

Example 8.20 (Binary Source) Let X be a binary random variable with

$$\Pr\{X = 0\} = 1 - \gamma \quad \text{and} \quad \Pr\{X = 1\} = \gamma.$$

Let $\hat{X} = \{0, 1\}$ and d be the Hamming distortion measure.

Determination of $R_I(D)$

1. First consider $0 \leq \gamma \leq \frac{1}{2}$. We will show that

$$R_I(D) = \begin{cases} h_b(\gamma) - h_b(D) & \text{if } 0 \leq D < \gamma \\ 0 & \text{if } D \geq \gamma. \end{cases}$$

2. Since $\gamma \leq 1/2$, $\hat{x}^* = 0$ and $D_{max} = Ed(X, 0) = \Pr\{X = 1\} = \gamma$.

3. Consider any \hat{X} and let $Y = d(X, \hat{X})$.

4. Conditioning on \hat{X} , X and Y determine each other, and so, $H(X|\hat{X}) = H(Y|\hat{X})$.

Then for any \hat{X} such that $Ed(X, \hat{X}) \leq D$, where $D < \gamma = D_{max}$,

$$\begin{aligned} I(X; \hat{X}) &= H(X) - H(X|\hat{X}) \\ &= h_b(\gamma) - H(Y|\hat{X}) \\ &\geq h_b(\gamma) - H(Y) \end{aligned} \tag{1}$$

$$\begin{aligned} &= h_b(\gamma) - h_b(\Pr\{X \neq \hat{X}\}) \\ &\geq h_b(\gamma) - h_b(D), \end{aligned} \tag{2}$$

because $\Pr\{X \neq \hat{X}\} = Ed(X, \hat{X}) \leq D$ and $h_b(a)$ is increasing for $0 \leq a \leq \frac{1}{2}$.

5. Therefore,

$$R_I(D) = \min_{\hat{X}: Ed(X, \hat{X}) \leq D} I(X; \hat{X}) \geq h_b(\gamma) - h_b(D).$$

Now need to construct \hat{X} which is tight for (1) and (2), so that the above bound is achieved.

6. Observe that

- (1) tight $\Leftrightarrow Y$ independent of \hat{X}
- (2) tight $\Leftrightarrow \Pr\{X \neq \hat{X}\} = D$

The required \hat{X} can be specified by the following reverse BSC:

Note that

$$\left(\frac{1 - \gamma - D}{1 - 2D} \right) D + \left(\frac{\gamma - D}{1 - 2D} \right) (1 - D) = \gamma$$

$$0 \leq \frac{\gamma - D}{1 - 2D} \leq 1, \text{ because } D < \gamma \leq 1/2$$