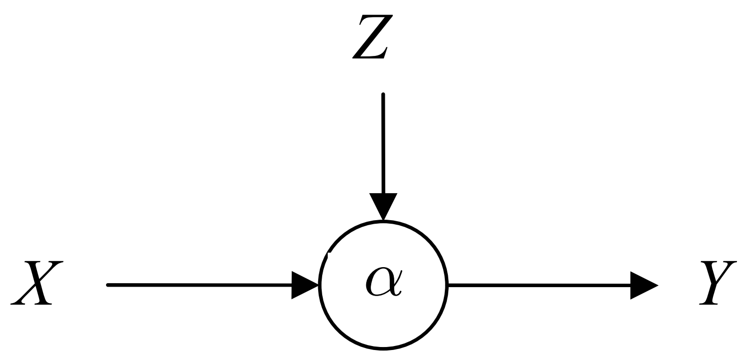

Definition 11.3 Two continuous channels $f(y|x)$ and (α, Z) are equivalent if for every input distribution $F(x)$,

$$\Pr\{X \leq x, \alpha(X, Z) \leq y\} = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{Y|X}(v|u) dv \, dF_X(u)$$

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$F_{XY}(X \le x, Y \le y)$

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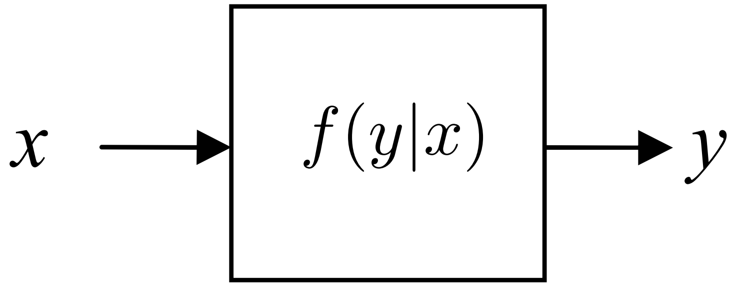

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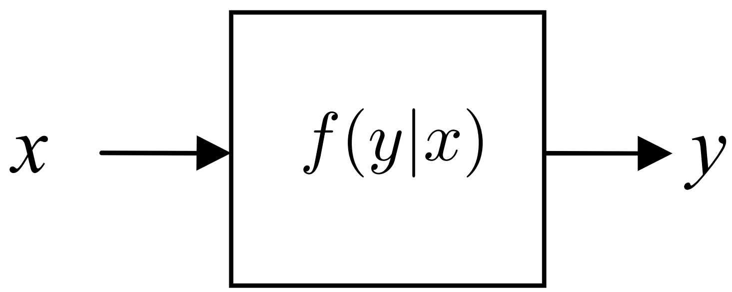
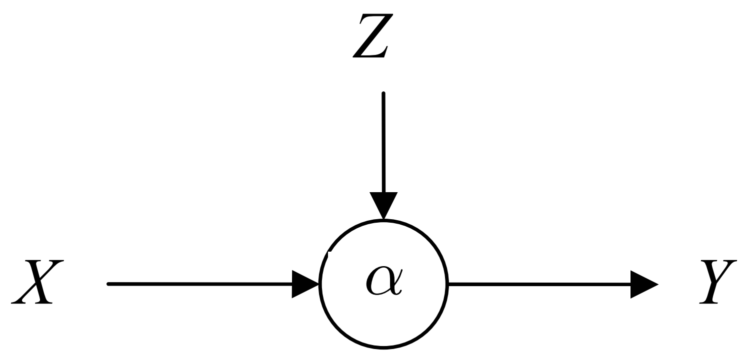


Definition 11.2



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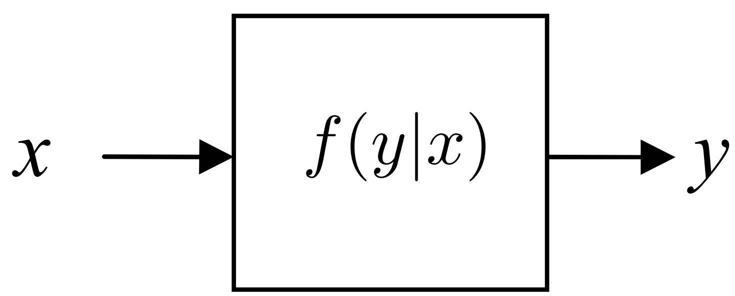

