香港中文大學
The Chinese University of Hong Kong

Chapter 10
Differential Entropy

The Chinese University of Hong Kong

In this chapter

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- Real-valued random vectors

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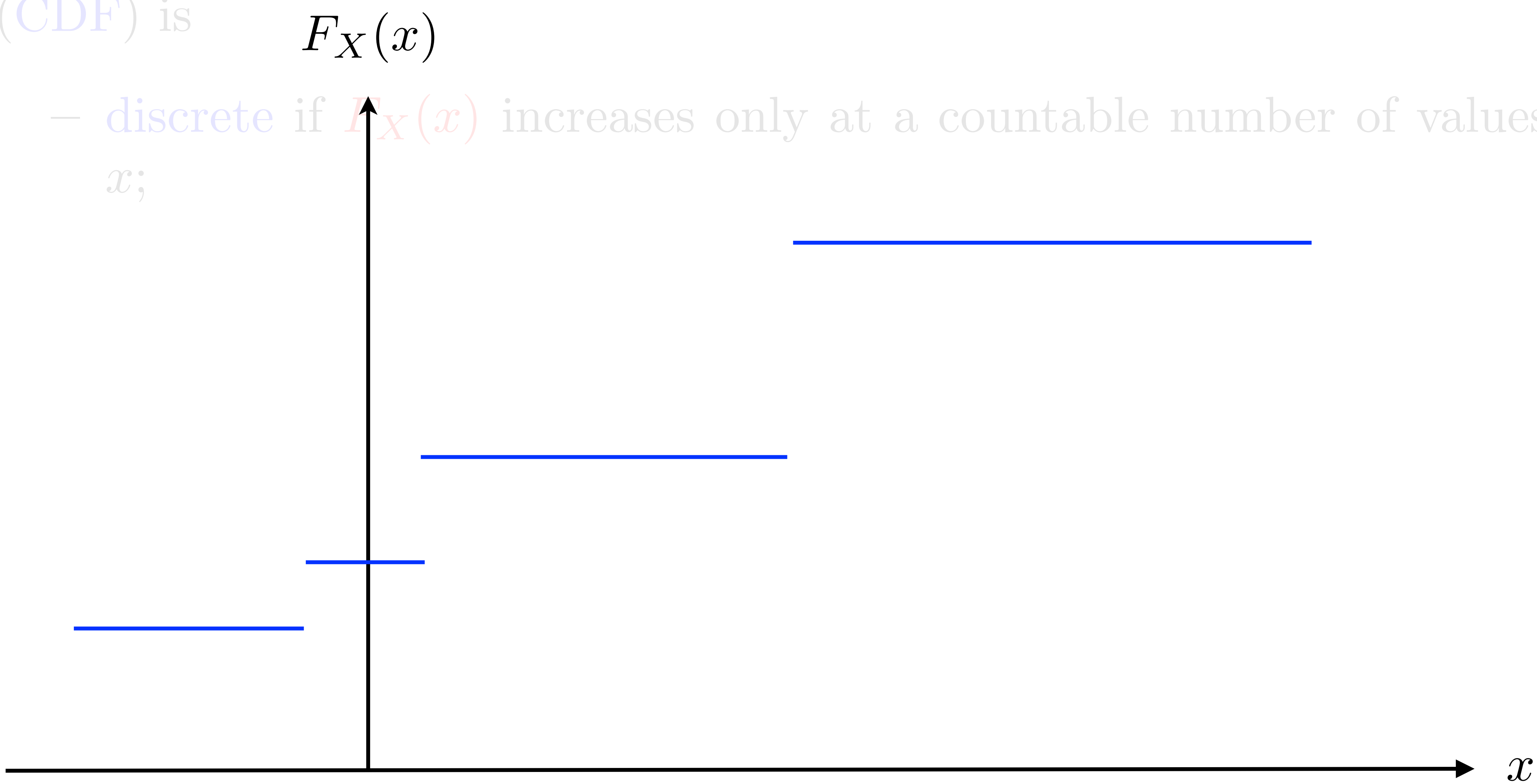

