Chapter 9
The Blahut-Arimoto Algorithms

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Single-Letter Characterization

- For a DMC $p(y|x)$, the capacity

$$C = \max_{r(x)} I(X;Y),$$

where $r(x)$ is the input distribution, gives the maximum asymptotically achievable rate for reliable communication as the blocklength $n \to \infty$.

- This characterization of C, in the form of an optimization problem, is called a single-letter characterization because it involves only $p(y|x)$ but not n.

- Similarly, the rate-distortion function

$$R(D) = \min_{Q(\hat{x}|x): Ed(X,\hat{X}) \leq D} I(X;\hat{X})$$

for an i.i.d. information source $\{X_k\}$ is a single-letter characterization.

Numerical Methods

- When the alphabets are finite, C and $R(D)$ are given as solutions of finite-dimensional optimization problems.
- However, these quantities cannot be expressed in closed-forms except for very special cases.
- Even computing these quantities is not straightforward because the associated optimization problems are nonlinear.
- So we have to resort to numerical methods.
- The BA algorithms are iterative algorithms devised for this purpose.

9.1 Alternating Optimization

Consider the double supremum

$$\sup_{\mathbf{u}_1 \in A_1} \sup_{\mathbf{u}_2 \in A_2} f(\mathbf{u}_1, \mathbf{u}_2).$$

- A_i is a convex subset of $\Re^{n_i}$ for $i = 1, 2$.
- $f : A_1 \times A_2 \to \Re$ is bounded from above, such that
  - f is continuous and has continuous partial derivatives on $A_1 \times A_2$;
  - For all $\mathbf{u}_2 \in A_2$, there exists a unique $c_1(\mathbf{u}_2) \in A_1$ such that

$$f(c_1(\mathbf{u}_2), \mathbf{u}_2) = \max_{\mathbf{u}_1' \in A_1} f(\mathbf{u}_1', \mathbf{u}_2),$$

and for all $\mathbf{u}_1 \in A_1$, there exists a unique $c_2(\mathbf{u}_1) \in A_2$ such that

$$f(\mathbf{u}_1, c_2(\mathbf{u}_1)) = \max_{\mathbf{u}_2' \in A_2} f(\mathbf{u}_1, \mathbf{u}_2').$$

- Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and $A = A_1 \times A_2$. Then the double supremum can be written as
$$\sup_{\mathbf{u} \in A} f(\mathbf{u}).$$

- In other words, the supremum of f is taken over a subset of $\Re^{n_1+n_2}$ which is equal to the Cartesian product of two convex subsets of $\Re^{n_1}$ and $\Re^{n_2}$, respectively.

- Let f^* denote this supremum.

An Alternating Optimization Algorithm for Computing f*

- Let $\mathbf{u}^{(k)} = (\mathbf{u}_1^{(k)}, \mathbf{u}_2^{(k)})$ for $k \geq 0$, defined as follows.
- Let $\mathbf{u}_1^{(0)}$ be an arbitrarily chosen vector in A_1, and let $\mathbf{u}_2^{(0)} = c_2(\mathbf{u}_1^{(0)})$.
- For $k \geq 1$, $\mathbf{u}^{(k)}$ is defined by