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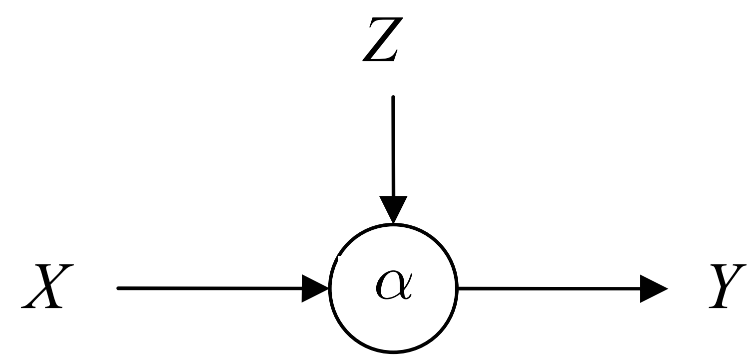

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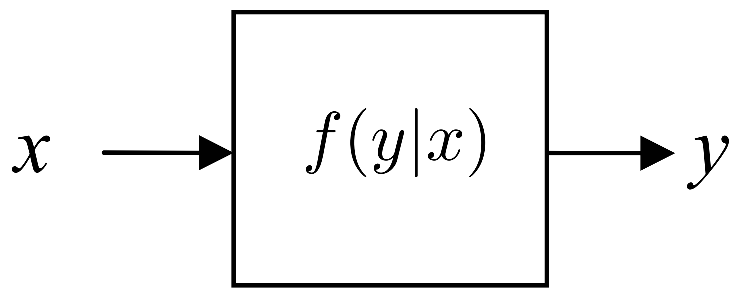

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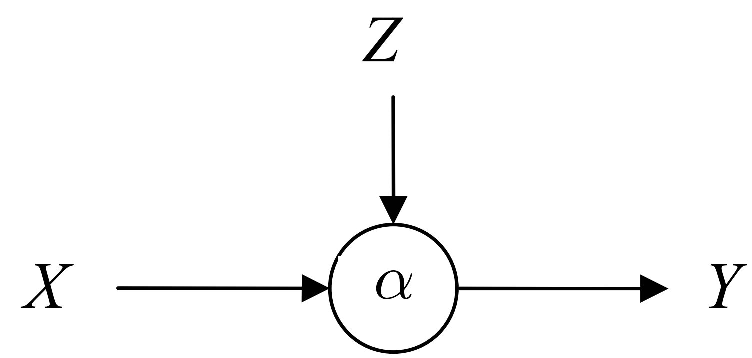
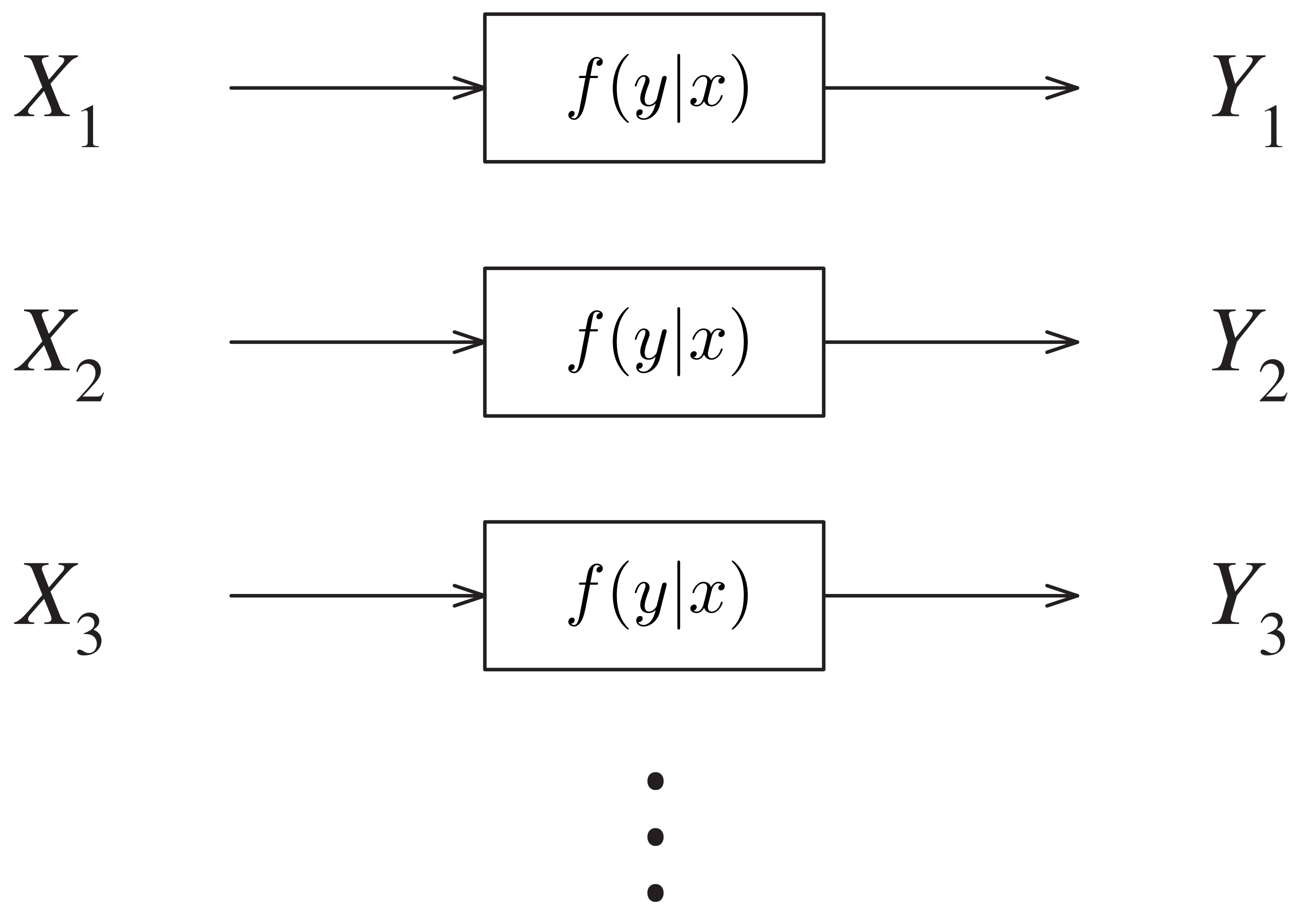


