Example 10.13 (Gaussian Distribution) Let $X \sim \mathcal{N}(0, \sigma^2)$ and let e be the base of the logarithm. Then

$$\begin{aligned}
h(X) &= -\int f(x) \ln f(x) dx \\
&= -\int f(x) \left(-\frac{x^2}{2\sigma^2} - \ln \sqrt{2\pi\sigma^2} \right) dx \\
&= \frac{1}{2\sigma^2} \int x^2 f(x) dx + \ln \sqrt{2\pi\sigma^2} \int f(x) dx \\
&= \frac{EX^2}{2\sigma^2} + \frac{1}{2} \ln(2\pi\sigma^2) \\
&= \frac{\mathrm{var} X + (EX)^2}{2\sigma^2} + \frac{1}{2} \ln(2\pi\sigma^2) \\
&= \frac{\sigma^2 + 0}{2\sigma^2} + \frac{1}{2} \ln(2\pi\sigma^2) \\
&= \frac{1}{2} + \frac{1}{2} \ln(2\pi\sigma^2) \\
&= \frac{1}{2} \ln e + \frac{1}{2} \ln(2\pi\sigma^2) \\
&= \frac{1}{2} \ln(2\pi e\sigma^2)
\end{aligned}$$

in nats. Changing the base of the logarithm to any chosen positive value, we obtain

$$h(X) = \frac{1}{2} \log(2\pi e\sigma^2).$$