$$\mathbf{u}_1^{(k)} = c_1(\mathbf{u}_2^{(k-1)})$$

and

$$\mathbf{u}_2^{(k)} = c_2(\mathbf{u}_1^{(k)}).$$

- Let

$$f^{(k)} = f(\mathbf{u}^{(k)}).$$

- Then

$$f^{(k)} \geq f^{(k-1)}.$$

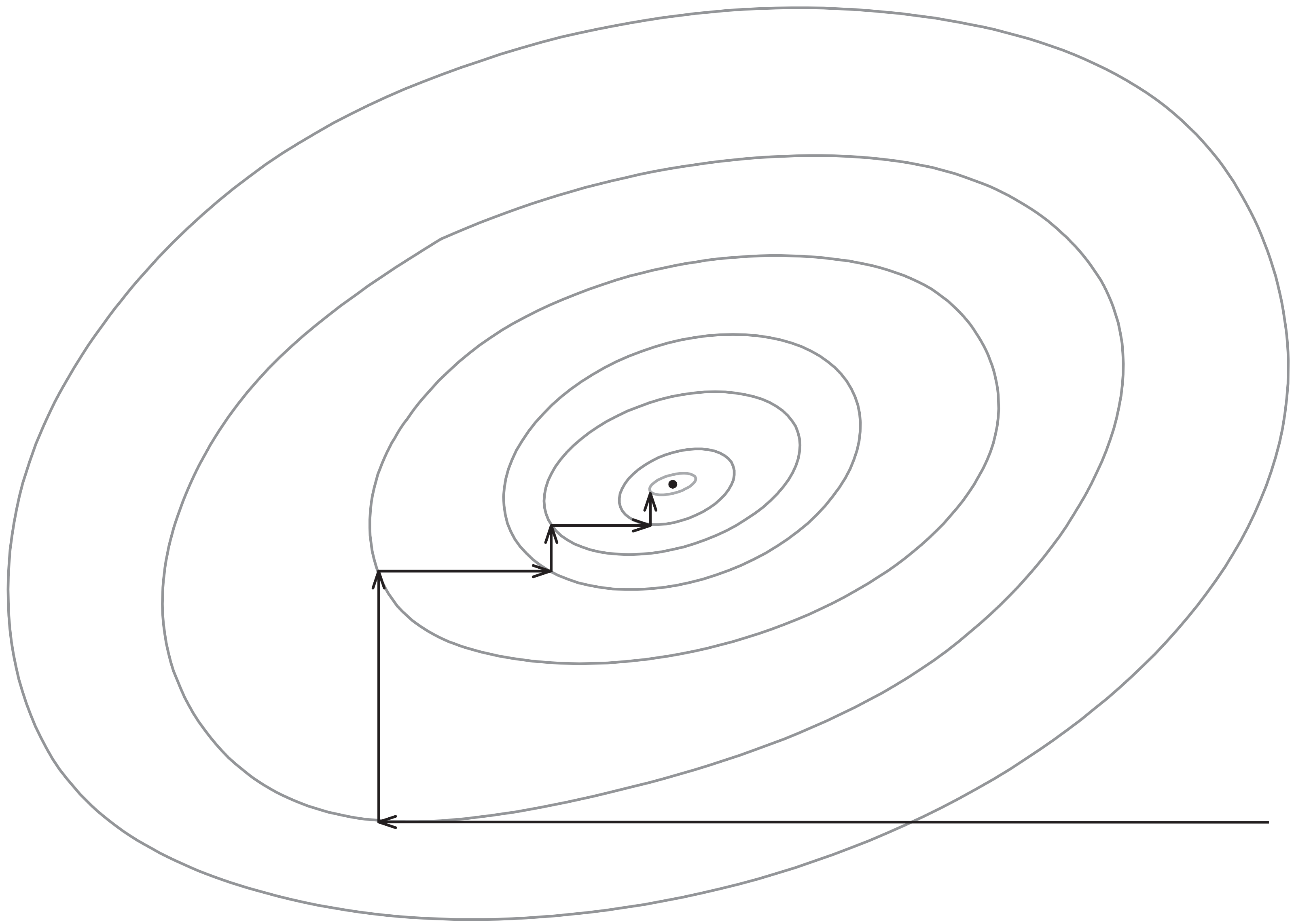
- Since the sequence $f^{(k)}$ is non-decreasing, it must converge because f is bounded from above.

- We will show that $f^{(k)} \to f^*$ if f is concave.

- Replacing f by $-f$, the double supremum becomes the double infimum

$$\inf_{\mathbf{u}_1 \in A_1} \inf_{\mathbf{u}_2 \in A_2} f(\mathbf{u}_1, \mathbf{u}_2).$$

- The same alternating optimization algorithm can be applied to compute this infimum.



9.2 The Algorithms

- The alternating optimization algorithm is specialized for computing C and $R(D)$.

9.2.1 Channel Capacity

Lemma 9.1 Let $r(x)p(y|x)$ be a given joint distribution on $\mathcal{X} \times \mathcal{Y}$ such that $\mathbf{r} > 0$, and let $\mathbf{q}$ be a transition matrix from $\mathcal{Y}$ to $\mathcal{X}$. Then

$$\max_{\mathbf{q}} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)} = \sum_x \sum_y r(x)p(y|x) \log \frac{q^*(x|y)}{r(x)},$$

where the maximization is taken over all $\mathbf{q}$ such that

$$q(x|y) = 0 \quad \text{if and only if} \quad p(y|x) = 0, \tag{1}$$

and

$$q^*(x|y) = \frac{r(x)p(y|x)}{\sum_{x'} r(x')p(y|x')}, \tag{2}$$

i.e., the maximizing $\mathbf{q}$ is the one which corresponds to the input distribution $\mathbf{r}$ and the transition matrix $p(y|x)$.

Proof

1. In (2), let
$$w(y) = \sum_{x'} r(x')p(y|x').$$
2. Assume w.l.o.g. that for all $y \in \mathcal{Y}$, $p(y|x) > 0$ for some $x \in \mathcal{X}$.
3. Since $\mathbf{r} > 0$, $w(y) > 0$ for all y, and hence $q^*(x|y)$ is well-defined.
4. Rearranging (2), we have
$$r(x)p(y|x) = w(y)q^*(x|y).$$

5. Consider

$$
\begin{aligned}
&\sum_x \sum_y r(x)p(y|x) \log \frac{q^*(x|y)}{r(x)} - \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)} \\
&= \sum_x \sum_y r(x)p(y|x) \log \frac{q^*(x|y)}{q(x|y)} \\
&= \sum_y \sum_x w(y)q^*(x|y) \log \frac{q^*(x|y)}{q(x|y)} \\
&= \sum_y w(y) \sum_x q^*(x|y) \log \frac{q^*(x|y)}{q(x|y)} \\
&= \sum_y w(y) D(q^*(x|y) \| q(x|y)) \\
&\geq 0.
\end{aligned}
$$

6. The proof is completed by noting in (2) that $\mathbf{q}^*$ satisfies (1) because $\mathbf{r} > 0$.

Theorem 9.2 For a discrete memoryless channel $p(y|x)$,

$$C = \sup_{\mathbf{r}>0} \max_{\mathbf{q}} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)},$$

where the maximization is taken over all $\mathbf{q}$ that satisfies (1) in Lemma 9.1.

