

**Corollary 8.19** If  $R_I(0) > 0$ , then  $R_I(D)$  is strictly decreasing for  $0 \leq D \leq D_{max}$ , and the inequality constraint in the definition of  $R_I(D)$  can be replaced by an equality constraint.

**Proof**

1. Show that  $R_I(D) > 0$  for  $0 \leq D < D_{max}$  by contradiction.

a. Suppose  $R_I(D') = 0$  for some  $0 \leq D' < D_{max}$ , and let  $R_I(D')$  be achieved by some  $\hat{X}$ . Then

$$R_I(D') = I(X; \hat{X}) = 0$$

implies that  $X$  and  $\hat{X}$  are independent.

b. Show that such an  $\hat{X}$  which is independent of  $X$  cannot do better than the constant estimate  $\hat{x}^*$ , i.e.,  $Ed(X, \hat{X}) \geq Ed(X, \hat{x}^*) = D_{max}$ . This is done by considering

$$\begin{aligned} D' &\geq Ed(X, \hat{X}) \\ &= \sum_x \sum_{\hat{x}} p(x, \hat{x}) d(x, \hat{x}) \\ &= \sum_x \sum_{\hat{x}} p(x) p(\hat{x}) d(x, \hat{x}) \\ &= \sum_{\hat{x}} p(\hat{x}) \sum_x p(x) d(x, \hat{x}) \\ &= \sum_{\hat{x}} p(\hat{x}) Ed(X, \hat{x}) \\ &\geq \sum_{\hat{x}} p(\hat{x}) Ed(X, \hat{x}^*) \\ &= \sum_{\hat{x}} p(\hat{x}) D_{max} \\ &= D_{max}. \end{aligned}$$

c. This is a contradiction because

$$0 \leq D' < D_{max}.$$

2.  $R_I(D)$  must be strictly decreasing for  $0 \leq D \leq D_{max}$  because  $R_I(0) > 0$ ,  $R_I(D_{max}) = 0$ , and  $R_I(D)$  is non-increasing and convex.

3. Show that the inequality constraints in  $R_I(D)$  can be replaced by an equality constraint by contradiction.

a. Assume that  $R_I(D)$  is achieved by some  $\hat{X}^*$  such that  $Ed(X, \hat{X}^*) = D'' < D$ .

b. Then

$$\begin{aligned} R_I(D'') &= \min_{\hat{X}: Ed(X, \hat{X}) \leq D''} I(X; \hat{X}) \\ &\leq I(X; \hat{X}^*) \\ &= R_I(D), \end{aligned}$$

a contradiction because  $R_I(D)$  is strictly decreasing for  $0 \leq D \leq D_{max}$ .

c. Therefore,  $Ed(X, \hat{X}^*) = D$ .