香港中文大學
The Chinese University of Hong Kong

10.3 Joint Differential Entropy, Conditional (Differential) Entropy, and Mutual Information

Definition 10.17 The joint differential entropy $h(\mathbf{X})$ of a random vector $\mathbf{X}$ with joint pdf $f(\mathbf{x})$ is defined as

$$h(\mathbf{X}) = -\int_{\mathcal{S}} f(\mathbf{x}) \log f(\mathbf{x}) d\mathbf{x} = -E \log f(\mathbf{X}).$$

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$$h(\mathbf{X}) = \sum_{i=1}^{n} h(X_i).$$

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Theorem 10.19 (Scaling) $h(A\mathbf{X}) = h(\mathbf{X}) + \log |\det(A)|$.

Theorem 10.20 (Multivariate Gaussian Distribution) Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, K)$. Then

$$h(\mathbf{X}) = \frac{1}{2} \log \left[(2\pi e)^n |K| \right].$$

Theorem 10.20 (Multivariate Gaussian Distribution) Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, K)$. Then

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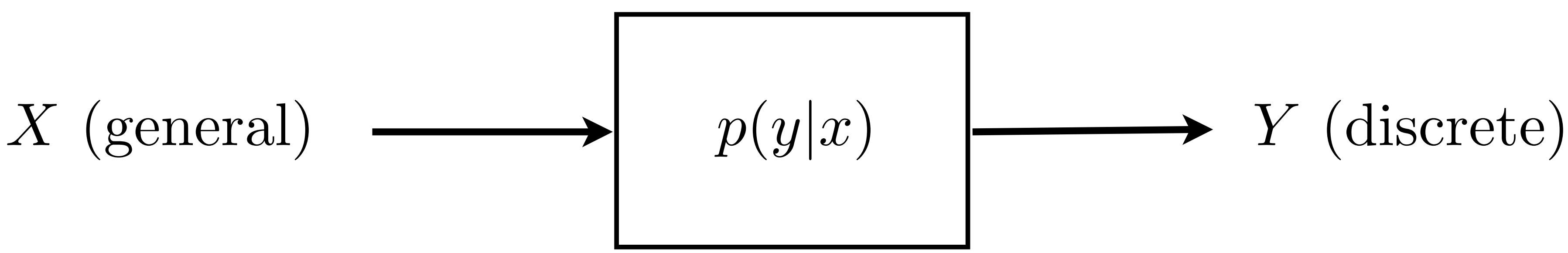

The Model of a "Channel" with Continuous Output

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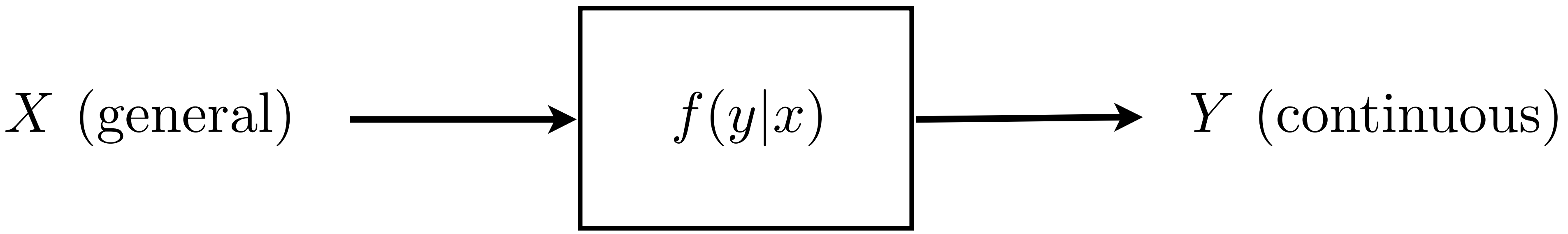

Conditional Differential Entropy

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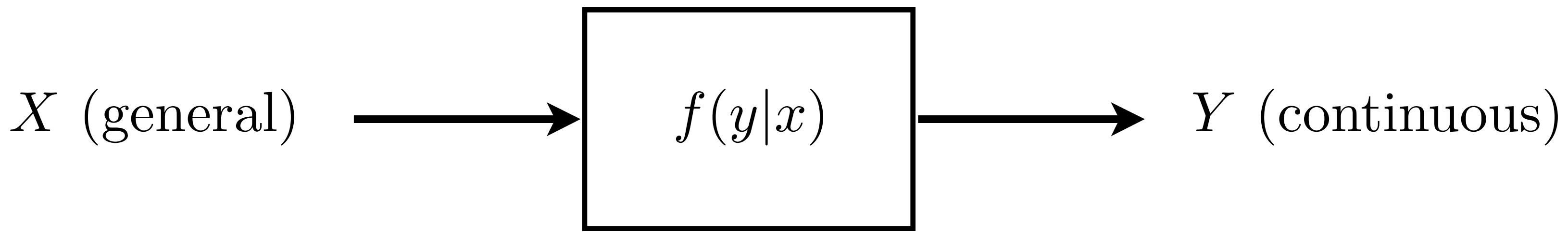

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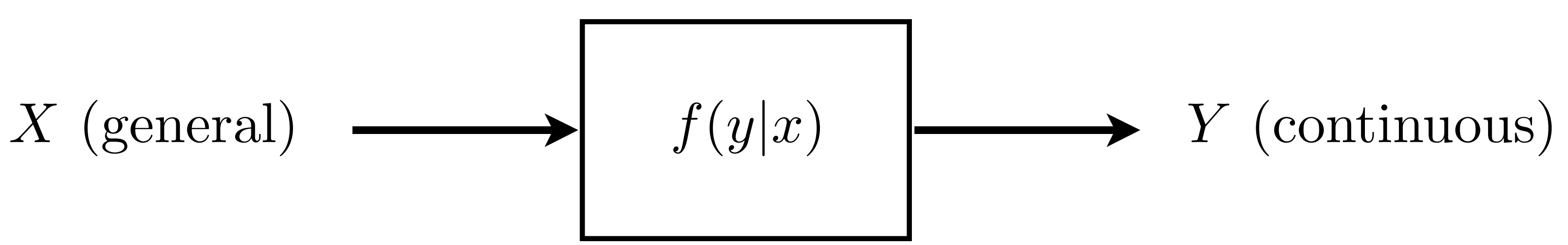

