

Theorem 10.20 (Multivariate Gaussian Distribution) Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, K)$. Then

$$h(\mathbf{X}) = \frac{1}{2} \log \left[(2\pi e)^n |K| \right].$$

Proof

1. Let K be diagonalizable as $Q\Lambda Q^\top$.
2. Write $\mathbf{X} = Q\mathbf{Y}$, where the random variables in \mathbf{Y} are uncorrelated with $\text{var } Y_i = \lambda_i$, the i th diagonal element of Λ (cf. Corollary 10.7).
3. Since \mathbf{X} is Gaussian, so is \mathbf{Y} .
4. Then the random variables in \mathbf{Y} are mutually independent because they are uncorrelated.

5. Now consider

$$\begin{aligned} h(\mathbf{X}) &= h(Q\mathbf{Y}) \\ &= h(\mathbf{Y}) + \log |\det(Q)| \\ &= h(\mathbf{Y}) + 0 \\ &= \sum_{i=1}^n h(Y_i) \\ &= \sum_{i=1}^n \frac{1}{2} \log(2\pi e \lambda_i) \\ &= \frac{1}{2} \log \left[(2\pi e)^n \prod_{i=1}^n \lambda_i \right] \\ &= \frac{1}{2} \log[(2\pi e)^n |\Lambda|] \\ &= \frac{1}{2} \log[(2\pi e)^n |K|]. \end{aligned}$$