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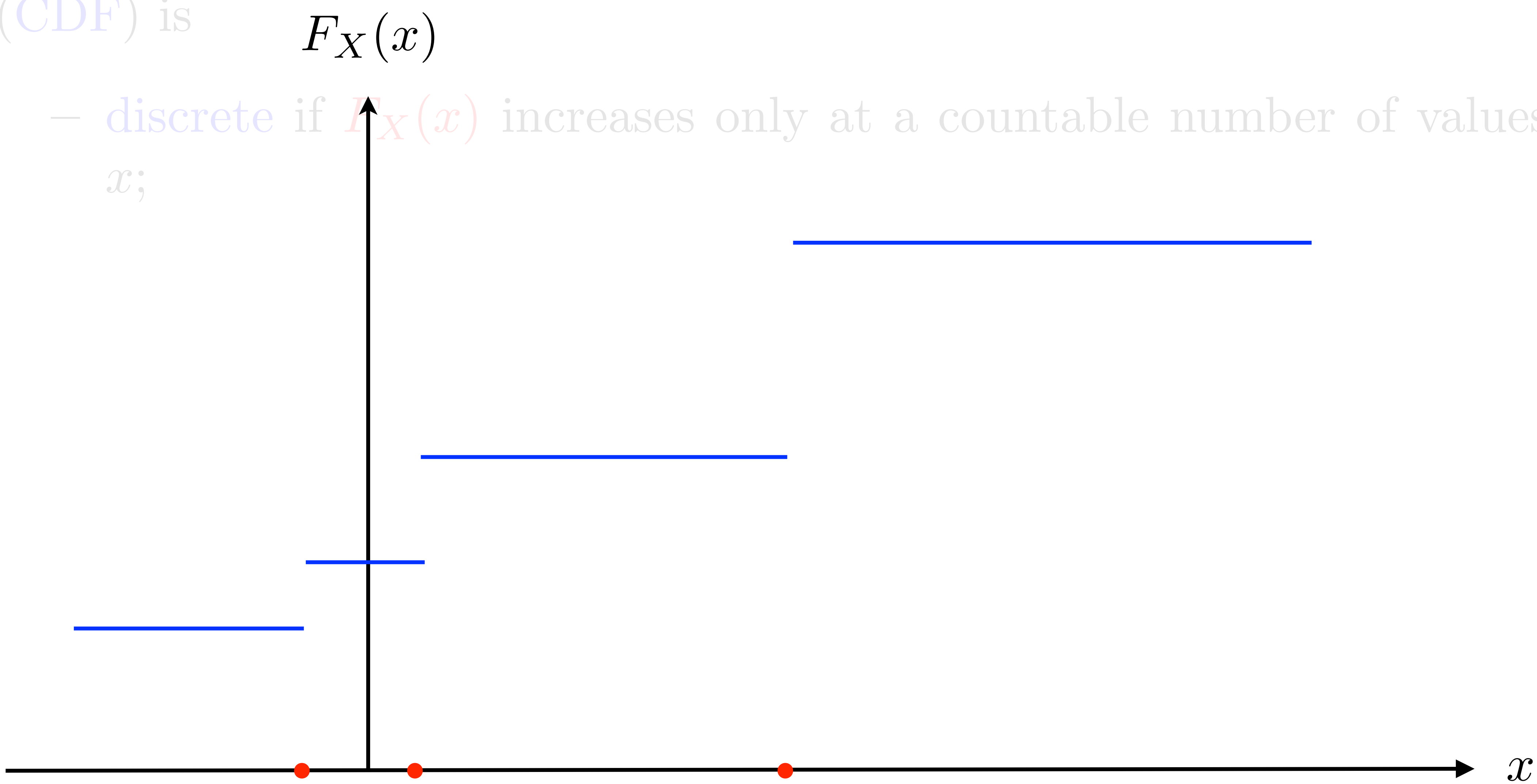

