**Theorem 3.11** If there is no constraint on $X_1, X_2, \cdots, X_n$, then $\mu^*$ can take any set of nonnegative values on the nonempty atoms of $\mathcal{F}_n$.

**Proof**

1. Let $Y_A, A \in \mathcal{A}$ be mutually independent random variables. Note that these r.v.'s are labeled by the nonempty atoms of $\mathcal{F}_n$.

2. Define the random variables $X_i, i = 1, 2, \cdots, n$ by

$$X_i = (Y_A : A \in \mathcal{A} \text{ and } A \subset \tilde{X}_i).$$

We will determine the $I$-Measure $\mu^*$ for $X_1, X_2, \cdots, X_n$ so defined.

3. Since $Y_A$ are mutually independent, for any nonempty subsets $G$ of $\mathcal{N}_n$, we have

$$\begin{aligned} H(X_G) &= H(X_i, i \in G) \\ &= H((Y_A : A \in \mathcal{A} \text{ and } A \subset \tilde{X}_i), i \in G) \\ &= H(Y_A : A \in \mathcal{A} \text{ and } A \subset \tilde{X}_G) \\ &= \sum_{A \in \mathcal{A}: A \subset \tilde{X}_G} H(Y_A). \end{aligned} \tag{1}$$

4. On the other hand, by set-additivity, we have

$$H(X_G) = \mu^*(\tilde{X}_G) = \sum_{A \in \mathcal{A}: A \subset \tilde{X}_G} \mu^*(A). \tag{2}$$

Equating the right hand sides of (1) and (2), we have

$$\sum_{A \in \mathcal{A}: A \subset \tilde{X}_G} H(Y_A) = \sum_{A \in \mathcal{A}: A \subset \tilde{X}_G} \mu^*(A).$$

5. Here $\mu^*(A)$ are the unknowns. Evidently, we can make the above equality hold for all nonempty subsets $G$ of $\mathcal{N}_n$ by taking

$$\mu^*(A) = H(Y_A)$$

for all $A \in \mathcal{A}$. By the uniqueness of $\mu^*$ (Theorem 3.9), this is also the only possibility for $\mu^*$.

6. Since $H(Y_A)$ can take any nonnegative value by Corollary 2.44, $\mu^*$ can take any set of nonnegative values on the nonempty atoms of $\mathcal{F}_n$. The theorem is proved.