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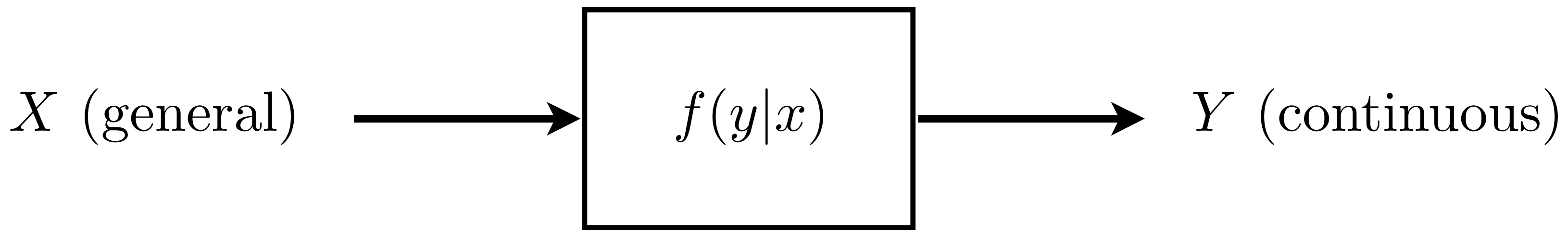

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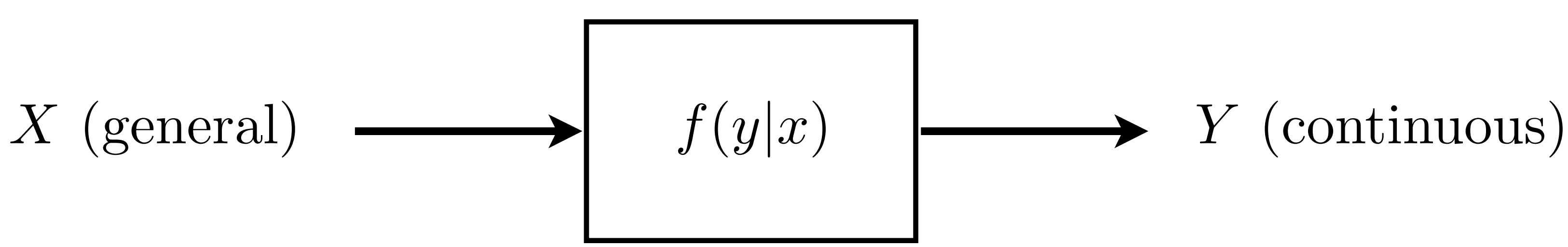
$f(x)\,dx$



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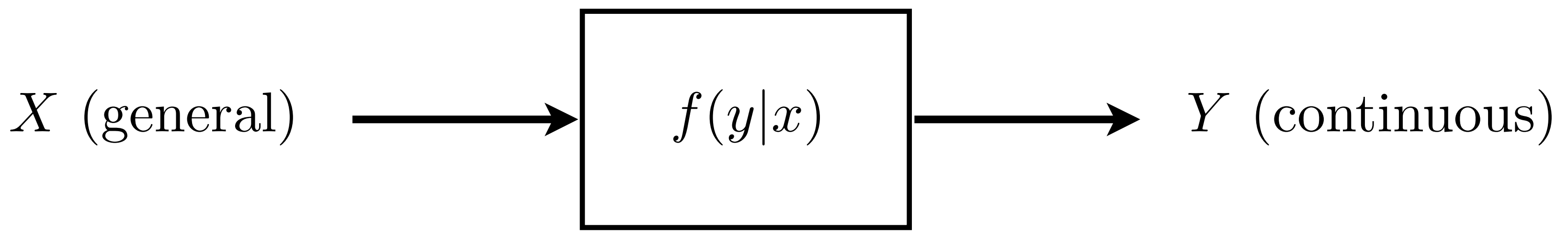

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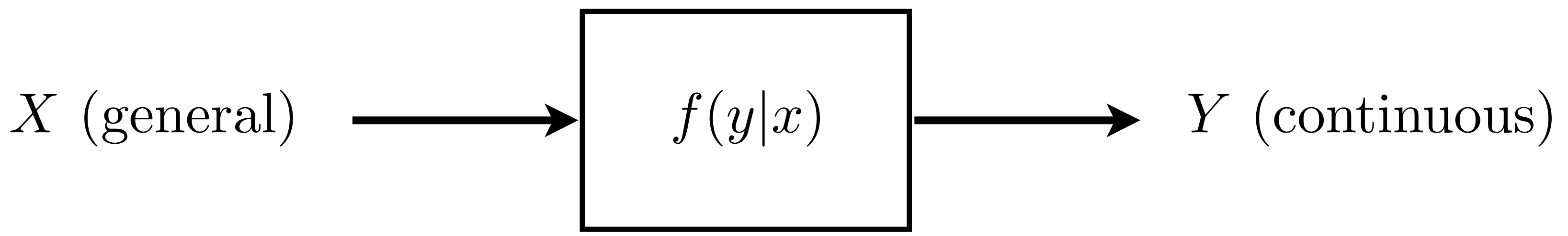

then $f(y)$ exists and is given by

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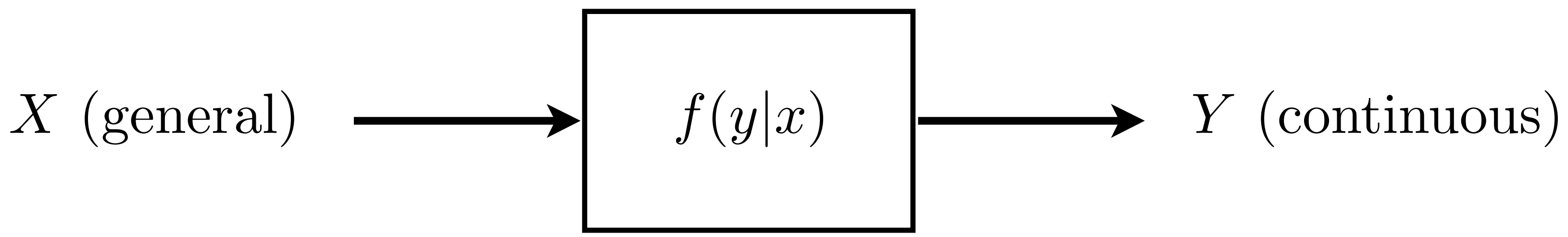

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Remark Proposition 10.24 says that the pdf of Y exists regardless of the distribution of X. The next proposition is its vector generalization.

