

**Lemma 11.18** Let  $(\mathbf{X}', \mathbf{Y}')$  be  $n$  i.i.d. copies of a pair of generic random variables  $(X', Y')$ , where  $X'$  and  $Y'$  are independent and have the same marginal distributions as  $X$  and  $Y$ , respectively. Then

$$\Pr\{(\mathbf{X}', \mathbf{Y}') \in \Psi_{[XY]\delta}^n\} \leq 2^{-n(I(X;Y)-\delta)}.$$

**Proof**

1. For any  $(\mathbf{x}, \mathbf{y}) \in \Psi_{[XY]\delta}^n$ , by definition,

$$\left| \frac{1}{n} \log \frac{f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} - I(X; Y) \right| \leq \delta.$$

2. Then

$$\frac{1}{n} \log \frac{f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} \geq I(X; Y) - \delta$$

which implies

$$\frac{f(\mathbf{y}|\mathbf{x})}{f(\mathbf{y})} \geq 2^{n(I(X;Y)-\delta)}$$

or

$$f(\mathbf{y}|\mathbf{x}) \geq f(\mathbf{y})2^{n(I(X;Y)-\delta)}.$$

3. Consider

$$\begin{aligned} 1 &\geq \Pr\{(\mathbf{X}, \mathbf{Y}) \in \Psi_{[XY]\delta}^n\} \\ &= \int \int_{\Psi_{[XY]\delta}^n} f(\mathbf{y}|\mathbf{x}) dF(\mathbf{x}) d\mathbf{y} \\ &\geq 2^{n(I(X;Y)-\delta)} \int \int_{\Psi_{[XY]\delta}^n} \textcolor{red}{f(\mathbf{y})} \textcolor{blue}{dF(\mathbf{x})} \textcolor{red}{d\mathbf{y}} \\ &= 2^{n(I(X;Y)-\delta)} \Pr\{(\mathbf{X}', \mathbf{Y}') \in \Psi_{[XY]\delta}^n\}. \end{aligned}$$

4. Hence

$$\Pr\{(\mathbf{X}', \mathbf{Y}') \in \Psi_{[XY]\delta}^n\} \leq 2^{-n(I(X;Y)-\delta)}.$$