Proposition 11.31 $Z'\left(\frac{i}{2W}\right)$, $-\infty < i < \infty$ are i.i.d. Gaussian random variables with zero mean and variance $N_0 W$.

Proof

1. $Z'(t)$ is a filtered version of $Z(t)$, so $Z'(t)$ is also a zero-mean Gaussian process.

2. $Z'\left(\frac{i}{2W}\right)$, $-\infty < i < \infty$ are zero-mean Gaussian random variables.

3. The power spectral density of $Z'(t)$ is a rectangular function:

$$\begin{aligned} S_{Z'}(f) &= \begin{cases} N_0/2 & -W \le f \le W \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{N_0}{2} \operatorname{rect}\left(\frac{f}{2W}\right). \end{aligned}$$

4. $R_{Z'}(\tau)$, the autocorrelation function of $Z'(t)$, is a sinc function:

$$S_{Z'}(f) \rightleftharpoons R_{Z'}(\tau) = N_0 W \operatorname{sinc}(2W\tau).$$

5. $R_{Z'}(\tau)$ vanishes at $\tau = \frac{i}{2W}$ for every $i \neq 0$:

$$R_{Z'}\left(\frac{i}{2W}\right) = \begin{cases} N_0 W & i = 0 \\ 0 & i \neq 0. \end{cases}$$

6. Then for all t and all $i \neq 0$, $Z'(t)$ and $Z'\left(t + \frac{i}{2W}\right)$ are uncorrelated because

$$E\left[Z'\left(t + \frac{i}{2W}\right) Z'(t)\right] = R_{Z'}\left(\frac{i}{2W}\right) = 0.$$

7. In particular, letting $t = \frac{j}{2W}$, we see that $Z'\left(\frac{j}{2W}\right)$ and $Z'\left(\frac{j}{2W} + \frac{i}{2W}\right) = Z'\left(\frac{j+i}{2W}\right)$ are uncorrelated.

8. Therefore, $Z'\left(\frac{i}{2W}\right)$, $-\infty < i < \infty$ are uncorrelated and hence independent because they are jointly Gaussian.

9. Since $Z'\left(\frac{i}{2W}\right)$ has zero mean, its variance is given by $R_{Z'}(0) = N_0 W$, because

$$\begin{aligned} R_{Z'}(0) &= E\left[Z'\left(\frac{i}{2W} + 0\right) Z'\left(\frac{i}{2W}\right)\right] \\ &= E\left[Z'\left(\frac{i}{2W}\right)\right]^2 \\ &= \operatorname{var}\left(Z'\left(\frac{i}{2W}\right)\right). \end{aligned}$$