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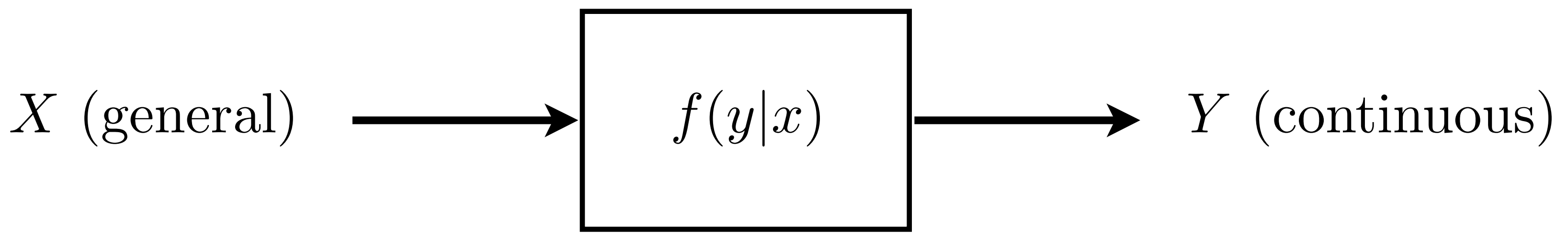

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Remark Proposition 10.24 says that the pdf of Y exists regardless of the distribution of X. The next proposition is its vector generalization.

Proposition 10.25 Let $\mathbf{X}$ and $\mathbf{Y}$ be jointly distributed random vectors where $\mathbf{Y}$ is continuous and is related to $\mathbf{X}$ through a conditional pdf $f(\mathbf{y}|\mathbf{x})$ defined for all $\mathbf{x}$. Then $f(\mathbf{y})$ exists and is given by

$$f(\mathbf{y}) = \int f(\mathbf{y}|\mathbf{x}) dF(\mathbf{x}).$$

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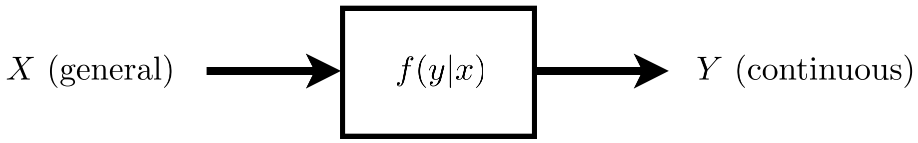

then $f(y)$ exists and is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, dF_X(x).$$

Proof

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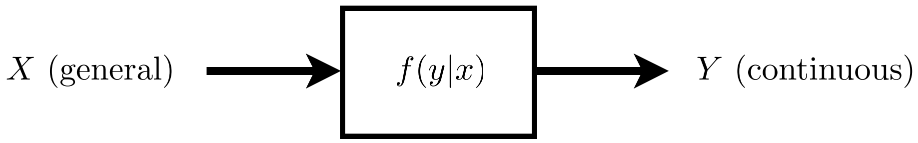

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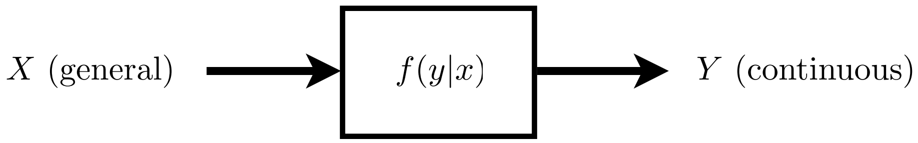

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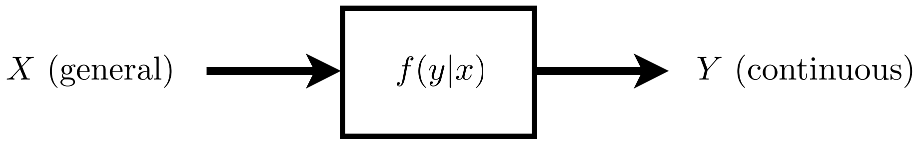

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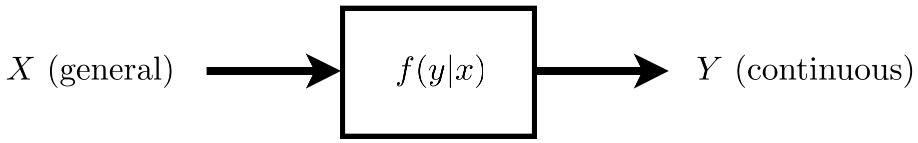

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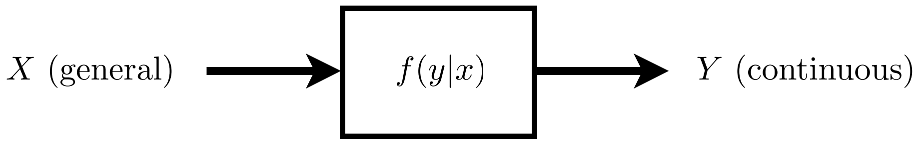

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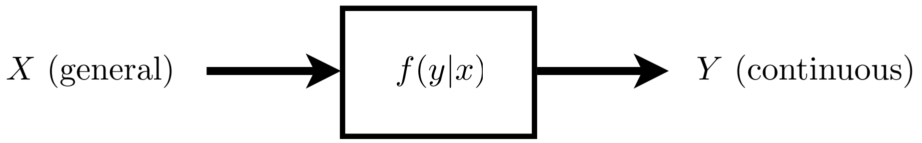

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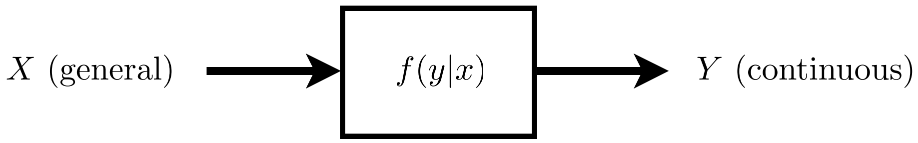

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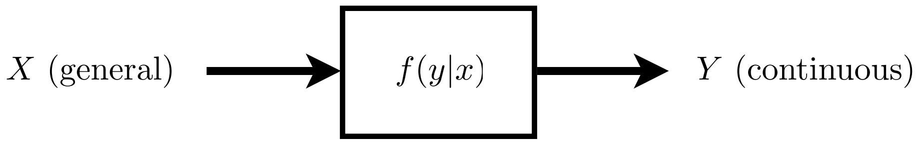

