

Theorem 8.18 The following properties hold for the information rate-distortion function $R_I(D)$:

1. $R_I(D)$ is non-increasing in D .
2. $R_I(D)$ is convex.
3. $R_I(D) = 0$ for $D \geq D_{max}$.
4. $R_I(0) \leq H(X)$.

Proof

1. For a larger D , the minimization is taken over a larger set.
2. Consider any $D^{(1)}, D^{(2)} \geq 0$ and $0 \leq \lambda \leq 1$. Let $\hat{X}^{(i)}$ achieves $R_I(D^{(i)})$ for $i = 1, 2$, i.e.,

$$R_I(D^{(i)}) = I(X; \hat{X}^{(i)}),$$

where

$$Ed(X, \hat{X}^{(i)}) \leq D^{(i)}.$$

Let $\hat{X}^{(\lambda)}$ be jointly distributed with X defined by

$$p_\lambda(\hat{x}|x) = \lambda p_1(\hat{x}|x) + \bar{\lambda} p_2(\hat{x}|x).$$

Then

$$\begin{aligned} Ed(X, \hat{X}^{(\lambda)}) &= \lambda Ed(X, \hat{X}^{(1)}) + \bar{\lambda} Ed(X, \hat{X}^{(2)}) \\ &\leq \lambda D^{(1)} + \bar{\lambda} D^{(2)} \\ &= D^{(\lambda)}. \end{aligned}$$

Finally consider

$$\begin{aligned} &\lambda R_I(D^{(1)}) + \bar{\lambda} R_I(D^{(2)}) \\ &= \lambda I(X; \hat{X}^{(1)}) + \bar{\lambda} I(X; \hat{X}^{(2)}) \\ &\geq I(X; \hat{X}^{(\lambda)}) \\ &\geq R_I(D^{(\lambda)}). \end{aligned}$$

3. Let $\hat{X} = \hat{x}^*$ w.p. 1 to show that $(0, D_{max})$ is achievable. Then for $D \geq D_{max}$, $R_I(D) \leq I(X; \hat{X}) = 0$, which implies $R_I(D) = 0$.

4. Let $\hat{X} = \hat{x}^*(X)$, so that $Ed(X, \hat{X}) = 0$ (since d is normal). Then

$$R_I(0) \leq I(X; \hat{X}) \leq H(X).$$