Theorem 9.5 If f is concave, then $f^{(k)} \to f^*$.

Proof

1. $f^{(k)}$ necessarily converges, say to f', because $f^{(k)}$ is nondecreasing and bounded from above.

2. Hence, for any $\epsilon > 0$ and all sufficiently large k,

$$f' - \epsilon \leq f^{(k)} \leq f'. \tag{1}$$

3. Let

$$\gamma = \min_{\mathbf{u} \in A'} \Delta f(\mathbf{u}),$$

where $A' = \{\mathbf{u} \in A : f' - \epsilon \leq f(\mathbf{u}) \leq f'\}$.

4. Since f has continuous partial derivatives, $\Delta f(\mathbf{u})$ is a continuous function of $\mathbf{u}$.

5. A' is compact because it is the inverse image of a closed interval under a continuous function and A is bounded. Therefore γ exists.

6. If $f' < f^*$, since f is concave, by Lemma 9.4, $\Delta f(\mathbf{u}) > 0$ for all $\mathbf{u} \in A'$ and hence $\gamma > 0$.

7. Since $f^{(k)} = f(\mathbf{u}^{(k)})$ satisfies (1), $\mathbf{u}^{(k)} \in A'$.

8. Thus for all sufficiently large k,

$$f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) \geq \gamma.$$

9. No matter how small γ is, $f^{(k)}$ will eventually be greater than f', which is a contradiction to $f^{(k)} \to f'$.

10. Hence, $f^{(k)} \to f^*$.