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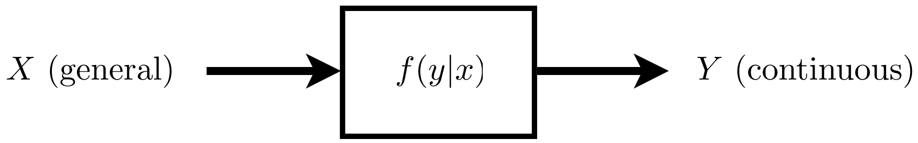

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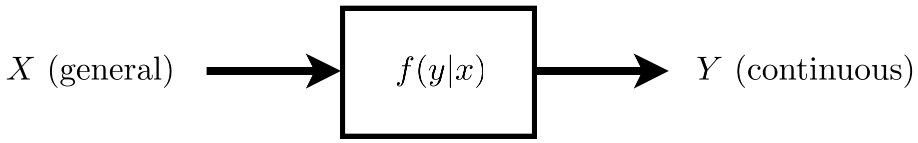

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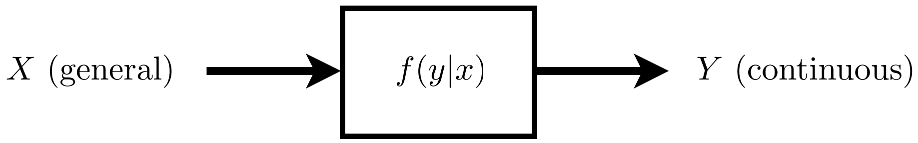

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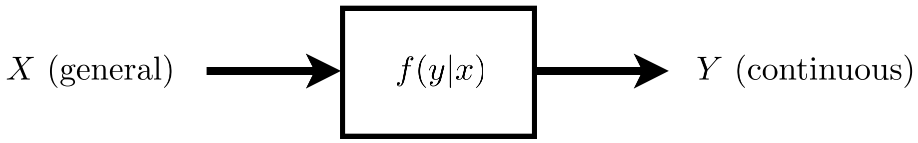

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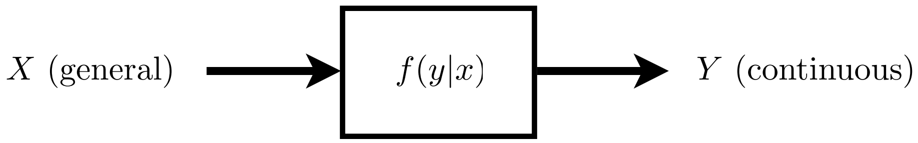

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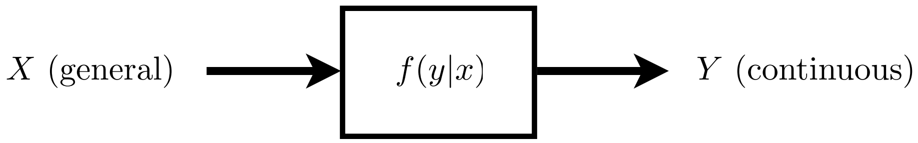

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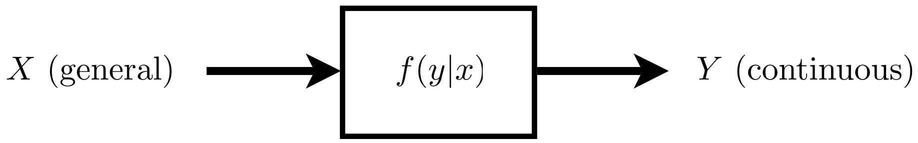

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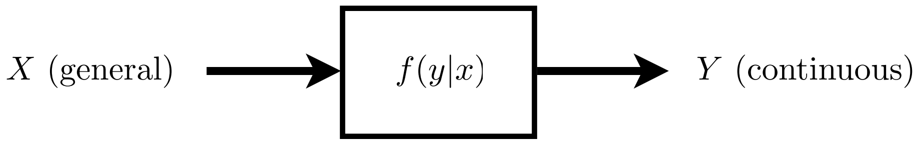

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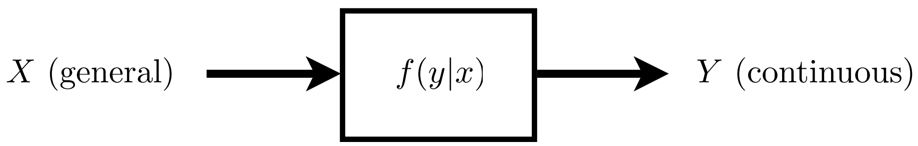

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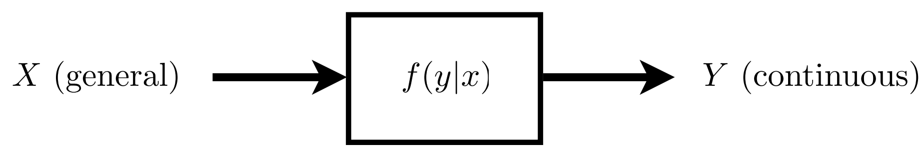

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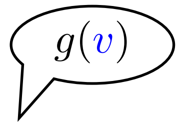

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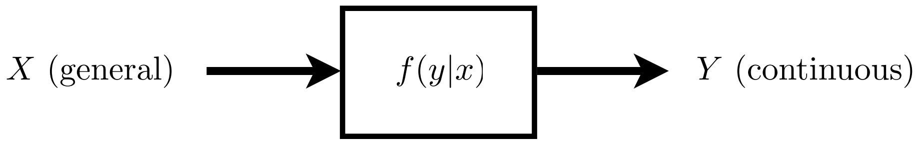
Proposition 10.24 If



then $f(y)$ exists and is given by

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Proof

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$f_{Y|X}(v|x)$ is absolutely integrable.

