Lemma 7.17 Let $(\mathbf{X}', \mathbf{Y}')$ be n i.i.d. copies of a pair of generic random variables (X', Y'), where $X' \sim X$, $Y' \sim Y$, and X' and Y' are independent. Then

$$\Pr\{(\mathbf{X}', \mathbf{Y}') \in T^n_{[XY]\delta}\} \leq 2^{-n(I(X;Y)-\tau)},$$

where $\tau \to 0$ as $\delta \to 0$.

Proof

1. Consider

$$\Pr\{(\mathbf{X}', \mathbf{Y}') \in T^n_{[XY]\delta}\} = \sum_{(\mathbf{x},\mathbf{y}) \in T^n_{[XY]\delta}} p(\mathbf{x})p(\mathbf{y}).$$

2. By the consistency of strong typicality, for $(\mathbf{x}, \mathbf{y}) \in T^n_{[XY]\delta}$, $\mathbf{x} \in T^n_{[X]\delta}$ and $\mathbf{y} \in T^n_{[Y]\delta}$.

3. By the strong AEP, all the $p(\mathbf{x})$ and $p(\mathbf{y})$ in the above summation satisfy

$$p(\mathbf{x}) \leq 2^{-n(H(X)-\eta)}$$

and

$$p(\mathbf{y}) \leq 2^{-n(H(Y)-\zeta)},$$

where $\eta, \zeta \to 0$ as $\delta \to 0$.

4. By the strong JAEP,

$$|T^n_{[XY]\delta}| \leq 2^{n(H(X,Y)+\xi)},$$

where $\xi \to 0$ as $\delta \to 0$.

5. Then we have

$$\begin{aligned}
&\Pr\{(\mathbf{X}', \mathbf{Y}') \in T^n_{[XY]\delta}\} \\
&\quad = \sum_{(\mathbf{x},\mathbf{y}) \in T^n_{[XY]\delta}} p(\mathbf{x})p(\mathbf{y}) \\
&\quad \leq 2^{n(H(X,Y)+\xi)} \cdot 2^{-n(H(X)-\eta)} \cdot 2^{-n(H(Y)-\zeta)} \\
&\quad = 2^{-n(H(X)+H(Y)-H(X,Y)-\xi-\eta-\zeta)} \\
&\quad = 2^{-n(I(X;Y)-\xi-\eta-\zeta)} \\
&\quad = 2^{-n(I(X;Y)-\tau)},
\end{aligned}$$

where

$$\tau = \xi + \eta + \zeta \to 0$$

as $\delta \to 0$. The lemma is proved.