Proof

- Write $I(X;Y)$ as $I(\mathbf{r},\mathbf{p})$ where $\mathbf{r}$ is the input distribution and $\mathbf{p}$ denotes the transition matrix of the generic channel $p(y|x)$. Then

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}).$$

- By Lemma 9.1, we need to prove that

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}) = \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}).$$

- Let $\mathbf{r}^*$ achieves C.

- If $\mathbf{r}^* > 0$, then

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}) = \max_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}) = \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}).$$

- Next, consider $\mathbf{r}^* \geq 0$. Since $I(\mathbf{r},\mathbf{p})$ is continuous in $\mathbf{r}$, for any $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\|\mathbf{r} - \mathbf{r}^*\| < \delta,$$

then

$$C - I(\mathbf{r},\mathbf{p}) < \epsilon,$$

- In particular, there exists $\tilde{\mathbf{r}} > 0$ such that $\|\tilde{\mathbf{r}} - \mathbf{r}^*\| < \delta$.

- Then

$$C = \max_{\mathbf{r}\geq 0} I(\mathbf{r},\mathbf{p}) \geq \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}) \geq I(\tilde{\mathbf{r}},\mathbf{p}) > C - \epsilon.$$

- Let $\epsilon \to 0$ to conclude that

$$C = \sup_{\mathbf{r}>0} I(\mathbf{r},\mathbf{p}).$$

Recall the double supremum in Section 9.1:

$$\sup_{\mathbf{u}_1 \in A_1} \sup_{\mathbf{u}_2 \in A_2} f(\mathbf{u}_1, \mathbf{u}_2).$$

- A_i is a convex subset of $\Re^{n_i}$ for $i = 1, 2$.
- $f : A_1 \times A_2 \to \Re$ is bounded from above, such that
  - f is continuous and has continuous partial derivatives on $A_1 \times A_2$;
  - For all $\mathbf{u}_2 \in A_2$, there exists a unique $c_1(\mathbf{u}_2) \in A_1$ such that

    $$f(c_1(\mathbf{u}_2), \mathbf{u}_2) = \max_{\mathbf{u}_1' \in A_1} f(\mathbf{u}_1', \mathbf{u}_2),$$

    and for all $\mathbf{u}_1 \in A_1$, there exists a unique $c_2(\mathbf{u}_1) \in A_2$ such that

    $$f(\mathbf{u}_1, c_2(\mathbf{u}_1)) = \max_{\mathbf{u}_2' \in A_2} f(\mathbf{u}_1, \mathbf{u}_2').$$

Cast the computation of C into this optimization problem:

- Let
$$f(\mathbf{r}, \mathbf{q}) = \sum_x \sum_y r(x) p(y|x) \log \frac{q(x|y)}{r(x)},$$
where $\mathbf{u}_1 \leftarrow \mathbf{r}$ and $\mathbf{u}_2 \leftarrow \mathbf{q}$.

- Let
$$A_1 = \{(r(x), x \in \mathcal{X}) : \ r(x) > 0 \text{ and } \textstyle\sum_x r(x) = 1\} \subset \Re^{|\mathcal{X}|}$$
and
$$\begin{aligned} A_2 &= \{(q(x|y), (x,y) \in \mathcal{X} \times \mathcal{Y}) : \ q(x|y) > 0 \text{ iff } p(y|x) > 0, \\ &\quad \text{and } \textstyle\sum_x q(x|y) = 1 \text{ for all } y \in \mathcal{Y}\} \\ &\subset \Re^{|\mathcal{X}||\mathcal{Y}|}. \end{aligned}$$

Remarks

- Both A_1 and A_2 are convex.
- f is bounded from above.
- In $f(\mathbf{r}, \mathbf{q})$, the double summation by convention is over all x such that $r(x) > 0$ and all y such that $p(y|x) > 0$.
- Since $q(x|y) > 0$ whenever $p(y|x) > 0$, all the probabilities involved in the double summation are positive.
- Therefore, f is continuous and has continuous partial derivatives on $A = A_1 \times A_2$.

- The double supremum now becomes

$$\sup_{\mathbf{r}\in A_1} \sup_{\mathbf{q}\in A_2} \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)} = \sup_{\mathbf{r}\in A_1} \sup_{\mathbf{q}\in A_2} f(\mathbf{r}, \mathbf{q}),$$

where the supremum over all $\mathbf{q} \in A_2$ is in fact a maximum, and

$$f^* = \sup_{\mathbf{r}\in A_1} \sup_{\mathbf{q}\in A_2} f(\mathbf{r}, \mathbf{q}) = C.$$

Algorithm Details

- By Lemma 9.1, for any given $\mathbf{r} \in A_1$, there exists a unique $\mathbf{q} \in A_2$ that maximizes f.

- By Lagrange multiplies, it can be shown that for a given $\mathbf{q} \in A_2$, the input distribution $\mathbf{r}$ that maximizes f is given by

$$r(x) = \frac{\prod_y q(x|y)^{p(y|x)}}{\sum_{x'} \prod_y q(x'|y)^{p(y|x')}},$$

  where the product is over all y such that $p(y|x) > 0$, and $q(x|y) > 0$ for all such y. This implies $\mathbf{r} > 0$ and hence $\mathbf{r} \in A_1$.

