

Theorem 11.8

$$C(P) = \sup_{F(x): E\kappa(X) \leq P} I(X; Y) \quad (1)$$

is non-decreasing, concave, and left-continuous.

Proof $C(P)$ is non-decreasing

Note that in (1), the supremum is taken over a larger set for a larger P .

 $C(P)$ is concave

1. Let $j = 1, 2$. For an input distribution $F_j(x)$, denote the corresponding input and output random variables by X_j and Y_j , respectively.

2. Then for any P_j , for all $\epsilon > 0$, there exists $F_j(x)$ such that

$$E\kappa(X_j) \leq P_j$$

and

$$I(X_j; Y_j) \geq C(P_j) - \epsilon.$$

3. For $0 \leq \lambda \leq 1$, let $\bar{\lambda} = 1 - \lambda$ and define the random variable

$$X^{(\lambda)} \sim \lambda F_1(x) + \bar{\lambda} F_2(x).$$

Then

$$E\kappa(X^{(\lambda)}) = \lambda E\kappa(X_1) + \bar{\lambda} E\kappa(X_2) \leq \lambda P_1 + \bar{\lambda} P_2.$$

4. By the concavity of mutual information with respect to the input distribution, we have

$$\begin{aligned} I(X^{(\lambda)}; Y^{(\lambda)}) &\geq \lambda I(X_1; Y_1) + \bar{\lambda} I(X_2; Y_2) \\ &\geq \lambda(C(P_1) - \epsilon) + \bar{\lambda}(C(P_2) - \epsilon) \\ &= \lambda C(P_1) + \bar{\lambda} C(P_2) - \epsilon. \end{aligned}$$

5. Then

$$C(\lambda P_1 + \bar{\lambda} P_2) \geq I(X^{(\lambda)}; Y^{(\lambda)}) \geq \lambda C(P_1) + \bar{\lambda} C(P_2) - \epsilon.$$

6. Letting $\epsilon \rightarrow 0$, we have

$$C(\lambda P_1 + \bar{\lambda} P_2) \geq \lambda C(P_1) + \bar{\lambda} C(P_2), \quad (2)$$

proving that $C(P)$ is concave.

 $C(P)$ is left-continuous

1. Let $P_1 < P_2$, so that $P_2 \geq \lambda P_1 + \bar{\lambda} P_2$. Since $C(P)$ is non-decreasing, we have

$$C(P_2) \geq C(\lambda P_1 + \bar{\lambda} P_2) \geq \lambda C(P_1) + \bar{\lambda} C(P_2).$$

2. Letting $\lambda \rightarrow 0$, we have

$$C(P_2) \geq \lim_{\lambda \rightarrow 0} C(\lambda P_1 + \bar{\lambda} P_2) \geq C(P_2),$$

which implies

$$\lim_{\lambda \rightarrow 0} C(\lambda P_1 + \bar{\lambda} P_2) = C(P_2).$$

3. Hence, we conclude that

$$\lim_{P \uparrow P_2} C(P) = C(P_2),$$

i.e., $C(P)$ is left-continuous.