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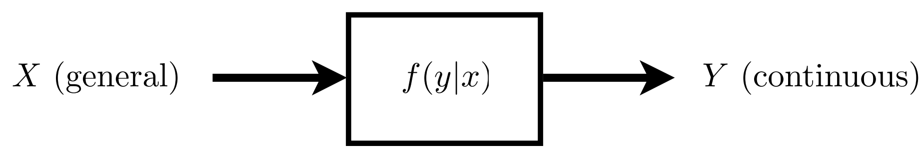
Fundamental theorem of calculus

Proposition 10.24 If



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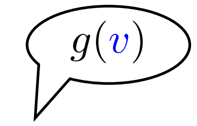

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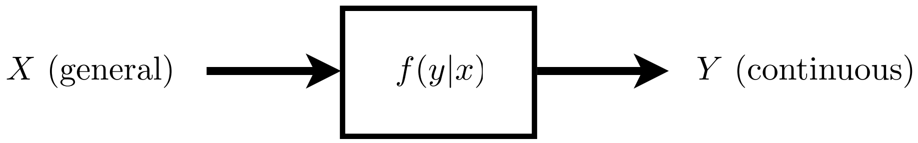

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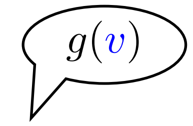

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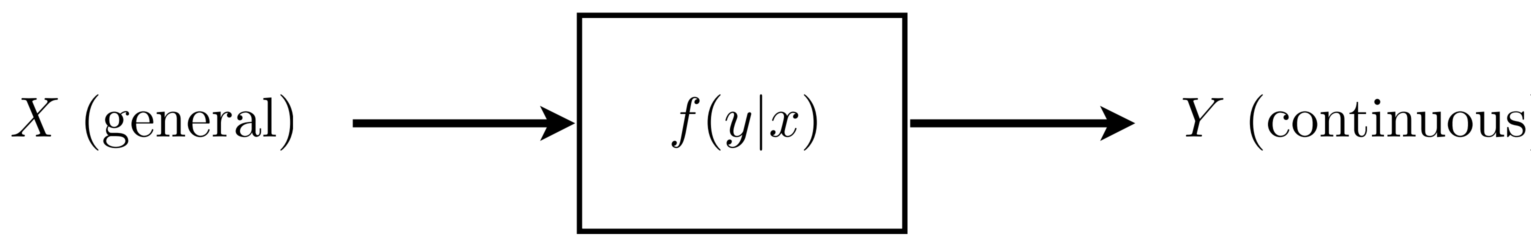
Mutual Information

Definition 10.26 Let



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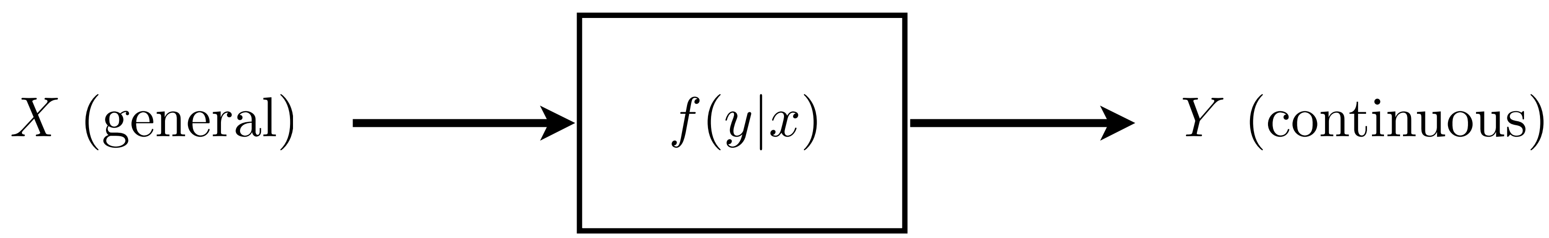

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$$I(X;Y) \quad = \quad E \log \frac{f(Y|X)}{f(Y)}$$

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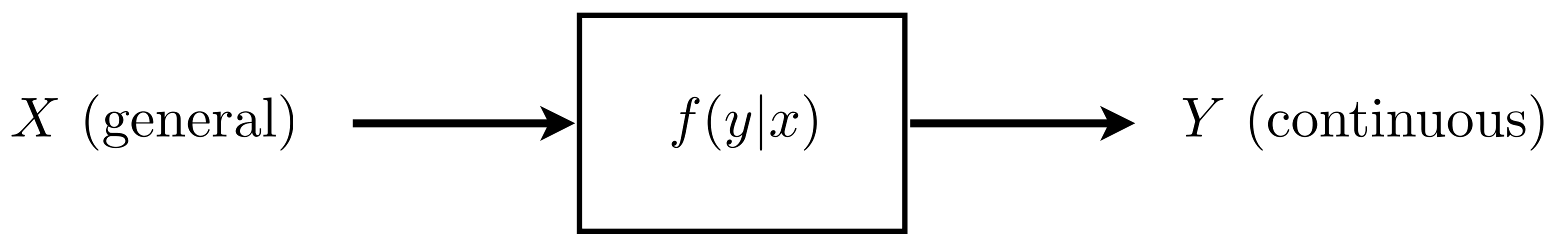

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I(X;Y) &= E \log \frac{f(Y|X)}{f(Y)} \\
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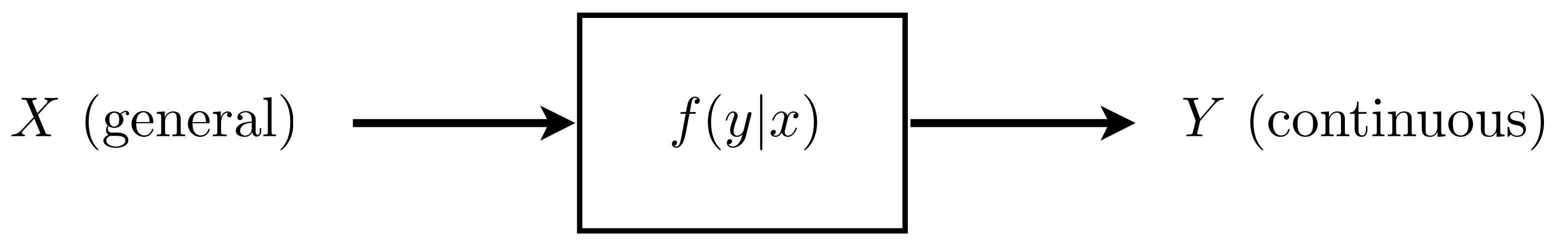

