Theorem 11.32 For a fixed zero-mean Gaussian random vector $\mathbf{X}^*$, let

$$\mathbf{Y} = \mathbf{X}^* + \mathbf{Z},$$

where the joint pdf of $\mathbf{Z}$ exists and $\mathbf{Z}$ is independent of $\mathbf{X}^*$. Under the constraint that the correlation matrix of $\mathbf{Z}$ is equal to K, where K is any symmetric positive definite matrix, $I(\mathbf{X}^*; \mathbf{Y})$ is minimized if and only if $\mathbf{Z} = \mathbf{Z}^* \sim \mathcal{N}(0, K)$.

Proof

1. Since $E\mathbf{Z}^* = 0$, $\tilde{K}_{\mathbf{Z}^*} = K_{\mathbf{Z}^*} = K$. Therefore, $\mathbf{Z}^*$ and $\mathbf{Z}$ have the same correlation matrix.

2. By noting that $\mathbf{X}^*$ has zero mean, we apply Lemma 11.33 to see that $\mathbf{Y}^*$ and $\mathbf{Y}$ have the same correlation matrix.

3. The theorem is proved by considering

$$\begin{aligned}
& I(\mathbf{X}^*; \mathbf{Y}^*) - I(\mathbf{X}^*; \mathbf{Y}) \\
= \quad & h(\mathbf{Y}^*) - h(\mathbf{Z}^*) - h(\mathbf{Y}) + h(\mathbf{Z}) \\
= \quad & -\int f_{\mathbf{Y}^*}(\mathbf{y}) \log f_{\mathbf{Y}^*}(\mathbf{y}) d\mathbf{y} + \int f_{\mathbf{Z}^*}(\mathbf{z}) \log f_{\mathbf{Z}^*}(\mathbf{z}) d\mathbf{z} \\
& + \int f_{\mathbf{Y}}(\mathbf{y}) \log f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{\mathcal{S}_{\mathbf{Z}}} f_{\mathbf{Z}}(\mathbf{z}) \log f_{\mathbf{Z}}(\mathbf{z}) d\mathbf{z} \\
= \quad & -\int f_{\mathbf{Y}}(\mathbf{y}) \log f_{\mathbf{Y}^*}(\mathbf{y}) d\mathbf{y} + \int_{\mathcal{S}_{\mathbf{Z}}} f_{\mathbf{Z}}(\mathbf{z}) \log f_{\mathbf{Z}^*}(\mathbf{z}) d\mathbf{z} \\
& + \int f_{\mathbf{Y}}(\mathbf{y}) \log f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{\mathcal{S}_{\mathbf{Z}}} f_{\mathbf{Z}}(\mathbf{z}) \log f_{\mathbf{Z}}(\mathbf{z}) d\mathbf{z} \\
= \quad & \int \log \left(\frac{f_{\mathbf{Y}}(\mathbf{y})}{f_{\mathbf{Y}^*}(\mathbf{y})} \right) f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{\mathcal{S}_{\mathbf{Z}}} \log \left(\frac{f_{\mathbf{Z}}(\mathbf{z})}{f_{\mathbf{Z}^*}(\mathbf{z})} \right) f_{\mathbf{Z}}(\mathbf{z}) d\mathbf{z} \\
= \quad & D(f_{\mathbf{Y}} \| f_{\mathbf{Y}^*}) - D(f_{\mathbf{Z}} \| f_{\mathbf{Z}^*}) \\
\leq \quad & 0.
\end{aligned}$$