Lemma 9.4 Let f be concave. If $f^{(k)} < f^*$, then $f^{(k+1)} > f^{(k)}$.

Proof

It suffices to prove that $\Delta f(\mathbf{u}) > 0$ for any $\mathbf{u} \in A$ such that $f(\mathbf{u}) < f^*$. Then if $f^{(k)} = f(\mathbf{u}^{(k)}) < f^*$, we have

$$f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) > 0,$$

proving the lemma.

1. First, prove that if $\Delta f(\mathbf{u}) = 0$, then $\mathbf{u}_1 = c_1(\mathbf{u}_2)$ and $\mathbf{u}_2 = c_2(\mathbf{u}_1)$.

a. Consider

$$f(c_1(\mathbf{u}_2), c_2(c_1(\mathbf{u}_2))) \overset{i)}{\geq} f(c_1(\mathbf{u}_2), \mathbf{u}_2) \overset{ii)}{\geq} f(\mathbf{u}_1, \mathbf{u}_2).$$

If $\Delta f(\mathbf{u}) = 0$, then both $i)$ and $ii)$ are tight.

b. Due to the uniqueness of $c_2(\cdot)$ and $c_1(\cdot)$,

$$\begin{aligned} ii) \text{ is tight} \quad &\Rightarrow \quad \mathbf{u}_1 = c_1(\mathbf{u}_2) \\ i) \text{ is tight} \quad &\Rightarrow \quad \mathbf{u}_2 = c_2(c_1(\mathbf{u}_2)) = c_2(\mathbf{u}_1). \end{aligned}$$

c. This also implies that if

$$f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) = 0,$$

then $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)}$.