香港中文大學
The Chinese University of Hong Kong

10.2 Definition

Definition 10.10 The differential entropy $h(X)$ of a continuous random variable X with pdf $f(x)$ is defined as

$$h(X) = -\int_{\mathcal{S}} f(x) \log f(x) dx = -E \log f(X)$$

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Remarks

1. Differential entropy is not a measure of the average amount of information contained in a continuous r.v.

2. A continuous random variable generally contains an infinite amount of information.

Example 10.11 Let X be uniformly distributed on $[0, 1)$. Then we can write

$$X = .X_1 X_2 X_3 \cdots ,$$

the dyadic expansion of X, where $X_1, X_2, X_3, \cdots$ is a sequence of fair bits.

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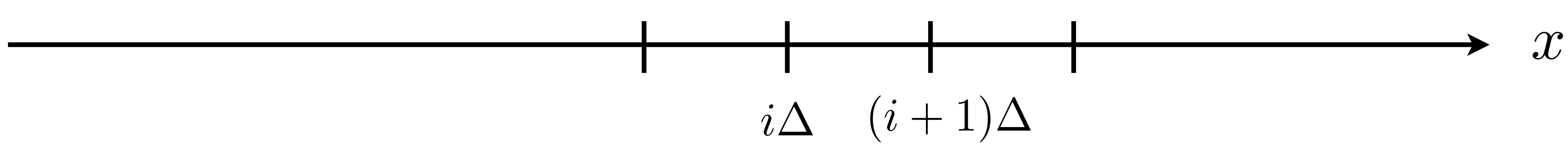

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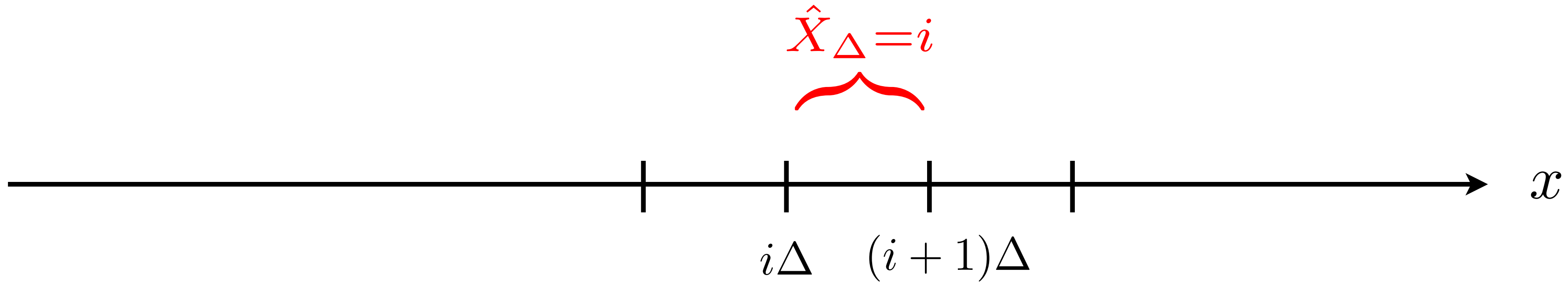

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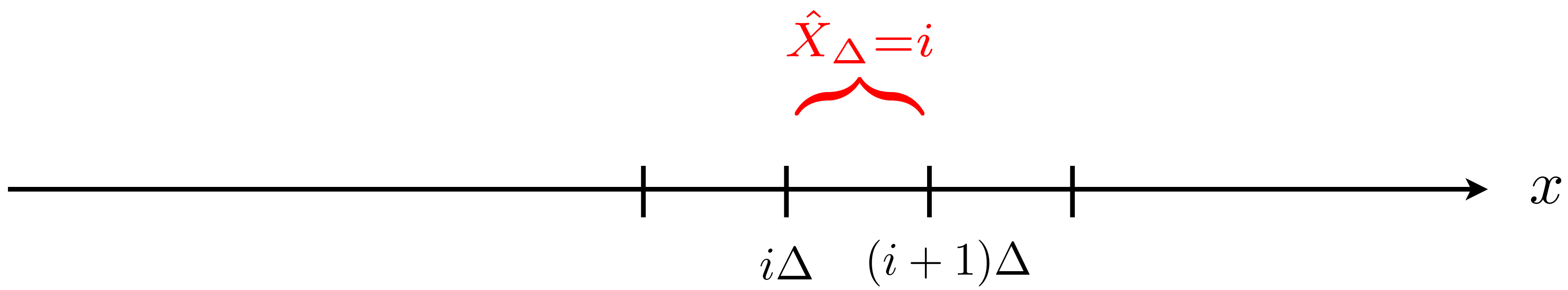

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$$p_i = \Pr\{\hat{X}_\Delta = i\} \approx f(x_i)\Delta$$

