Lemma 11.22 Let $Y = X + Z$. Then

$$h(Y|X) = h(Z|X)$$

provided that $f_{Z|X}(z|x)$ exists for all $x \in \mathcal{S}_X$.

Theorem 11.21 (Capacity of a Memoryless Gaussian Channel) The capacity of a memoryless Gaussian channel with noise power N and input power constraint P is

$$\frac{1}{2}\log\left(1+\frac{P}{N}\right).$$

The capacity is achieved by the input distribution $\mathcal{N}(0, P)$.

Proof

1. Let $F(x)$ be the CDF of the input random variable X such that $EX^2 \leq P$, where X is not necessarily continuous.

2. Since $Z \sim \mathcal{N}(0, N)$, f_Z exists. Then $f_{Z|X}(z|x)$ exists and is equal to $f_Z(z)$ because Z is independent of X.

3. From the proof of Lemma 11.22,

$$f_{Y|X}(y|x) = f_{Z|X}(y-x|x) = f_Z(y-x),$$

and by Proposition 10.24,

$$f_Y(y) = \int f_{Y|X}(y|x)\, dF_X(x).$$

Therefore $f_Y(y)$ exists and hence $h(Y)$ is defined.

4. Therefore, by Lemma 11.22,

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) \\ &= h(Y) - h(Z|X) \\ &= h(Y) - h(Z). \end{aligned}$$

5. Since Z is independent of X and Z is zero-mean,

$$\begin{aligned} EY^2 &= E(X+Z)^2 \\ &= EX^2 + EZ^2 + 2(EXZ) \\ &= EX^2 + EZ^2 + 2(EX)(EZ) \\ &\leq P + N. \end{aligned}$$

6. By Theorem 10.43,

$$h(Y) \leq \frac{1}{2}\log[2\pi e(P+N)]$$

with equality if $Y \sim \mathcal{N}(0, P+N)$. This is achieved with $X \sim \mathcal{N}(0, P)$.

7. Hence,

$$\begin{aligned} C &= \max_{F(x):EX^2 \leq P} h(Y) - h(Z) \\ &= \frac{1}{2}\log[2\pi e(P+N)] - \frac{1}{2}\log(2\pi e N) \\ &= \frac{1}{2}\log\left(1+\frac{P}{N}\right). \end{aligned}$$