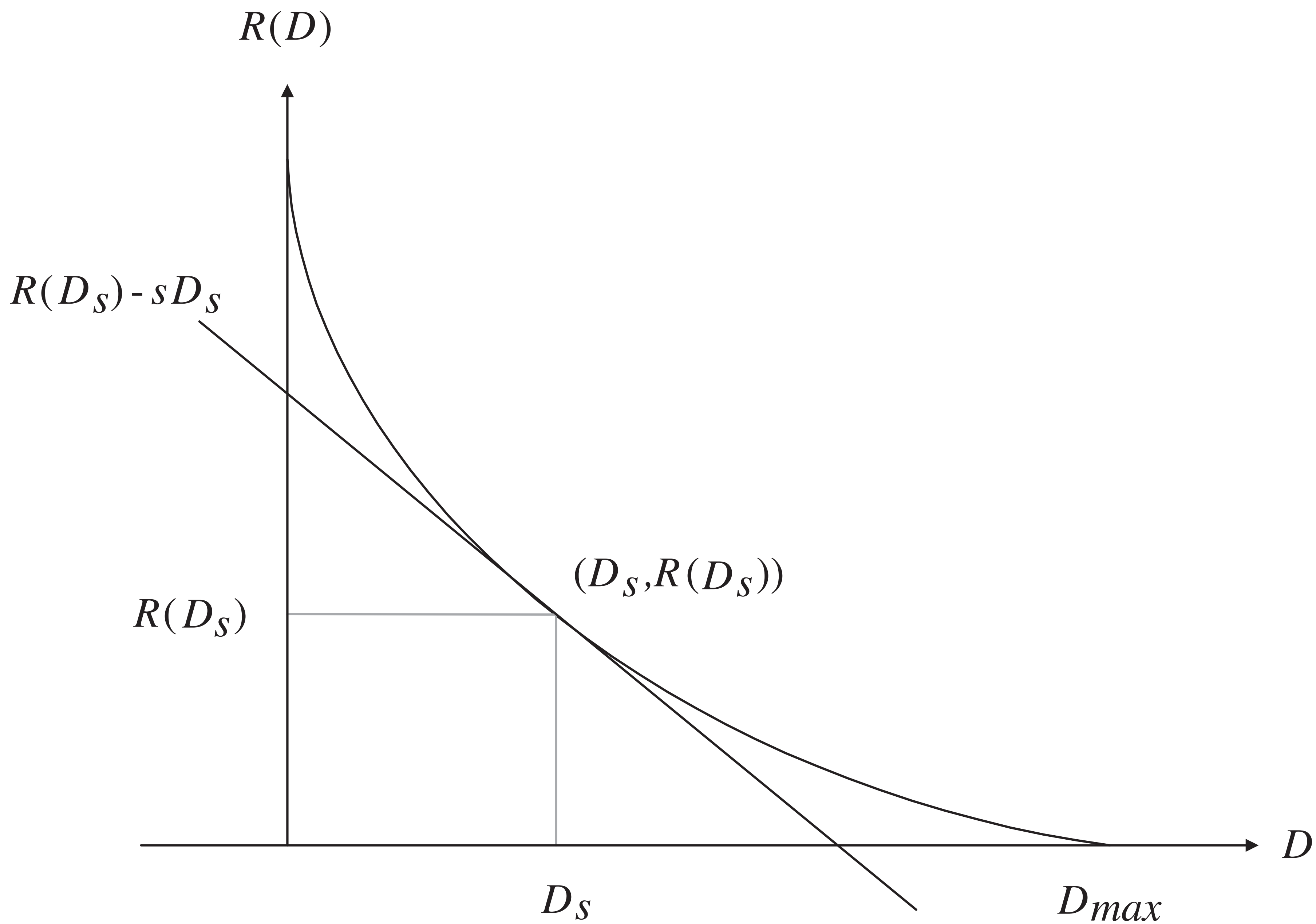
- Let $\mathbf{r}^{(0)}$ an arbitrarily chosen strictly positive input distribution in A_1. Then $\mathbf{q}^{(0)} \in A_2$ can be determined accordingly. This forms $(\mathbf{r}^{(0)}, \mathbf{q}^{(0)})$.

- Compute $(\mathbf{r}^{(k)}, \mathbf{q}^{(k)}), k \geq 1$ iteratively.

- It will be shown in Section 9.3 that $f^{(k)} = f(\mathbf{r}^{(k)}, \mathbf{q}^{(k)}) \to f^* = C$.

9.2.2 The Rate-Distortion Function

- Assume $R(0) > 0$, so that $R(D)$ is strictly decreasing for $0 \leq D \leq D_{max}$.
- Since $R(D)$ is convex, for any $s \leq 0$, there exists a point on the $R(D)$ curve for $0 \leq D \leq D_{max}$ such that the slope of a tangent to the $R(D)$ curve at that point is equal to s.
- Denote such a point on the $R(D)$ curve by $(D_s, R(D_s))$, which is not necessarily unique.
- Then this tangent intersects with the ordinate at $R(D_s) - sD_s$.



$R(D)$
$R(D_s) - sD_s$
$(D_s, R(D_s))$
$R(D_s)$
D
D_s
D_{max}

- Write $I(X;\hat{X})$ and $Ed(X,\hat{X})$ as $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$, respectively, where $\mathbf{p}$ is the distribution for X and $\mathbf{Q}$ is the transition matrix from $\mathcal{X}$ to $\hat{\mathcal{X}}$ defining $\hat{X}$.