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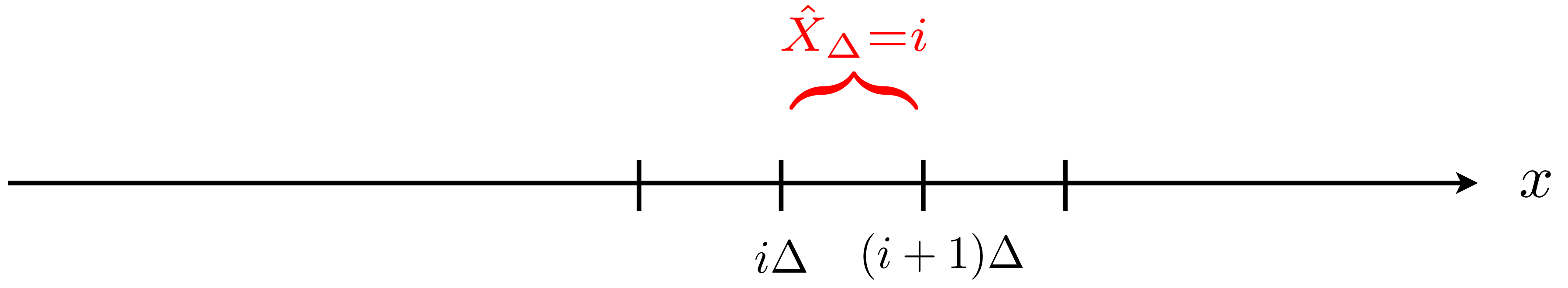

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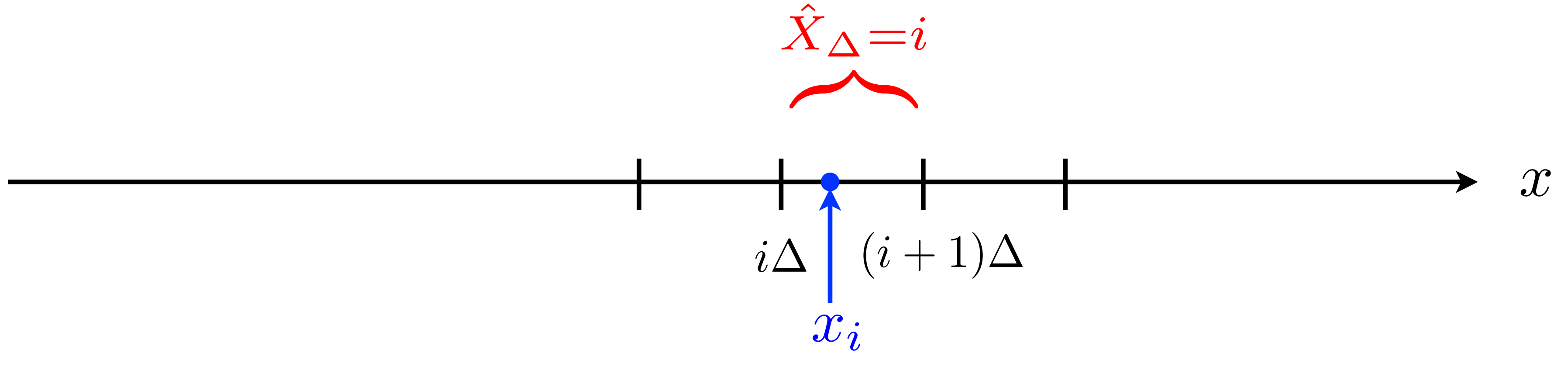

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\end{aligned}$$

- Note that $H(\hat{X}_\Delta) \to \infty$ as $\Delta \to 0$.

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$$h(X) = -\int_0^a \frac{1}{a} \log \frac{1}{a} dx = \log a.$$

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Example 10.13 (Gaussian Distribution) Let $X \sim \mathcal{N}(0, \sigma^2)$. Then

$$h(X) = \frac{1}{2} \log(2\pi e \sigma^2).$$

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\end{aligned}$$

Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.
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Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log |a|.$$

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&= -\int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\
&= -\int_{\mathcal{S}_Y} f_X\left(\frac{y}{a}\right) \log f_X\left(\frac{y}{a}\right) \frac{dy}{|a|} \\
&= -\int_{\mathcal{S}_Y} \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \left[\log\left(\frac{1}{|a|} f_X\left(\frac{y}{a}\right)\right) + \log |a|\right] dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy - \log |a| \int_{\mathcal{S}_Y} f_Y(y) dy
\end{aligned}$$

Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.
2. Then
$$f_Y(y) = f_X(y - c)$$
and
$$\mathcal{S}_Y = \{x + c : x \in \mathcal{S}_X\}.$$
3. Letting $x = y - c$, we have

$$\begin{aligned} h(X) &= -\int_{\mathcal{S}_X} f_X(x)\log f_X(x)dx \\ &= -\int_{\mathcal{S}_Y} f_X(y-c)\log f_X(y-c)dy \\ &= -\int_{\mathcal{S}_Y} f_Y(y)\log f_Y(y)dy \\ &= h(Y) \\ &= h(X+c). \end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log|a|.$$