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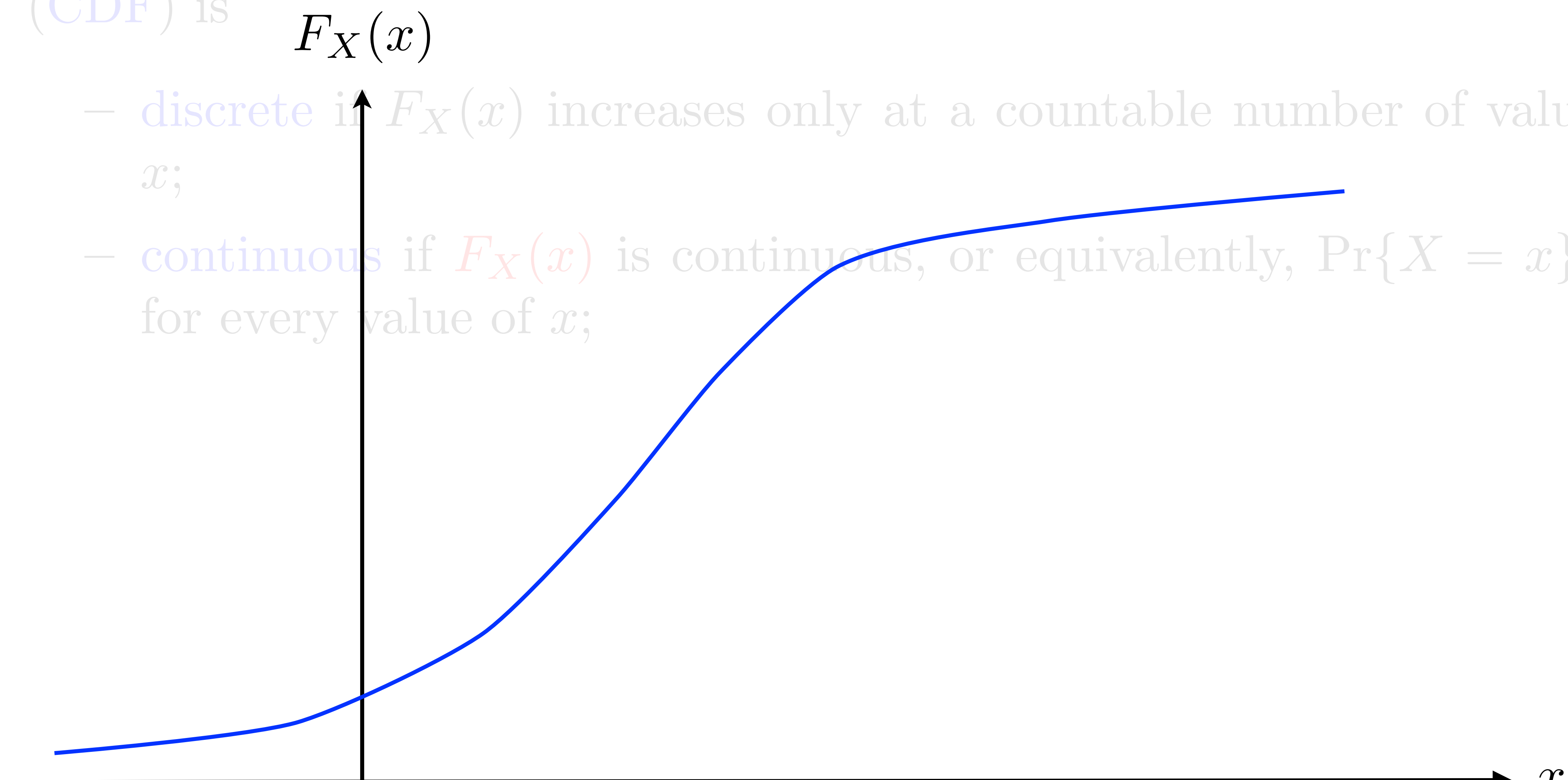

