

1. Divide  $[0, W]$  into  $k$  subintervals, each with width  $\Delta_k = \frac{W}{k}$ .

2. Let the  $i$ th subinterval be  $[f_l^i, f_h^i]$ ,  $1 \leq i \leq k$ .

3. As an approximation, assume that the noise power over the  $i$ th subinterval is a constant  $S_{Z,i}$ .

4. Then the channel consists of  $k$  sub-channels, with the  $i$ th sub-channel being a **bandpass white Gaussian channel** occupying the frequency band  $[f_l^i, f_h^i]$ .

5. Let  $P_i$  be the power allocated to the  $i$ th sub-channel. Then the capacity of the  $i$ th sub-channel is

$$\Delta_k \log \left( 1 + \frac{P_i}{2S_{Z,i}\Delta_k} \right).$$

6. The noise process  $Z'_i(t)$  of the  $i$ th sub-channel is obtained by passing  $Z(t)$  through the corresponding ideal bandpass filter bandlimited to  $[f_l^i, f_h^i]$ .

7. It can be shown (see Problem 9) that the noise processes  $Z'_i(t)$ ,  $1 \leq i \leq k$  are independent.

8. By sampling the sub-channels at the Nyquist rate  $2\Delta_k$ , the  $k$  sub-channels can be regarded as a system of **parallel Gaussian channels**.

9. Thus the capacity of the channel is equal to the sum of the capacities of the individual sub-channels when the power allocation among the  $k$  sub-channels is optimal.

10. Let  $P_i^*$  be the optimal power allocation for the  $i$ th sub-channel.

11. The channel capacity is equal to

$$\sum_{i=1}^k \Delta_k \log \left( 1 + \frac{P_i^*}{2S_{Z,i}\Delta_k} \right) = \sum_{i=1}^k \Delta_k \log \left( 1 + \frac{\frac{P_i^*}{2\Delta_k}}{S_{Z,i}} \right)$$

where by Proposition 11.23,

$$\frac{P_i^*}{2\Delta_k} = (\nu - S_{Z,i})^+ \quad \text{with} \quad \sum_{i=1}^k P_i^* = P.$$

12. As  $k \rightarrow \infty$ ,

$$\begin{aligned} & \sum_{i=1}^k \Delta_k \log \left( 1 + \frac{\frac{P_i^*}{2\Delta_k}}{S_{Z,i}} \right) \\ & \rightarrow \int_0^W \log \left( 1 + \frac{(\nu - S_Z(f))^+}{S_Z(f)} \right) df \\ & = \frac{1}{2} \int_{-W}^W \log \left( 1 + \frac{(\nu - S_Z(f))^+}{S_Z(f)} \right) df \\ & \quad \text{bits per unit time} \end{aligned}$$

since  $S_Z(-f) = S_Z(f)$  (see Problem 8).