- For any $\mathbf{Q}$, $(D(\mathbf{p},\mathbf{Q}), I(\mathbf{p},\mathbf{Q}))$ is a point in the rate-distortion region, and the line with slope s passing through $(D(\mathbf{p},\mathbf{Q}), I(\mathbf{p},\mathbf{Q}))$ intersects the ordinate at $I(\mathbf{p},\mathbf{Q}) - sD(\mathbf{p},\mathbf{Q})$.

- Then

$$R(D_s) - sD_s = \min_{\mathbf{Q}}[I(\mathbf{p},\mathbf{Q}) - sD(\mathbf{p},\mathbf{Q})]. \tag{1}$$

- By varying over all $s \leq 0$, we can then trace out the whole $R(D)$ curve.

Lemma 9.3 Let $p(x)Q(\hat{x}|x)$ be a given joint distribution on $\mathcal{X} \times \hat{\mathcal{X}}$ such that $\mathbf{Q} > 0$, and let $\mathbf{t}$ be any distribution on $\hat{\mathcal{X}}$ such that $\mathbf{t} > 0$. Then

$$\min_{\mathbf{t}>0} \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} = \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t^*(\hat{x})}, \quad (2)$$

where

$$t^*(\hat{x}) = \sum_x p(x)Q(\hat{x}|x),$$

i.e., the minimizing $\mathbf{t}$ is the one which corresponds to the input distribution $\mathbf{p}$ and the transition matrix $\mathbf{Q}$.

Remarks:

- $\mathbf{t}^* > 0$ because $\mathbf{Q} > 0$.
- The right-hand side of (2) is equal to $I(\mathbf{p}, \mathbf{Q})$.

- Since $I(\mathbf{p},\mathbf{Q})$ and $D(\mathbf{p},\mathbf{Q})$ are continuous in $\mathbf{Q}$, the minimum over all $\mathbf{Q}$ in (1) can be replaced by the infimum over all $\mathbf{Q} > 0$.
- Note that

$$D(\mathbf{p},\mathbf{Q}) = \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x)d(x,\hat{x}).$$

- Then

$$\begin{aligned}
& R(D_s) - sD_s \\
&\quad = \min_{\mathbf{Q}} [I(\mathbf{p},\mathbf{Q}) - sD(\mathbf{p},\mathbf{Q})] \\
&\quad = \inf_{\mathbf{Q}>0} \left[\min_{\mathbf{t}>0} \sum_{x,\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_{x,\hat{x}} p(x)Q(\hat{x}|x)d(x,\hat{x}) \right] \\
&\quad = \inf_{\mathbf{Q}>0} \min_{\mathbf{t}>0} \left[\sum_{x,\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_{x,\hat{x}} p(x)Q(\hat{x}|x)d(x,\hat{x}) \right].
\end{aligned}$$

Recall the double infimum in Section 9.1:

$$\inf_{\mathbf{u}_1 \in A_1} \inf_{\mathbf{u}_2 \in A_2} f(\mathbf{u}_1, \mathbf{u}_2).$$

- A_i is a convex subset of $\Re^{n_i}$ for $i = 1, 2$.
- $f : A_1 \times A_2 \to \Re$ is bounded from below, such that
  - f is continuous and has continuous partial derivatives on $A_1 \times A_2$;
  - For all $\mathbf{u}_2 \in A_2$, there exists a unique $c_1(\mathbf{u}_2) \in A_1$ such that

    $$f(c_1(\mathbf{u}_2), \mathbf{u}_2) = \min_{\mathbf{u}_1' \in A_1} f(\mathbf{u}_1', \mathbf{u}_2),$$

    and for all $\mathbf{u}_1 \in A_1$, there exists a unique $c_2(\mathbf{u}_1) \in A_2$ such that

    $$f(\mathbf{u}_1, c_2(\mathbf{u}_1)) = \min_{\mathbf{u}_2' \in A_2} f(\mathbf{u}_1, \mathbf{u}_2').$$

Cast the computation of $R(D_s) - sD_s$ into this optimization problem:

- Let

$$f(\mathbf{Q},\mathbf{t}) = \sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_x \sum_{\hat{x}} p(x) Q(\hat{x}|x) d(x,\hat{x}),$$

where $\mathbf{u}_1 \leftarrow \mathbf{Q}$ and $\mathbf{u}_2 \leftarrow \mathbf{t}$.

- Let

$$\begin{aligned} A_1 &= \left\{ (Q(\hat{x}|x), (x,\hat{x}) \in \mathcal{X} \times \hat{\mathcal{X}}) : Q(\hat{x}|x) > 0, \ \sum_{\hat{x}} Q(\hat{x}|x) = 1 \text{ for all } x \in \mathcal{X} \right\} \\ &\subset \Re^{|\mathcal{X}||\hat{\mathcal{X}}|} \end{aligned}$$

and

$$A_2 = \{(t(\hat{x}), \hat{x} \in \hat{\mathcal{X}}) : t(\hat{x}) > 0.$$

Remarks

- Both A_1 and A_2 are convex.
- f is bounded from below.
- In $f(\mathbf{Q}, \mathbf{t})$, the double summation by convention is over all x such that $p(x) > 0$ and all $\hat{x}$ such that $Q(\hat{x}|x) > 0$.
- Therefore, f is continuous and has continuous partial derivatives on $A = A_1 \times A_2$.
- The double infimum now becomes

