Theorem 2.31 (Divergence Inequality)

$$D(p\|q) \geq 0 \tag{1}$$

with equality if and only if $p = q$.

Proof

1. For simplicity, assume $\mathcal{S}_p = \mathcal{S}_q$. For a proof without this assumption, see the textbook.

2. Consider

$$\begin{aligned}
D(p\|q) &= \sum_{x \in \mathcal{S}_p} p(x) \log \frac{p(x)}{q(x)} \\
&= (\log e) \sum_{x \in \mathcal{S}_p} p(x) \ln \frac{p(x)}{q(x)} \\
&\geq (\log e) \sum_{x \in \mathcal{S}_p} p(x) \left(1 - \frac{q(x)}{p(x)}\right) \qquad (2) \\
&= (\log e) \left[\sum_{x \in \mathcal{S}_p} p(x) - \sum_{x \in \mathcal{S}_p} q(x)\right] \\
&= (\log e) \left[\sum_{x \in \mathcal{S}_p} p(x) - \sum_{x \in \mathcal{S}_q} q(x)\right] \\
&= (\log e)[1 - 1] \\
&= 0.
\end{aligned}$$

This proves (1).

3. For equality to hold in (2), we see from Corollary 2.30 that this is the case if and only if

$$\frac{p(x)}{q(x)} = 1 \text{ or } p(x) = q(x) \quad \text{for all } x \in \mathcal{S}_p.$$

This proves the theorem.