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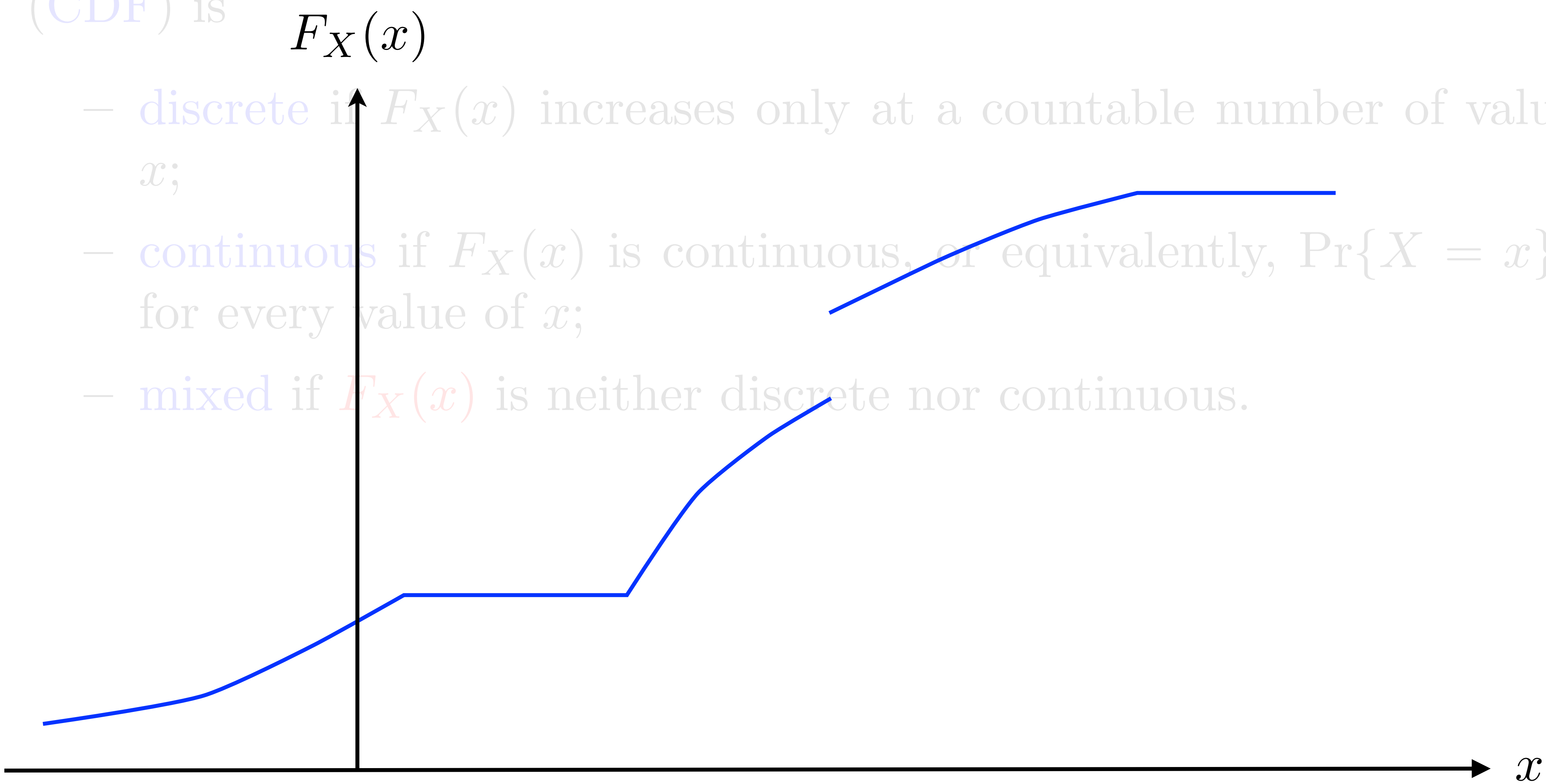