- Definition 11.2 is more general than Definition 11.1 because the former does not require the existence of $f(y|x)$.
- We confine our discussion to channels defined by Definition 11.1.

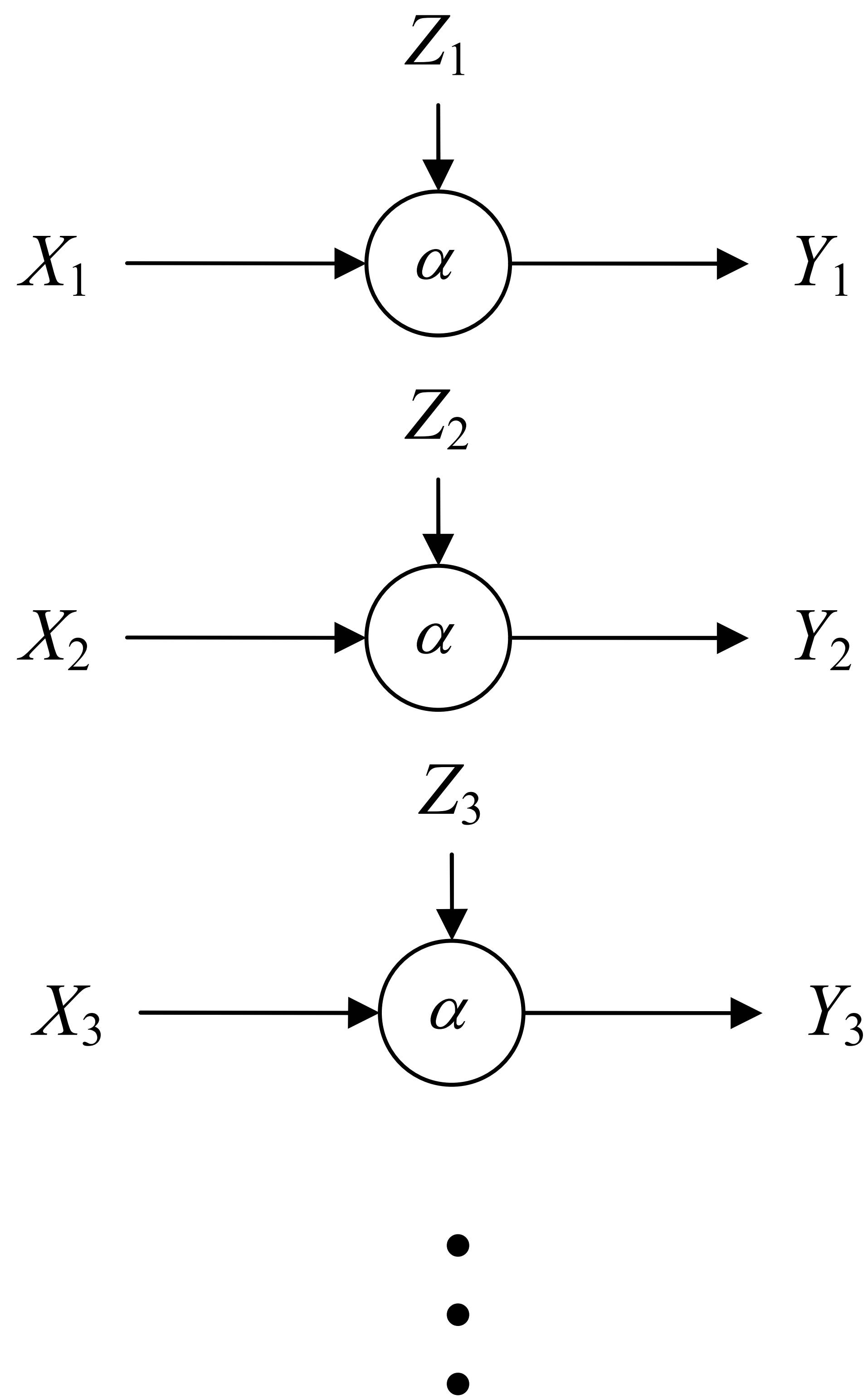
Definition 11.4 (CMC I) A continuous memoryless channel (CMC) $f(y|x)$ is a sequence of replicates of a generic continuous channel $f(y|x)$. These continuous channels are indexed by a discrete-time index i, where $i \geq 1$, with the ith channel being available for transmission at time i. Transmission through a channel is assumed to be instantaneous. Let X_i and Y_i be respectively the input and the output of the CMC at time i, and let T_{i-} denote all the random variables that are generated in the system before X_i. The Markov chain $T_{i-} \to X_i \to Y_i$ holds, and

