Example 3.10

1. In this example, all entropies are in the base 2.

2. Let X_1 and X_2 be independent binary random variables with uniform distribution, i.e.,

$$\Pr\{X_i = 0\} = \Pr\{X_i = 1\} = 0.5,$$

$i = 1, 2$. Let

$$X_3 = (X_1 + X_2) \bmod 2. \tag{1}$$

3. It is easy to check that X_3 also has a uniform distribution. Thus,

$$H(X_i) = 1$$

for $i = 1, 2, 3$.

4. It is also easy to check that X_1, X_2, and X_3 are pairwise independent. Therefore,

$$H(X_i, X_j) = 2$$

and

$$I(X_i; X_j) = 0$$

for $1 \leq i < j \leq 3$.

5. We see from (1) that X_3 is a function of X_1 and X_2, so that

$$H(X_3|X_1, X_2) = 0.$$

Then by the chain rule for entropy, we have

$$\begin{aligned} &H(X_1, X_2, X_3) \\ &\quad = H(X_1, X_2) + H(X_3|X_1, X_2) \\ &\quad = 2 + 0 \\ &\quad = 2. \end{aligned}$$

6. Now for distinct $1 \leq i, j, k \leq 3$,

$$\begin{aligned} &I(X_i; X_j|X_k) \\ &\quad = H(X_i, X_k) + H(X_j, X_k) \\ &\qquad -H(X_1, X_2, X_3) - H(X_k) \\ &\quad = 2 + 2 - 2 - 1 \\ &\quad = 1. \end{aligned}$$

7. It then follows that

$$\begin{aligned} &\mu^*(\tilde{X}_1 \cap \tilde{X}_2 \cap \tilde{X}_3) \\ &\quad = \mu^*(\tilde{X}_1 \cap \tilde{X}_2) - \mu^*(\tilde{X}_1 \cap \tilde{X}_2 - \tilde{X}_3) \\ &\quad = I(X_1; X_2) - I(X_1; X_2|X_3) \\ &\quad = 0 - 1 \\ &\quad = -1 \\ &\quad < 0. \end{aligned}$$