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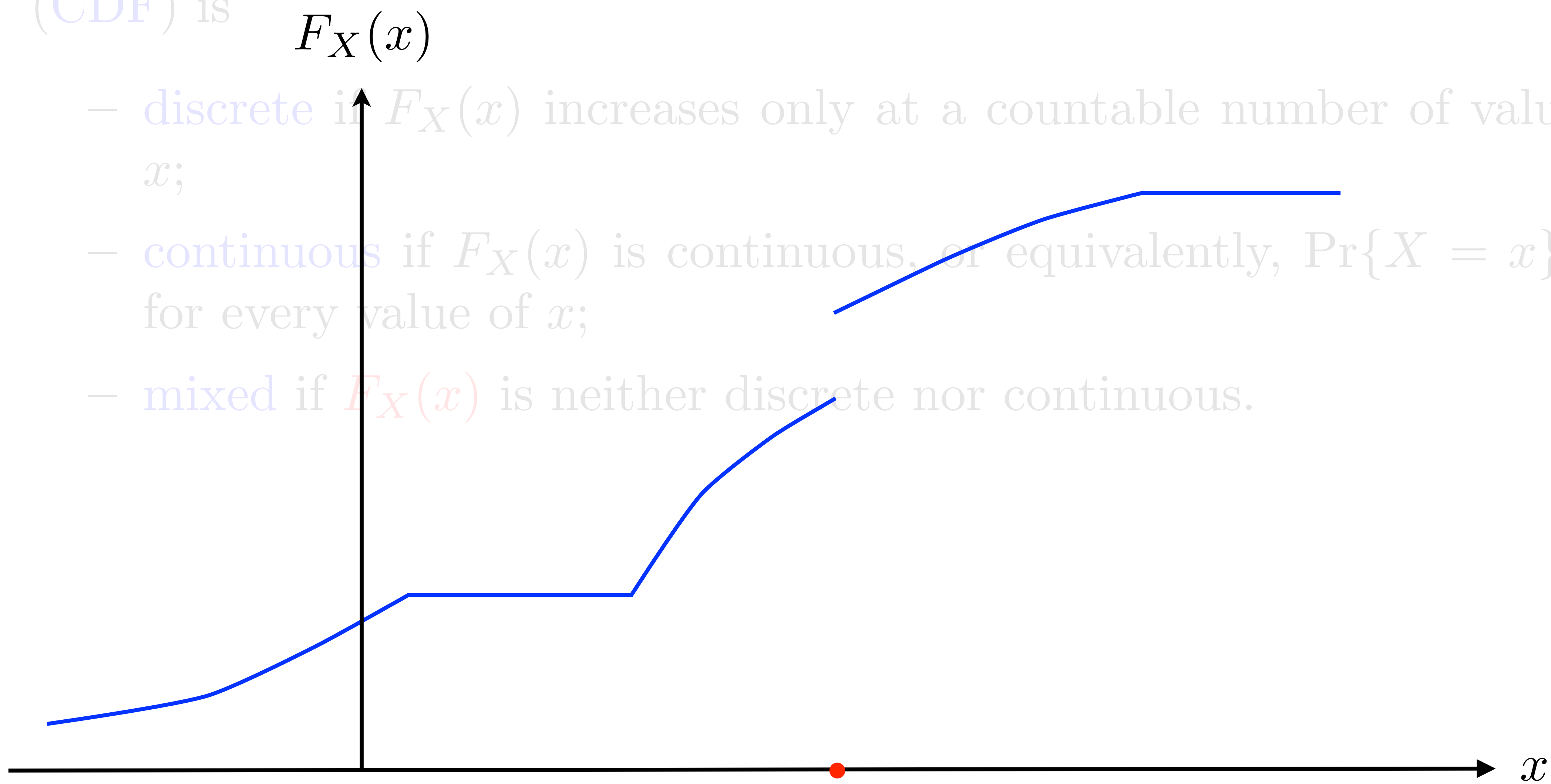