$$\begin{aligned} &\inf_{\mathbf{Q}\in A_1} \inf_{\mathbf{t}\in A_2} \left[\sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x) \log \frac{Q(\hat{x}|x)}{t(\hat{x})} - s \sum_x \sum_{\hat{x}} p(x)Q(\hat{x}|x)d(x,\hat{x}) \right] \\ &= \inf_{\mathbf{Q}\in A_1} \inf_{\mathbf{t}\in A_2} f(\mathbf{Q}, \mathbf{t}), \end{aligned}$$

where the infimum over all $\mathbf{t} \in A_2$ is in fact a minimum, and

$$f^* = \inf_{\mathbf{Q}\in A_1} \inf_{\mathbf{t}\in A_2} f(\mathbf{Q}, \mathbf{t}) = R(D_s) - sD_s.$$

Algorithm Details

- By Lemma 9.3, for any given $\mathbf{Q} \in A_1$, there exists a unique $\mathbf{t} \in A_2$ that minimizes f.

- By Lagrange multipliers, it can be shown that for a given $\mathbf{t} \in A_2$, the transition matrix $\mathbf{Q}$ that minimizers f is given by

$$Q(\hat{x}|x) = \frac{t(\hat{x}) e^{sd(x,\hat{x})}}{\sum_{\hat{x}'} t(\hat{x}') e^{sd(x,\hat{x}')}} > 0,$$

  and so $\mathbf{Q} \in A_1$.

- Let $\mathbf{Q}^{(0)}$ an arbitrarily chosen strictly positive transition matrix in A_1. Then $\mathbf{t}^{(0)} \in A_2$ can be determined accordingly. This forms $(\mathbf{Q}^{(0)}, \mathbf{t}^{(0)})$.

- Compute $(\mathbf{Q}^{(k)}, \mathbf{t}^{(k)}), k \geq 1$ iteratively.

- It will be shown in Section 9.3 that $f^{(k)} = f(\mathbf{Q}^{(k)}, \mathbf{t}^{(k)}) \to f^* = R(D_s) - sD_s$.

9.3 Convergence

- Consider the double supremum optimization problem in Section 9.1.
- We first prove in general that if f is concave, then $f^{(k)} \to f^*$.
- We then apply this sufficient condition to prove the convergence of the BA algorithm for computing C.
- The convergence of the BA algorithm for computing $R(D_s) - sD_s$ can be proved likewise.

9.3.1 A Sufficient Condition

- In the alternating optimization algorithm, we have

$$\mathbf{u}^{(k+1)} = (\mathbf{u}_1^{(k+1)}, \mathbf{u}_2^{(k+1)}) = (c_1(\mathbf{u}_2^{(k)}), c_2(c_1(\mathbf{u}_2^{(k)})))$$

  for $k \geq 0$.

- Define

$$\Delta f(\mathbf{u}) = f(c_1(\mathbf{u}_2), c_2(c_1(\mathbf{u}_2))) - f(\mathbf{u}_1, \mathbf{u}_2).$$

- Then

$$\begin{aligned} f^{(k+1)} - f^{(k)} &= f(\mathbf{u}^{(k+1)}) - f(\mathbf{u}^{(k)}) \\ &= f(c_1(\mathbf{u}_2^{(k)}), c_2(c_1(\mathbf{u}_2^{(k)}))) - f(\mathbf{u}_1^{(k)}, \mathbf{u}_2^{(k)}) \\ &= \Delta f(\mathbf{u}^{(k)}). \end{aligned}$$

We first prove that if f is concave, then the algorithm cannot be trapped at $\mathbf{u}$ if $f(\mathbf{u}) < f^*$.

Lemma 9.4 Let f be concave. If $f^{(k)} < f^*$, then $f^{(k+1)} > f^{(k)}$.

Proof First, prove that if $\Delta f(\mathbf{u}) = 0$, then $\mathbf{u}_1 = c_1(\mathbf{u}_2)$ and $\mathbf{u}_2 = c_2(\mathbf{u}_1)$.

1. It suffices to prove that $\Delta f(\mathbf{u}) > 0$ for any $\mathbf{u} \in A$ such that $f(\mathbf{u}) < f^*$. Then if $f^{(k)} = f(\mathbf{u}^{(k)}) < f^*$, we have

$$f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) > 0,$$

   proving the lemma.

2. Consider

$$f(c_1(\mathbf{u}_2), c_2(c_1(\mathbf{u}_2))) \overset{a)}{\geq} f(c_1(\mathbf{u}_2), \mathbf{u}_2) \overset{b)}{\geq} f(\mathbf{u}_1, \mathbf{u}_2).$$

   If $\Delta f(\mathbf{u}) = 0$, then both a) and b) are tight.

3. Due to the uniqueness of $c_2(\cdot)$ and $c_1(\cdot)$,

$$\begin{aligned} \text{b) is tight} \quad &\Rightarrow \quad \mathbf{u}_1 = c_1(\mathbf{u}_2) \\ \text{a) is tight} \quad &\Rightarrow \quad \mathbf{u}_2 = c_2(c_1(\mathbf{u}_2)) = c_2(\mathbf{u}_1). \end{aligned}$$

4. This also implies that if $f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) = 0$, then $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)}$.

