

Proposition 10.24 If

3. By Fubini's theorem, the order of integration in $F_Y(y)$ can be exchanged, and so

$$F_Y(\textcolor{red}{y}) = \int_{-\infty}^{\textcolor{red}{y}} \left[\int_{-\infty}^{\infty} f_{Y|X}(\textcolor{blue}{v}|x) dF_X(x) \right] d\textcolor{blue}{v}.$$

then $f(y)$ exists and is given by

$$f_Y(\textcolor{red}{y}) = \int_{-\infty}^{\infty} f_{Y|X}(\textcolor{red}{y}|x) dF_X(x).$$

Proof

1.

$$\begin{aligned} F_Y(\textcolor{red}{y}) &= F_{XY}(\infty, \textcolor{red}{y}) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\textcolor{red}{y}} f_{Y|X}(v|x) dv dF_X(x). \end{aligned}$$

2. Since

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\textcolor{red}{y}} |f_{Y|X}(v|x)| dv dF_X(x) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\textcolor{red}{y}} f_{Y|X}(v|x) dv dF_X(x) \\ &= F_{XY}(\infty, \textcolor{red}{y}) \\ &= F_Y(\textcolor{red}{y}) \\ &\leq 1, \end{aligned}$$

$f_{Y|X}(v|x)$ is absolutely integrable.

4. Then

$$\begin{aligned} f_Y(\textcolor{red}{y}) &= \frac{d}{d\textcolor{red}{y}} F_Y(\textcolor{red}{y}) \\ &= \frac{d}{d\textcolor{red}{y}} \int_{-\infty}^{\textcolor{red}{y}} g(\textcolor{blue}{v}) d\textcolor{blue}{v} \\ &= g(\textcolor{red}{y}) \\ &= \int_{-\infty}^{\infty} f_{Y|X}(\textcolor{red}{y}|x) dF_X(x). \end{aligned}$$