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$f(x)\,dx$

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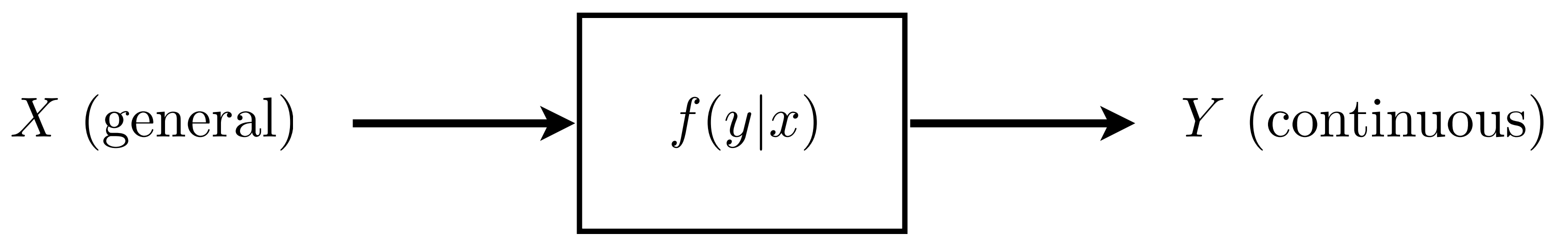

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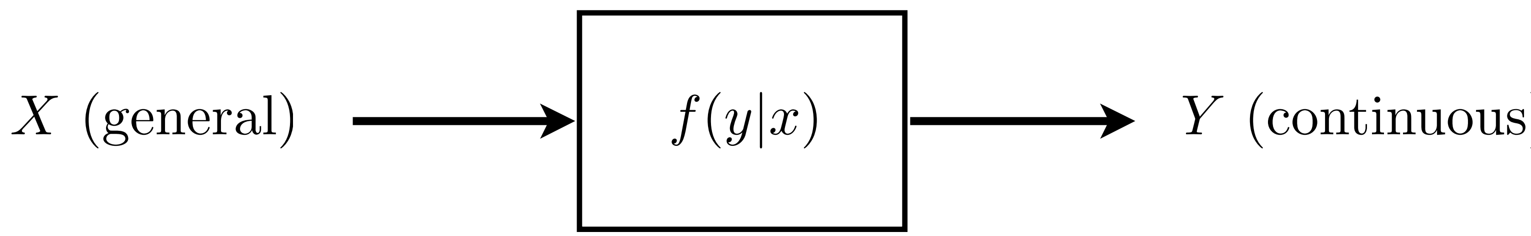

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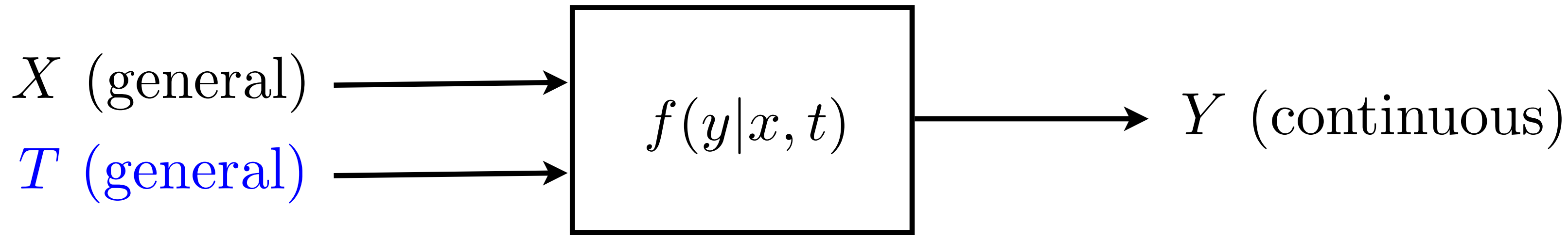
- Thus the mutual information is defined when each of the r.v.'s is either discrete or continuous.

Conditional Mutual Information

Definition 10.27 Let



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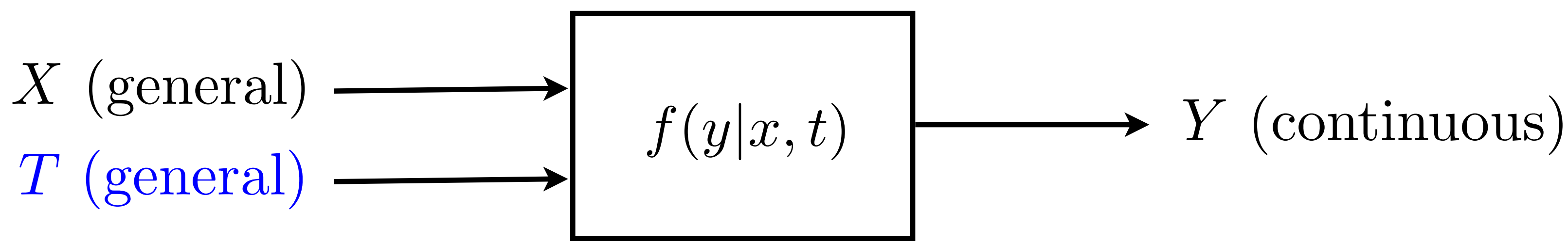

The mutual information between X and Y given T is defined as

$$I(X;Y|T) = \int_{\mathcal{S}_T} I(X;Y|T=t)dF(t) = E\log\frac{f(Y|X,T)}{f(Y|T)}$$

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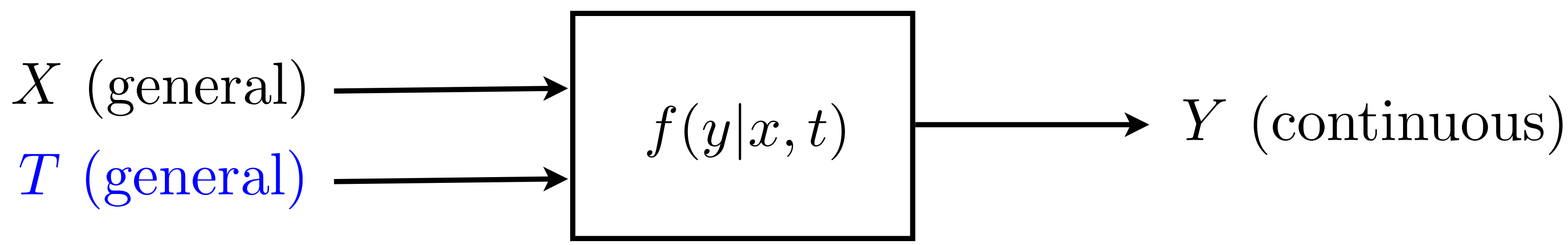

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Interpretation of I(X;Y)

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1. Assume $f(x, y)$ exists and is continuous.