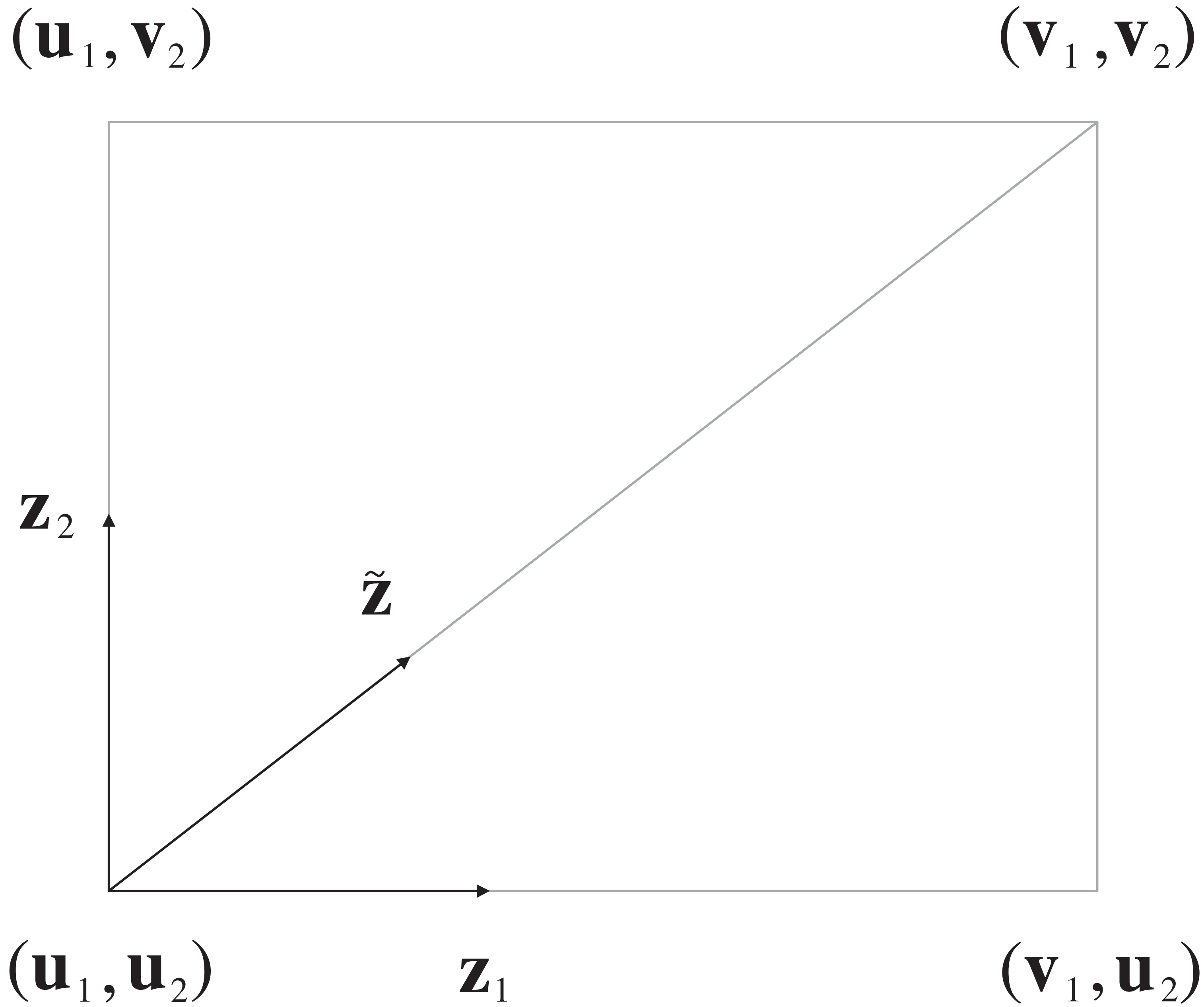
Second, consider any $\mathbf{u} \in A$ such that $f(\mathbf{u}) < f^*$. Prove by contradiction that $\Delta f(\mathbf{u}) > 0$.

1. Assume that $\Delta f(\mathbf{u}) = 0$. Then $\mathbf{u}_1 = c_1(\mathbf{u}_2)$ and $\mathbf{u}_2 = c_2(\mathbf{u}_1)$, i.e., $\mathbf{u}_1$ maximizes f for a fixed $\mathbf{u}_2$, and $\mathbf{u}_2$ maximizes f for a fixed $\mathbf{u}_1$.

2. Since $f(\mathbf{u}) < f^*$, there exists $\mathbf{v} \in A$ such that $f(\mathbf{u}) < f(\mathbf{v})$.

3. Let

$$\begin{aligned} \tilde{\mathbf{z}} \quad & \text{unit vector in the direction of } \mathbf{v} - \mathbf{u} \\ \mathbf{z}_1 \quad & \text{unit vector in the direction of } (\mathbf{v}_1 - \mathbf{u}_1, 0) \\ \mathbf{z}_2 \quad & \text{unit vector in the direction of } (0, \mathbf{v}_2 - \mathbf{u}_2). \end{aligned}$$



$(\mathbf{u}_1, \mathbf{v}_2)$
$(\mathbf{v}_1, \mathbf{v}_2)$
$\mathbf{z}_2$
$\tilde{\mathbf{z}}$
$(\mathbf{u}_1, \mathbf{u}_2)$
$\mathbf{z}_1$
$(\mathbf{v}_1, \mathbf{u}_2)$

4. Then $\tilde{\mathbf{z}} = \alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2$, where

$$\alpha_i = \frac{\|\mathbf{v}_i - \mathbf{u}_i\|}{\|\mathbf{v} - \mathbf{u}\|}, \quad i = 1, 2.$$

5. Since f is continuous and has continuous partial derivatives, the directional derivative of f at $\mathbf{u}$ in the direction of $\mathbf{z}_1$ is given by $\bigtriangledown f \cdot \mathbf{z}_1$.

