Lemma 11.18 Let $(\mathbf{X}', \mathbf{Y}')$ be n i.i.d. copies of a pair of generic random variables (X', Y'), where X' and Y' are independent and have the same marginal distributions as X and Y, respectively. Then

$$\Pr\{(\mathbf{X}', \mathbf{Y}') \in \Psi^n_{[XY]\delta}\} \leq 2^{-n(I(X;Y)-\delta)}.$$

Proof

1. For any $(\mathbf{x}, \mathbf{y}) \in \Psi^n_{[XY]\delta}$, by definition,

$$\left| \frac{1}{n} \log \frac{f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} - I(X;Y) \right| \leq \delta.$$

2. Then
$$\frac{1}{n} \log \frac{f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} \geq I(X;Y) - \delta$$

which implies

$$\frac{f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} \geq 2^{n(I(X;Y)-\delta)}$$

or
$$f(\mathbf{y}|\mathbf{x}) \geq f(\mathbf{y}) 2^{n(I(X;Y)-\delta)}.$$

3. Consider

$$\begin{aligned} 1 &\geq \Pr\{(\mathbf{X}, \mathbf{Y}) \in \Psi^n_{[XY]\delta})\} \\ &= \int\int_{\Psi^n_{[XY]\delta}} f(\mathbf{y}|\mathbf{x}) dF(\mathbf{x})\, d\mathbf{y} \\ &\geq 2^{n(I(X;Y)-\delta)} \int\int_{\Psi^n_{[XY]\delta}} f(\mathbf{y}) dF(\mathbf{x}) d\mathbf{y} \\ &= 2^{n(I(X;Y)-\delta)} \Pr\{(\mathbf{X}', \mathbf{Y}') \in \Psi^n_{[XY]\delta}\}. \end{aligned}$$

4. Hence

$$\Pr\{(\mathbf{X}', \mathbf{Y}') \in \Psi^n_{[XY]\delta}\} \leq 2^{-n(I(X;Y)-\delta)}.$$