Theorem 11.30 Let

$$\psi_i(t) = \sqrt{2W}\text{sinc}(2Wt - i).$$

Then $\psi_i(t)$, $-\infty < i < \infty$ form an orthonormal basis for signals which are bandlimited to $[0, W]$.

Proof

1. Consider

$$\psi_i(t) = \sqrt{2W}\text{sinc}\left(2W\left(t - \frac{i}{2W}\right)\right)$$

and $\psi_0(t) = \sqrt{2W}\text{sinc}\,(2Wt)$. Therefore,

$$\psi_i(t) = \psi_0\left(t - \frac{i}{2W}\right),$$

and so $\psi_i(t)$ and $\psi_0(t)$ have the same energy.

2. Consider

$$\text{sinc}(2Wt) \rightleftharpoons \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right),$$

where

$$\text{rect}(f) = \begin{cases} 1 & -\frac{1}{2} \le f \le \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

3. Then by Rayleigh's energy theorem, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \text{sinc}^2(2Wt)dt &= \left(\frac{1}{2W}\right)^2 \int_{-\infty}^{\infty} \text{rect}^2\left(\frac{f}{2W}\right) df \\ &= \left(\frac{1}{2W}\right)^2 (2W) \\ &= \frac{1}{2W}.\end{aligned}$$

4. It then follows that

$$\int_{-\infty}^{\infty} \psi_i^2(t)dt = \int_{-\infty}^{\infty} \psi_0^2(t)dt = 1. \tag{1}$$

5. Since (1) implies that both $\text{sinc}(2Wt - i)$ and $\text{sinc}(2Wt - i')$ have finite energy, we can consider their cross-correlation function

$$R_{ii'}(\tau) = \int_{-\infty}^{\infty} \text{sinc}(2Wt - i)\text{sinc}(2W(t - \tau) - i')dt.$$

In particular,

$$R_{ii'}(0) = \int_{-\infty}^{\infty} \text{sinc}(2Wt - i)\text{sinc}(2Wt - i')dt.$$

6. Now

$$\begin{aligned}&\text{sinc}(2Wt - i) \\ &\rightleftharpoons \quad e^{-j2\pi f\left(\frac{i}{2W}\right)}\left(\frac{1}{2W}\right)\text{rect}\left(\frac{f}{2W}\right) := G_i(f)\end{aligned}$$