Theorem 10.14 (Translation)

$$h(X + c) = h(X).$$

Proof

1. Let $Y = X + c$.
2. Then

$$f_Y(y) = f_X(y - c)$$

and

$$\mathcal{S}_Y = \{x + c : x \in \mathcal{S}_X\}.$$

3. Letting $x = y - c$, we have

$$\begin{aligned}
h(X) &= -\int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\
&= -\int_{\mathcal{S}_Y} f_X(y - c) \log f_X(y - c) dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy \\
&= h(Y) \\
&= h(X + c).
\end{aligned}$$

Theorem 10.15 (Scaling) For $a \neq 0$,

$$h(aX) = h(X) + \log|a|.$$

Proof

1. Let $Y = aX$.
2. Then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$$

and

$$\mathcal{S}_Y = \{ax : x \in \mathcal{S}_X\}.$$

3. Letting $x = \frac{y}{a}$, we have

$$\begin{aligned}
&h(X) \\
&= -\int_{\mathcal{S}_X} f_X(x) \log f_X(x) dx \\
&= -\int_{\mathcal{S}_Y} f_X\left(\frac{y}{a}\right) \log f_X\left(\frac{y}{a}\right) \frac{dy}{|a|} \\
&= -\int_{\mathcal{S}_Y} \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \left[\log\left(\frac{1}{|a|} f_X\left(\frac{y}{a}\right)\right) + \log|a|\right] dy \\
&= -\int_{\mathcal{S}_Y} f_Y(y) \log f_Y(y) dy - \log|a| \int_{\mathcal{S}_Y} f_Y(y) dy \\
&= h(Y) - \log|a| \\
&= h(aX) - \log|a|.
\end{aligned}$$

4. Hence,

$$h(aX) = h(X) + \log|a|.$$