Example 7.8 (Binary Erasure Channel)

Binary Erasure Channel:

- The output symbol e denotes "erasure".
- γ : Erasure probability, $0 \le \gamma \le 1$
- With probability $1-\gamma$, $Y = X$ (regardless of whether $X = 0$ or $X = 1$).
- With probability γ, $Y = e$ (erasure).
- If $X = 0$, Y cannot be 1. If $X = 1$, Y cannot be 0.

Determination of C

1. Consider

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) \\ &= \max_{p(x)} (H(Y) - H(Y|X)) \\ &= \max_{p(x)} H(Y) - h_b(\gamma). \end{aligned}$$

Thus we only have to maximize $H(Y)$ over all $p(x)$.

2. Let

$$p(0) = a$$

and define a binary random variable E by

$$E = \begin{cases} 0 & \text{if } Y \neq e \\ 1 & \text{if } Y = e. \end{cases}$$

The random variable E indicates whether an erasure has occurred, and it is a function of Y.

3. Then

$$\begin{aligned} H(Y) &= H(Y,E) \\ &= H(E) + H(Y|E) \\ &= h_b(\gamma) + (1-\gamma) h_b(a). \end{aligned}$$

4. Hence,

$$\begin{aligned} C &= \max_{p(x)} H(Y) - h_b(\gamma) \\ &= \max_{a} [h_b(\gamma) + (1-\gamma) h_b(a)] - h_b(\gamma) \\ &= (1-\gamma) \max_{a} h_b(a) \\ &= (1-\gamma) \text{ bit per use,} \end{aligned}$$

where the capacity is achieved by letting $a = 0.5$, i.e., the input distribution is uniform.

5. Therefore, $C = 1 - \gamma$.

Exercise: Show that $H(Y|E) = (1-\gamma) h_b(a)$.