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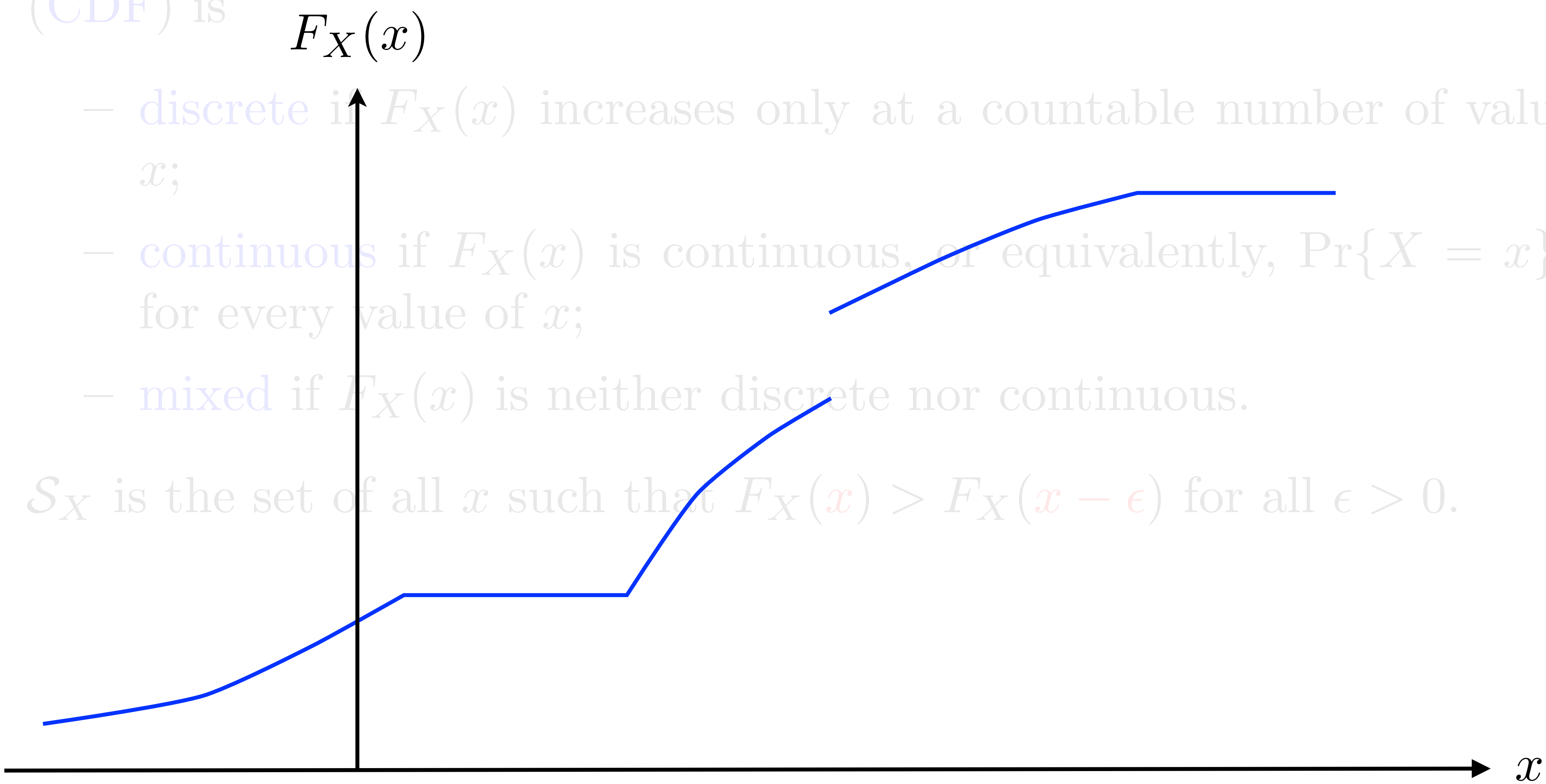