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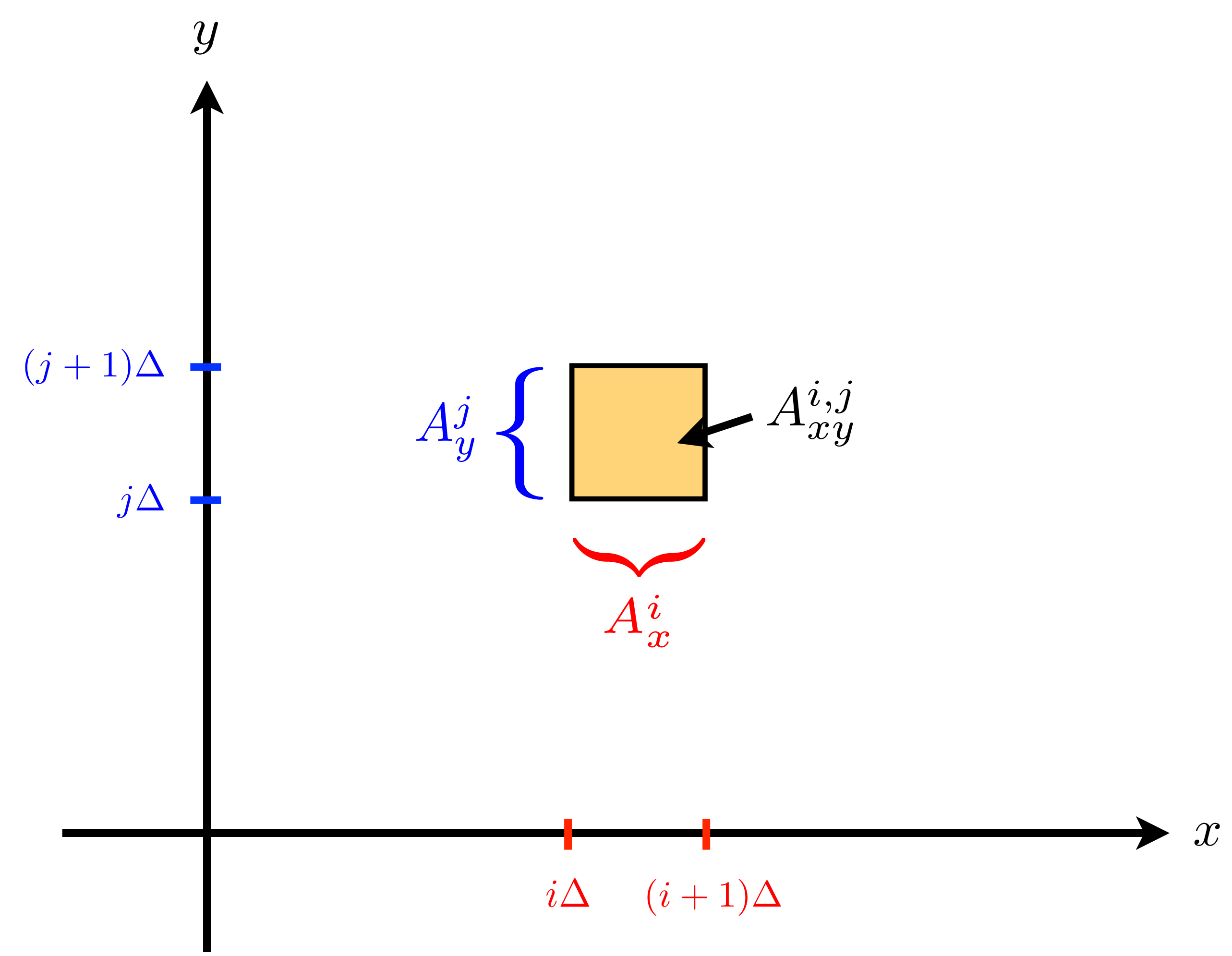

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1. Assume $f(x, y)$ exists and is continuous.

2. For a fixed Δ, for all integer i and j, define the intervals

$$A_x^i = [\, i\Delta, (i+1)\Delta) \quad \text{and} \quad A_y^j = [\, j\Delta, (j+1)\Delta)$$

and the rectangle

$$A_{xy}^{i,j} = A_x^i \times A_y^j.$$

3. Define discrete r.v.'s

$$\begin{cases} \hat{X}_\Delta = i & \text{if } X \in A_x^i \\ \hat{Y}_\Delta = j & \text{if } Y \in A_y^j \end{cases}$$

4. $\hat{X}_\Delta$ and $\hat{Y}_\Delta$ are quantizations of X and Y, respectively.

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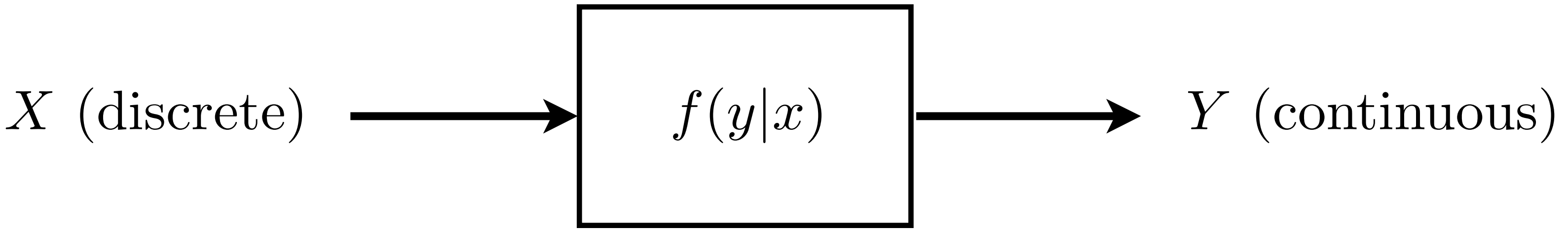
8. This interpretation continues to be valid for general distribution for X and Y.

