Theorem 2.32 (Log-Sum Inequality) For positive numbers $a_1, a_2, \cdots$ and nonnegative numbers $b_1, b_2, \cdots$ such that $\sum_i a_i < \infty$ and $0 < \sum_i b_i < \infty$,

$$\sum_i a_i \log \frac{a_i}{b_i} \geq \left(\sum_i a_i \right) \log \frac{\sum_i a_i}{\sum_i b_i} \qquad (1)$$

Moreover, equality holds if and only if $\frac{a_i}{b_i} = constant$ for all i.

Proof

1. Let $a_i' = a_i / \sum_j a_j$ and $b_i' = b_i / \sum_j b_j$. Then $\{a_i'\}$ and $\{b_i'\}$ are probability distributions.

2. Using the divergence inequality, we have

$$\begin{aligned}
0 &\leq \sum_i a_i' \log \frac{a_i'}{b_i'} \\
&= \sum_i \frac{a_i}{\sum_j a_j} \log \frac{a_i / \sum_j a_j}{b_i / \sum_j b_j} \\
&= \frac{1}{\sum_j a_j} \left[\sum_i a_i \log \frac{a_i / \sum_j a_j}{b_i / \sum_j b_j} \right] \\
&= \frac{1}{\sum_j a_j} \left[\sum_i a_i \log \frac{a_i}{b_i} - \sum_i a_i \log \frac{\sum_j a_j}{\sum_j b_j} \right] \\
&= \frac{1}{\sum_j a_j} \left[\sum_i a_i \log \frac{a_i}{b_i} - \left(\sum_i a_i \right) \log \frac{\sum_j a_j}{\sum_j b_j} \right],
\end{aligned}$$

which implies (1).

3. Equality holds if and only if for all i,

$$a_i' = b_i' \quad \text{or} \quad \frac{a_i}{b_i} = constant.$$

The theorem is proved.