Proposition 11.23 The problem

For given $\lambda_i \geq 0$, maximize $\sum_{i=1}^{k} \log(a_i + \lambda_i)$ subject to

$$\begin{aligned} \sum_i a_i &\leq P \qquad (1) \\ -a_i &\leq 0. \qquad (2) \end{aligned}$$

has the solution

$$a_i^* = (\nu - \lambda_i)^+, \quad 1 \leq i \leq k,$$

where ν satisfies

$$\sum_{i=1}^{k} (\nu - \lambda_i)^+ = P.$$

Proof

1. We will prove the proposition by verifying that the proposed solution satisfies the KKT condition. This is done by finding nonnegative μ and μ_i satisfying the equations

$$\begin{aligned} \frac{\log e}{(a_i^* + \lambda_i)} - \mu + \mu_i &= 0 \qquad (3) \\ \mu \left(P - \sum_{i=1}^{k} a_i^* \right) &= 0 \qquad (4) \\ \mu_i a_i^* &= 0, \quad 1 \leq i \leq k, \qquad (5) \end{aligned}$$

where μ and μ_i are the multipliers associated with the constraints in (1) and (2), respectively.

2. To avoid triviality, assume $P > 0$ so that $\nu > 0$, and observe that there exists at least one i such that $a_i^* > 0$.

3. For i such that $a_i^* > 0$:

- (5) implies $\mu_i = 0$
- $a_i^* = (\nu - \lambda_i)^+ = \nu - \lambda_i$, or $a_i^* + \lambda_i = \nu$
- from (3), we obtain $\mu = \frac{\log e}{\nu} > 0$.

4. For i such that $a_i^* = 0$,

- $\nu \leq \lambda_i$
- following (3), we have

$$\begin{aligned} \frac{\log e}{(a_i^* + \lambda_i)} - \mu + \mu_i &= 0 \\ \frac{\log e}{\lambda_i} - \frac{\log e}{\nu} + \mu_i &= 0 \end{aligned}$$

which implies

$$\mu_i = (\log e) \left(\frac{1}{\nu} - \frac{1}{\lambda_i} \right) \geq 0.$$

5. Thus we have obtained nonnegative μ and μ_i satisfying the KKT condition.