Lemma 9.4 Let f be concave. If $f^{(k)} < f^*$, then $f^{(k+1)} > f^{(k)}$.

Proof

It suffices to prove that $\Delta f(\mathbf{u}) > 0$ for any $\mathbf{u} \in A$ such that $f(\mathbf{u}) < f^*$. Then if $f^{(k)} = f(\mathbf{u}^{(k)}) < f^*$, we have

$$f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) > 0,$$

proving the lemma.

1. First, prove that if $\Delta f(\mathbf{u}) = 0$, then $\mathbf{u}_1 = c_1(\mathbf{u}_2)$ and $\mathbf{u}_2 = c_2(\mathbf{u}_1)$.

2. Second, consider any $\mathbf{u} \in A$ such that $f(\mathbf{u}) < f^*$. Prove by contradiction that $\Delta f(\mathbf{u}) > 0$.

a. Assume that $\Delta f(\mathbf{u}) = 0$. Then $\mathbf{u}_1 = c_1(\mathbf{u}_2)$ and $\mathbf{u}_2 = c_2(\mathbf{u}_1)$, i.e., $\mathbf{u}_1$ maximizes f for a fixed $\mathbf{u}_2$, and $\mathbf{u}_2$ maximizes f for a fixed $\mathbf{u}_1$.

b. Since $f(\mathbf{u}) < f^*$, there exists $\mathbf{v} \in A$ such that $f(\mathbf{u}) < f(\mathbf{v})$.

c. Let

$\tilde{\mathbf{z}}$	unit vector in the direction of $\mathbf{v} - \mathbf{u}$
$\mathbf{z}_1$	unit vector in the direction of $(\mathbf{v}_1 - \mathbf{u}_1, 0)$
$\mathbf{z}_2$	unit vector in the direction of $(0, \mathbf{v}_2 - \mathbf{u}_2)$.

d. Then $\tilde{\mathbf{z}} = \alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2$, where

$$\alpha_i = \frac{\|\mathbf{v}_i - \mathbf{u}_i\|}{\|\mathbf{v} - \mathbf{u}\|}, \quad i = 1, 2.$$

e. Since f is continuous and has continuous partial derivatives, the directional derivative of f at $\mathbf{u}$ in the direction of $\mathbf{z}_1$ is given by $\nabla f \cdot \mathbf{z}_1$.

f. f attains its maximum value at $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ when $\mathbf{u}_2$ is fixed.

g. In particular, f attains its maximum value at $\mathbf{u}$ along the line passing through $(\mathbf{u}_1, \mathbf{u}_2)$ and $(\mathbf{v}_1, \mathbf{u}_2)$.

h. Therefore, by considering the line passing through $(\mathbf{u}_1, \mathbf{u}_2)$ and $(\mathbf{v}_1, \mathbf{u}_2)$, we see that $\nabla f \cdot \mathbf{z}_1 = 0$. Similarly, $\nabla f \cdot \mathbf{z}_2 = 0$.

i. Then $\nabla f \cdot \tilde{\mathbf{z}} = \alpha_1(\nabla f \cdot \mathbf{z}_1) + \alpha_2(\nabla f \cdot \mathbf{z}_2) = 0$.

j. Since f is concave along the line passing through $\mathbf{u}$ and $\mathbf{v}$, this implies $f(\mathbf{u}) \geq f(\mathbf{v})$, a contradiction.

k. Hence, $\Delta f(\mathbf{u}) > 0$.