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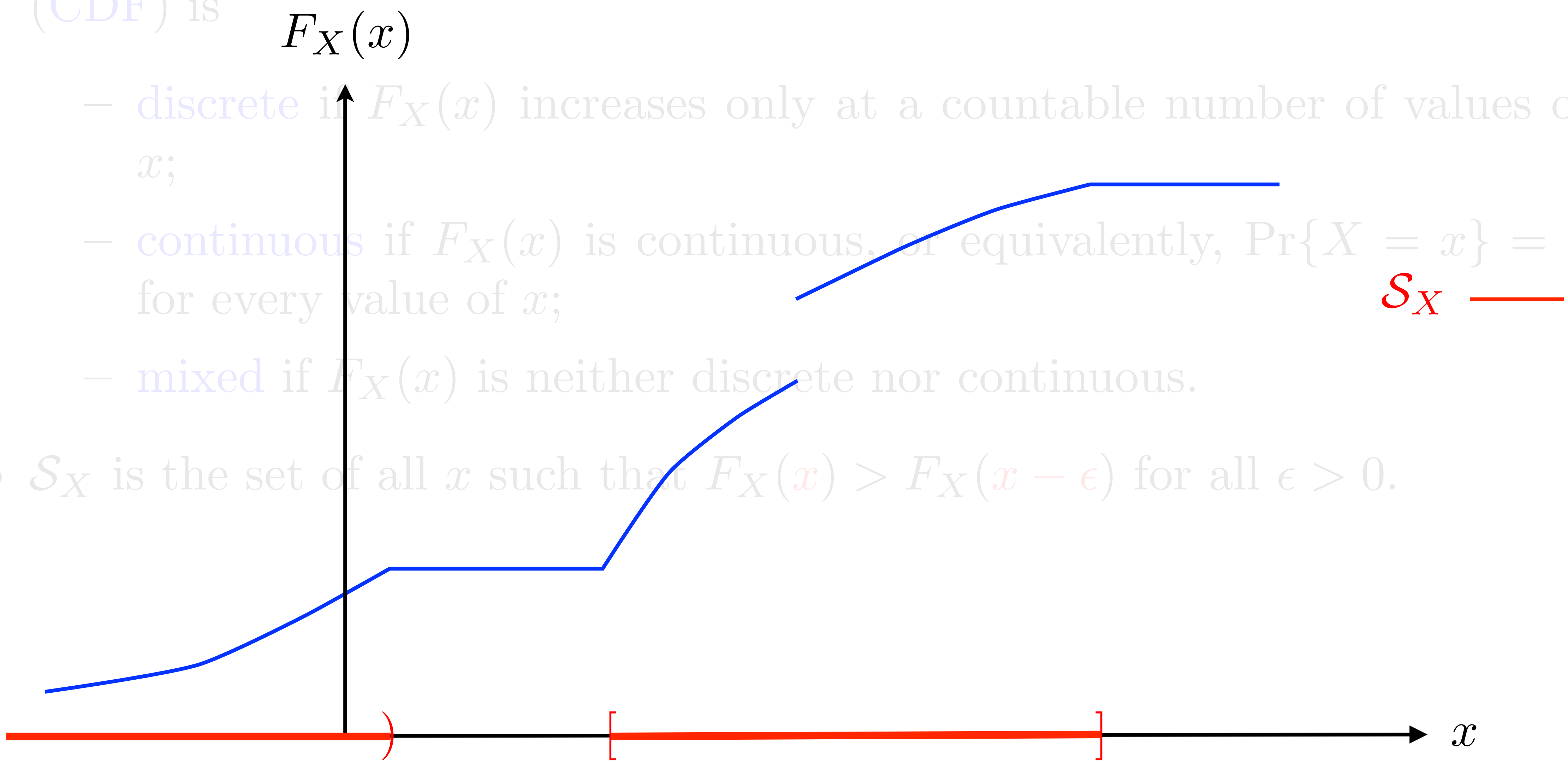

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- If X has a pdf, then X is continuous, but not vice versa.

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10.1 Preliminaries

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Proposition 10.3 A covariance matrix is both symmetric and positive semidefinite.

