

**Theorem 10.14 (Translation)**

$$h(X + c) = h(X).$$

**Proof**

1. Let  $Y = X + c$ .

2. Then

$$f_Y(y) = f_X(y - c)$$

and

$$\mathcal{S}_Y = \{x + c : x \in \mathcal{S}_X\}.$$

3. Letting  $x = y - c$ , we have

$$\begin{aligned} h(X) &= - \int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\ &= - \int_{\mathcal{S}_Y} f_X(y - c) \log f_X(y - c) dy \\ &= - \int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy \\ &= h(Y) \\ &= h(X + c). \end{aligned}$$

**Theorem 10.15 (Scaling)** For  $a \neq 0$ ,

$$h(aX) = h(X) + \log |a|.$$

**Proof**

1. Let  $Y = aX$ .

2. Then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$

and

$$\mathcal{S}_Y = \{ax : x \in \mathcal{S}_X\}.$$

3. Letting  $x = \frac{y}{a}$ , we have

$$\begin{aligned} h(X) &= - \int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\ &= - \int_{\mathcal{S}_Y} f_X\left(\frac{y}{a}\right) \log f_X\left(\frac{y}{a}\right) \frac{dy}{|a|} \\ &= - \int_{\mathcal{S}_Y} \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \left[ \log \left( \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \right) + \log |a| \right] dy \\ &= - \int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy - \log |a| \int_{\mathcal{S}_Y} f_Y(y) dy \\ &= h(Y) - \log |a| \\ &= h(aX) - \log |a|. \end{aligned}$$

4. Hence,

$$h(aX) = h(X) + \log |a|.$$