Proposition

$$f(\mathbf{r}, \mathbf{q}) = \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)}$$

is concave.

Proof

1. Consider $(\mathbf{r}_1, \mathbf{q}_1)$ and $(\mathbf{r}_2, \mathbf{q}_2)$ in A.
2. Let $0 \le \lambda \le 1$ and $\bar{\lambda} = 1 - \lambda$. An application of the log-sum inequality gives

$$(\lambda r_1(x) + \bar{\lambda} r_2(x)) \log \frac{\lambda r_1(x) + \bar{\lambda} r_2(x)}{\lambda q_1(x|y) + \bar{\lambda} q_2(x|y)} \le \lambda r_1(x) \log \frac{\lambda r_1(x)}{\lambda q_1(x|y)} + \bar{\lambda} r_2(x) \log \frac{\bar{\lambda} r_2(x)}{\bar{\lambda} q_2(x|y)}.$$

3. Taking reciprocal in the logarithms yields

$$(\lambda r_1(x) + \bar{\lambda} r_2(x)) \log \frac{\lambda q_1(x|y) + \bar{\lambda} q_2(x|y)}{\lambda r_1(x) + \bar{\lambda} r_2(x)} \ge \lambda r_1(x) \log \frac{q_1(x|y)}{r_1(x)} + \bar{\lambda} r_2(x) \log \frac{q_2(x|y)}{r_2(x)}.$$

4. Multiplying by $p(y|x)$ and summing over all x and y, we conclude that $f(\mathbf{r}, \mathbf{q})$ is concave.