

**Proposition 2.5** For random variables  $X$ ,  $Y$ , and  $Z$ ,  $X \perp Z|Y$  if and only if

$$p(x, y, z) = a(x, y)b(y, z)$$

for all  $x$ ,  $y$ , and  $z$  such that  $p(y) > 0$ .

**Proof**

B. 'If'

Refer to Definition 2.4.

1. Assume

$$p(x, y, z) = a(x, y)b(y, z)$$

for all  $x$ ,  $y$ , and  $z$  such that  $p(y) > 0$ .

2. Then for such  $x$ ,  $y$ , and  $z$ , we have

$$\begin{aligned} p(x, y) &= \sum_z p(x, y, z) \\ &= \sum_z a(x, y)b(y, z) \\ &= a(x, y) \sum_z b(y, z) \end{aligned}$$

3. Similarly,

$$\begin{aligned} p(y, z) &= \sum_x p(x, y, z) \\ &= \sum_x a(x, y)b(y, z) \\ &= b(y, z) \sum_x a(x, y). \end{aligned}$$

4. Furthermore,

$$p(y) = \sum_z p(y, z) = \left( \sum_x a(x, y) \right) \left( \sum_z b(y, z) \right) > 0.$$

5. Therefore,

$$\begin{aligned} &\frac{p(x, y)p(y, z)}{p(y)} \\ &= \frac{\left( a(x, y) \sum_z b(y, z) \right) \left( b(y, z) \sum_x a(x, y) \right)}{\left( \sum_x a(x, y) \right) \left( \sum_z b(y, z) \right)} \\ &= a(x, y)b(y, z) \\ &= p(x, y, z). \end{aligned}$$

6. For  $x$ ,  $y$ , and  $z$  such that  $p(y) = 0$ , since

$$0 \leq p(x, y, z) \leq p(y) = 0,$$

we have

$$p(x, y, z) = 0.$$

7. Hence,  $X \perp Z|Y$  according to Definition 2.4.