Definition 10.28 Let



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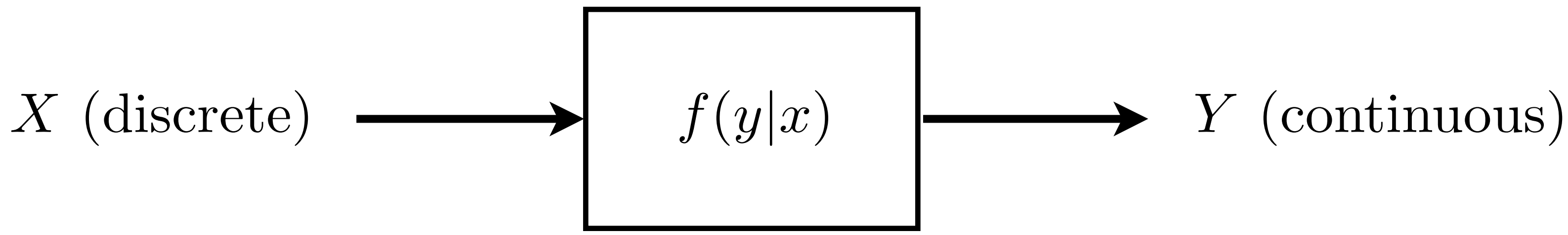

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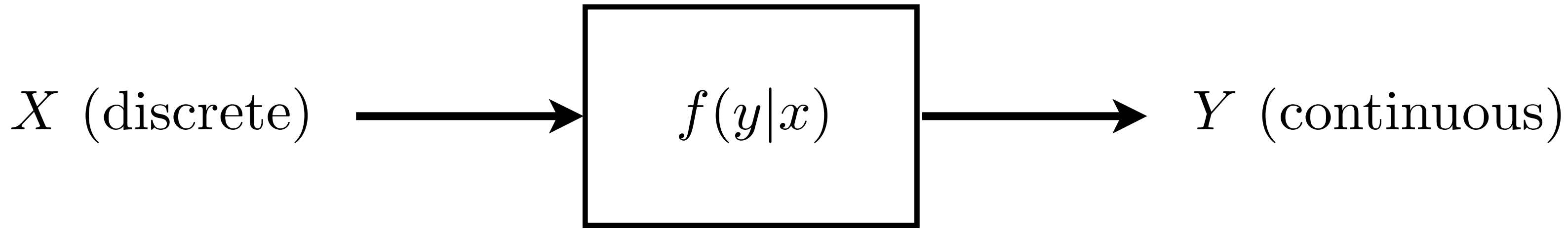

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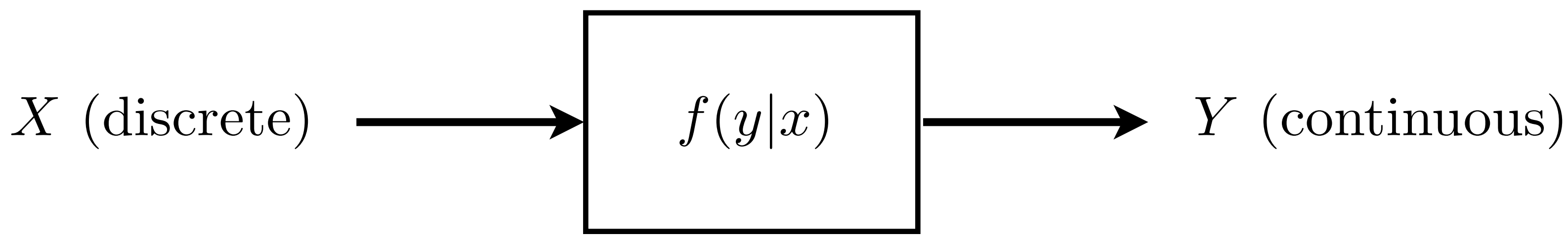

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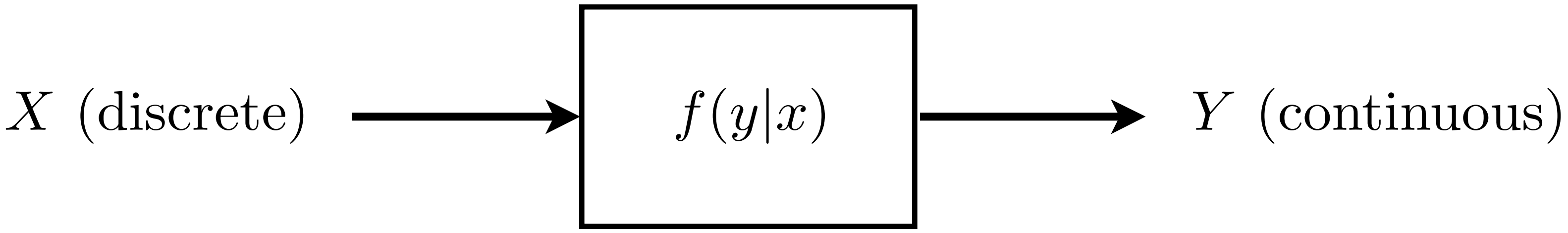

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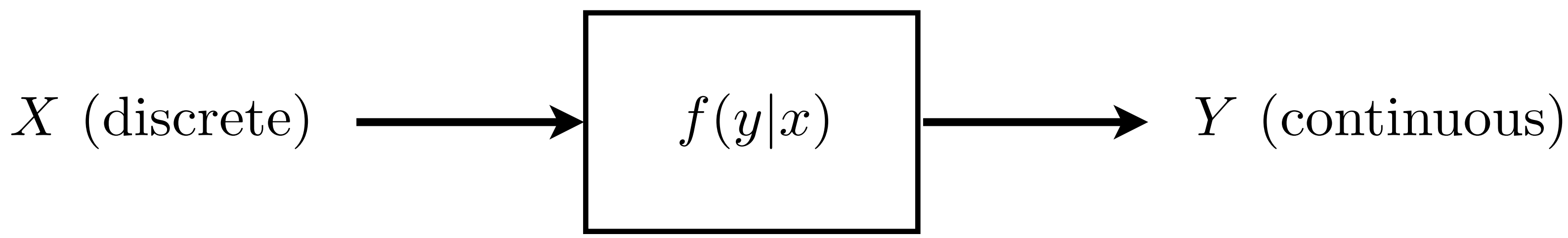

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Proposition 10.30 (Chain Rule)

$$h(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} h(X_i|X_1, \cdots, X_{i-1})$$

Theorem 10.31

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with equality if and only if X is independent of Y.

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2. $I(X;Y), I(X;Y|Z) \geq 0$ always hold.

Corollary 10.34 (Independence Bound for Differential Entropy)

$$h(X_1, X_2, \cdots, X_n) \leq \sum_{i=1}^{n} h(X_i)$$

with equality if and only if $i = 1, 2, \cdots, n$ are mutually independent.