6. f attains its maximum value at $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ when $\mathbf{u}_2$ is fixed.

7. In particular, f attains its maximum value at $\mathbf{u}$ along the line passing through $(\mathbf{u}_1, \mathbf{u}_2)$ and $(\mathbf{v}_1, \mathbf{u}_2)$.

8. It follows from the concavity of f along the line passing through $(\mathbf{u}_1, \mathbf{u}_2)$ and $(\mathbf{v}_1, \mathbf{u}_2)$ that $\bigtriangledown f \cdot \mathbf{z}_1 = 0$. Similarly, $\bigtriangledown f \cdot \mathbf{z}_2 = 0$.

9. Then $\bigtriangledown f \cdot \tilde{\mathbf{z}} = \alpha_1(\bigtriangledown f \cdot \mathbf{z}_1) + \alpha_2(\bigtriangledown f \cdot \mathbf{z}_2) = 0$.

10. Since f is concave along the line passing through $\mathbf{u}$ and $\mathbf{v}$, this implies $f(\mathbf{u}) \geq f(\mathbf{v})$, a contradiction.

11. Hence, $\Delta f(\mathbf{u}) > 0$.

Although $\Delta f(\mathbf{u}) > 0$ as long as $f(\mathbf{u}) < f^*$, $f^{(k)}$ does not necessarily converge to f^* because the increment in $f^{(k)}$ in each step may be arbitrarily small.

Theorem 9.5 If f is concave, then $f^{(k)} \to f^*$.

Proof

1. $f^{(k)}$ necessarily converges, say to f', because $f^{(k)}$ is nondecreasing and bounded from above.

2. Hence, for any $\epsilon > 0$ and all sufficiently large k,

$$f' - \epsilon \leq f^{(k)} \leq f'. \tag{1}$$

3. Let

$$\gamma = \min_{\mathbf{u} \in A'} \Delta f(\mathbf{u}),$$

   where $A' = \{\mathbf{u} \in A : f' - \epsilon \leq f(\mathbf{u}) \leq f'\}$.

4. Since f has continuous partial derivatives, $\Delta f(\mathbf{u})$ is a continuous function of $\mathbf{u}$.

5. A' is compact because it is the inverse image of a closed interval under a continuous function and A is bounded. Therefore γ exists.

6. Since f is concave, by Lemma 9.4, $\Delta f(\mathbf{u}) > 0$ for all $\mathbf{u} \in A'$ and hence $\gamma > 0$.

7. Since $f^{(k)} = f(\mathbf{u}^{(k)})$ satisfies (1), $\mathbf{u}^{(k)} \in A'$.

8. Thus for all sufficiently large k,

$$f^{(k+1)} - f^{(k)} = \Delta f(\mathbf{u}^{(k)}) \geq \gamma.$$

9. No matter how smaller γ is, $f^{(k)}$ will eventually be greater than f', which is a contradiction to $f^{(k)} \to f'$.

10. Hence, $f^{(k)} \to f^*$.

9.3 Convergence to the Channel Capacity

We only need to verify that

$$f(\mathbf{r}, \mathbf{q}) = \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}$$

is concave.

1. Consider $(\mathbf{r}_1, \mathbf{q}_1)$ and $(\mathbf{r}_2, \mathbf{q}_2)$ in A.
2. An application of the log-sum inequality gives

$$(\lambda r_1(x)+\bar{\lambda}r_2(x)) \log \frac{\lambda r_1(x)+\bar{\lambda}r_2(x)}{\lambda q_1(x|y)+\bar{\lambda}q_2(x|y)} \leq \lambda r_1(x) \log \frac{r_1(x)}{q_1(x|y)} + \bar{\lambda}r_2(x) \log \frac{r_2(x)}{q_2(x|y)}.$$

3. Taking reciprocal in the logarithms yields

$$(\lambda r_1(x)+\bar{\lambda}r_2(x)) \log \frac{\lambda q_1(x|y)+\bar{\lambda}q_2(x|y)}{\lambda r_1(x)+\bar{\lambda}r_2(x)} \geq \lambda r_1(x) \log \frac{q_1(x|y)}{r_1(x)} + \bar{\lambda}r_2(x) \log \frac{q_2(x|y)}{r_2(x)}.$$

4. Upon multiplying by $p(y|x)$ and summing over all x and y, we obtain

$$f(\lambda\mathbf{r}_1 + \bar{\lambda}\mathbf{r}_2, \lambda\mathbf{q}_1 + \bar{\lambda}\mathbf{q}_2) \geq \lambda f(\mathbf{r}_1, \mathbf{q}_1) + \bar{\lambda} f(\mathbf{r}_2, \mathbf{q}_2).$$

5. Hence, $f^{(k)} \to C$.