Proof

1. Let $Y = aX$.
2. Then
$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$
and
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3. Letting $x = \frac{y}{a}$, we have

$$\begin{aligned} h(X) & \\ &= -\int_{\mathcal{S}_X} f_X(x)\log f_X(x)dx \\ &= -\int_{\mathcal{S}_Y} f_X\left(\frac{y}{a}\right)\log f_X\left(\frac{y}{a}\right)\frac{dy}{|a|} \\ &= -\int_{\mathcal{S}_Y} \frac{1}{|a|} f_X\left(\frac{y}{a}\right)\left[\log\left(\frac{1}{|a|} f_X\left(\frac{y}{a}\right)\right) + \underline{\log|a|}\right] dy \\ &= -\int_{\mathcal{S}_Y} f_Y(y)\log f_Y(y)dy - \log|a|\int_{\mathcal{S}_Y} f_Y(y)dy \end{aligned}$$

Theorem 10.14 (Translation)

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1. Let $Y = X + c$.
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Theorem 10.14 (Translation)

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Proof

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Theorem 10.14 (Translation)

$$h(X+c) = h(X).$$

Proof

1. Let $Y = X + c$.
2. Then
$$f_Y(y) = f_X(y-c)$$
and
$$\mathcal{S}_Y = \{x + c : x \in \mathcal{S}_X\}.$$
3. Letting $x = y - c$, we have

$$\begin{aligned} h(X) &= -\int_{\mathcal{S}_X} f_X(x)\log f_X(x)dx \\ &= -\int_{\mathcal{S}_Y} f_X(y-c)\log f_X(y-c)dy \\ &= -\int_{\mathcal{S}_Y} f_Y(y)\log f_Y(y)dy \\ &= h(Y) \\ &= h(X+c). \end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log|a|.$$

Proof

1. Let $Y = aX$.
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Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.
2. Then
$$f_Y(y) = f_X(y - c)$$
and
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3. Letting $x = y - c$, we have

$$\begin{aligned}
h(X) &= -\int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\
&= -\int_{\mathcal{S}_Y} f_X(y - c) \log f_X(y - c) dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy \\
&= h(Y) \\
&= h(X + c).
\end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log |a|.$$

Proof

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&= -\int_{\mathcal{S}_Y} f_X\left(\frac{y}{a}\right) \log f_X\left(\frac{y}{a}\right) \frac{dy}{|a|} \\
&= -\int_{\mathcal{S}_Y} \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \left[\log\left(\frac{1}{|a|} f_X\left(\frac{y}{a}\right)\right) + \log |a|\right] dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy - \log |a| \int_{\mathcal{S}_Y} f_Y(y) dy
\end{aligned}$$

Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.

2. Then

$$f_Y(y) = f_X(y - c)$$

and

$$\mathcal{S}_Y = \{x + c : x \in \mathcal{S}_X\}.$$

3. Letting $x = y - c$, we have

$$\begin{aligned}
h(X) &= -\int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\
&= -\int_{\mathcal{S}_Y} f_X(y - c) \log f_X(y - c) dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy \\
&= h(Y) \\
&= h(X + c).
\end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log |a|.$$

Proof

1. Let $Y = aX$.

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$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$

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&= -\int_{\mathcal{S}_Y} \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \left[\log\left(\frac{1}{|a|} f_X\left(\frac{y}{a}\right)\right) + \log |a|\right] dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy - \log |a| \int_{\mathcal{S}_Y} f_Y(y) dy \\
&= \underline{h(Y)} - \log |a|
\end{aligned}$$

Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.

2. Then
$$f_Y(y) = f_X(y - c)$$
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Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log |a|.$$

Proof

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Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.
2. Then
$$f_Y(y) = f_X(y - c)$$
and
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3. Letting $x = y - c$, we have

$$\begin{aligned} h(X) &= -\int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\ &= -\int_{\mathcal{S}_Y} f_X(y - c) \log f_X(y - c) dy \\ &= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy \\ &= h(Y) \\ &= h(X + c). \end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log |a|.$$

Proof

1. Let $Y = aX$.
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Theorem 10.14 (Translation)

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Proof

1. Let $Y = X + c$.
2. Then
$$f_Y(y) = f_X(y-c)$$
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$$\begin{aligned} h(X) &= -\int_{\mathcal{S}_X} f_X(x)\log f_X(x)dx \\ &= -\int_{\mathcal{S}_Y} f_X(y-c)\log f_X(y-c)dy \\ &= -\int_{\mathcal{S}_Y} f_Y(y)\log f_Y(y)dy \\ &= h(Y) \\ &= h(X+c). \end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log|a|.$$

Proof

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Theorem 10.14 (Translation)

$$h(X+c) = h(X).$$

Proof

1. Let $Y = X + c$.
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4. Hence,

$$h(aX) = h(X) + \log|a|.$$