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$$QQ^\top = \begin{bmatrix} | & & | \\ \mathbf{q}_1 & \cdots & \mathbf{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & \mathbf{q}_1^\top & — \\ & \vdots & \\ — & \mathbf{q}_n^\top & — \end{bmatrix} = \underline{[\mathbf{q}_i \mathbf{q}_j^\top]} = I.$$

Diagonalization

- A symmetric matrix K can be diagonalized as

$$K = Q\Lambda Q^\top$$

where Λ is a diagonal matrix and Q (also $Q^\top$) is an orthogonal matrix, i.e.,

$$Q^{-1} = Q^\top.$$

Let $\mathbf{q}_i$ be the ith column of Q. Then

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Therefore $KQ = Q\Lambda$ is equivalent to

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for all i.

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- That is, $\mathbf{q}_i$ is an eigenvector of K with eigenvalue λ_i.

Proposition 10.4 The eigenvalues of a positive semidefinite matrix are non-negative.

Proof

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Remark Since a covariance matrix is both symmetric and positive semidefinite, it is diagonalizable and its eigenvalues are nonnegative.

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&= E(A\mathbf{X})(\mathbf{X}^\top A) - (EA\mathbf{X})(E\mathbf{X}^\top A) \\
&= E(A\mathbf{X}\mathbf{X}^\top A^\top) - A(E\mathbf{X})(E\mathbf{X}^\top)A^\top \\
&= A(E\mathbf{X}\mathbf{X}^\top)A^\top - A(E\mathbf{X})(E\mathbf{X}^\top)A^\top \\
&= A\left[\underline{E\mathbf{X}\mathbf{X}^\top - (E\mathbf{X})(E\mathbf{X})^\top}\right]A^\top
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Proposition 10.5 Let $\mathbf{Y} = A\mathbf{X}$. Then

$$K_{\mathbf{Y}} = AK_{\mathbf{X}}A^\top$$

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Lemma 10.6 Let $\mathbf{X}$ and $\mathbf{Y}$ be column vectors of n random variables such that

$$\mathbf{Y} = Q^\top \mathbf{X},$$

where $Q \Lambda Q^\top$ is a diagonalization of $K_{\mathbf{X}}$. Then $K_{\mathbf{Y}} = \Lambda$, i.e., the random variables in $\mathbf{Y}$ are uncorrelated and $\operatorname{var} Y_i = \lambda_i$, the ith diagonal element of Λ.

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$$\operatorname{cov}(Y_i, Y_j) = 0$$

for $i \neq j$.

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3. Furthermore, the variance of Y_i is given by the ith diagonal element of $K_\mathbf{Y} = \Lambda$, i.e., λ_i. The proposition is proved.

Proposition 10.5 Let $\mathbf{Y} = A\mathbf{X}$. Then

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Corollary 10.7 Any random vector $\mathbf{X}$ can be written as a linear transformation of an uncorrelated vector. Specifically, $\mathbf{X} = Q\mathbf{Y}$, where $K_{\mathbf{X}} = Q\Lambda Q^\top$.

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Proposition 10.8 Let $\mathbf{X}$ and $\mathbf{Z}$ be independent and $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$. Then

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Remarks

1. If $\mathbf{Y} = \sum_{i=1}^{n} \mathbf{X}_i$ where $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ are mutually independent, then

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2. When $\mathbf{X}_i$ are scalars, this reduces to

$$\operatorname{var} Y = \sum_{i=1}^{n} \operatorname{var} X_i.$$

Proposition 10.9 (Preservation of Energy) Let $\mathbf{Y} = Q\mathbf{X}$, where Q is an orthogonal matrix. Then

$$E \sum_{i=1}^{n} Y_i^2 = E \sum_{i=1}^{n} X_i^2.$$

Proposition 10.9 Let $\mathbf{Y} = Q\mathbf{X}$, where $\mathbf{X}$ and $\mathbf{Y}$ are column vectors of n random variables and Q is an orthogonal matrix. Then

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2. The proposition is proved upon taking expectation on both sides.