$$\Pr\{X_i \leq x, Y_i \leq y\} = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{Y|X}(v|u) dv \, dF_{X_i}(u).$$


X_1
$f(y|x)$
Y_1
X_2
$f(y|x)$
Y_2
X_3
$f(y|x)$
Y_3

Definition 11.5 (CMC II) A continuous memoryless channel (α, Z) is a sequence of replicates of a generic continuous channel (α, Z). These continuous channels are indexed by a discrete-time index i, where $i \geq 1$, with the ith channel being available for transmission at time i. Transmission through a channel is assumed to be instantaneous. Let X_i and Y_i be respectively the input and the output of the CMC at time i, and let T_{i-} denote all the random variables that are generated in the system before X_i. The noise variable Z_i for the transmission at time i is a copy of the generic noise variable Z, and is independent of (X_i, T_{i-}). The output of the CMC at time i is given by

$$Y_i = \alpha(X_i, Z_i).$$


Z_1
X_1
α
Y_1
Z_2
X_2
α
Y_2
Z_3
X_3
α
Y_3

Definition 11.6 Let κ be a real function. An average input constraint (κ, P) for a CMC is the requirement that for any codeword $(x_1, x_2, \cdots, x_n)$ transmitted over the channel,

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Definition 11.7 The capacity of a continuous memoryless channel $f(y|x)$ with input constraint (κ, P) is defined as

$$C(P) = \sup_{F(x):E\kappa(X)\leq P} I(X;Y).$$

Theorem 11.8 $C(P)$ is non-decreasing, concave, and left-continuous.

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Remarks

1. $C(P)$ is also right-continous (a consequence of concavity) but requires a separate proof.
2. This property of $C(P)$ is not used in this chapter.

Theorem 11.8

$$C(P) = \sup_{F(x):E\kappa(X)\leq P} I(X;Y) \tag{1}$$

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$$E\kappa(X_j) \leq P_j$$

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$$I(X_j;Y_j) \geq C(P_j) - \epsilon.$$

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$$\begin{aligned} I(X^{(\lambda)}; Y^{(\lambda)}) &\ge \lambda I(X_1; Y_1) + \bar{\lambda} I(X_2; Y_2) \\ &\ge \lambda (C(P_1) - \epsilon) + \bar{\lambda}(C(P_2) - \epsilon) \end{aligned}$$

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6. Letting $\epsilon \to 0$, we have

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proving that $C(P)$ is concave.

$C(P)$ is left-continuous

1. Let $P_1 < P_2$, so that $P_2 \ge \lambda P_1 + \bar{\lambda} P_2$. Since $C(P)$ is non-decreasing, we have

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<u>$C(P)$ is non-decreasing</u>

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