Index Coding

- Optimality of Fractional Coloring
& Minimal Necessity of Non-Shannon Inequalities

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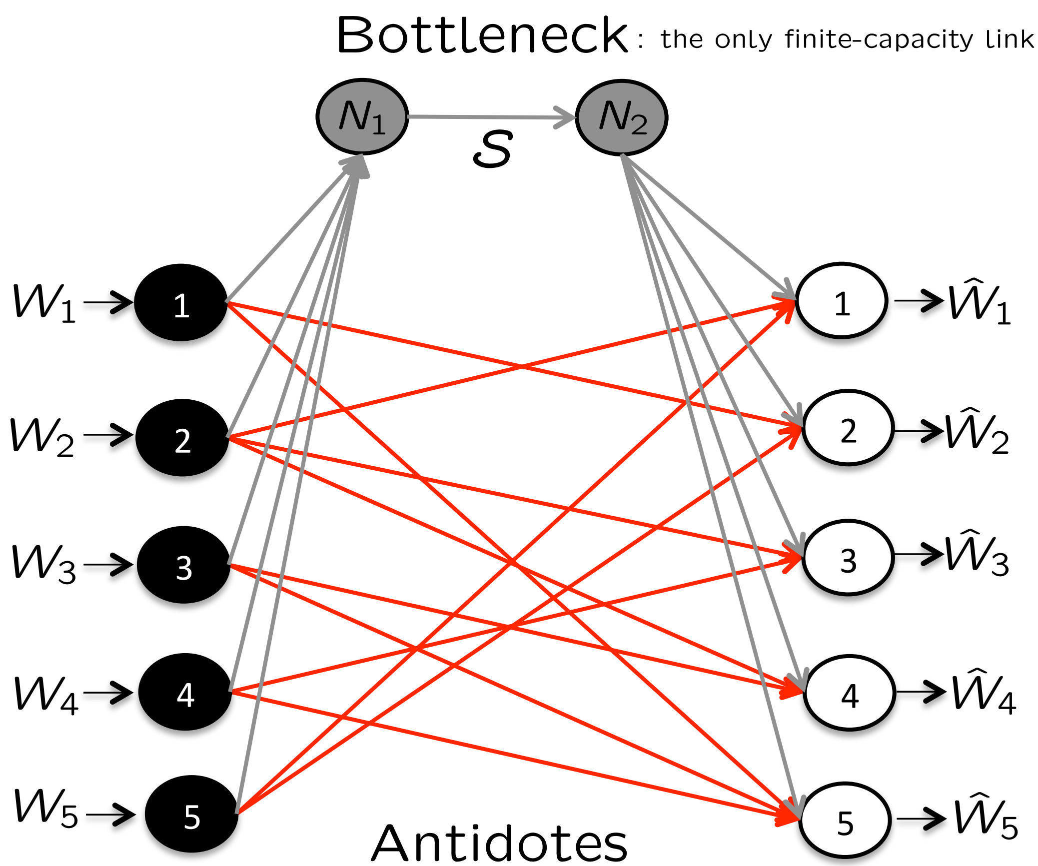
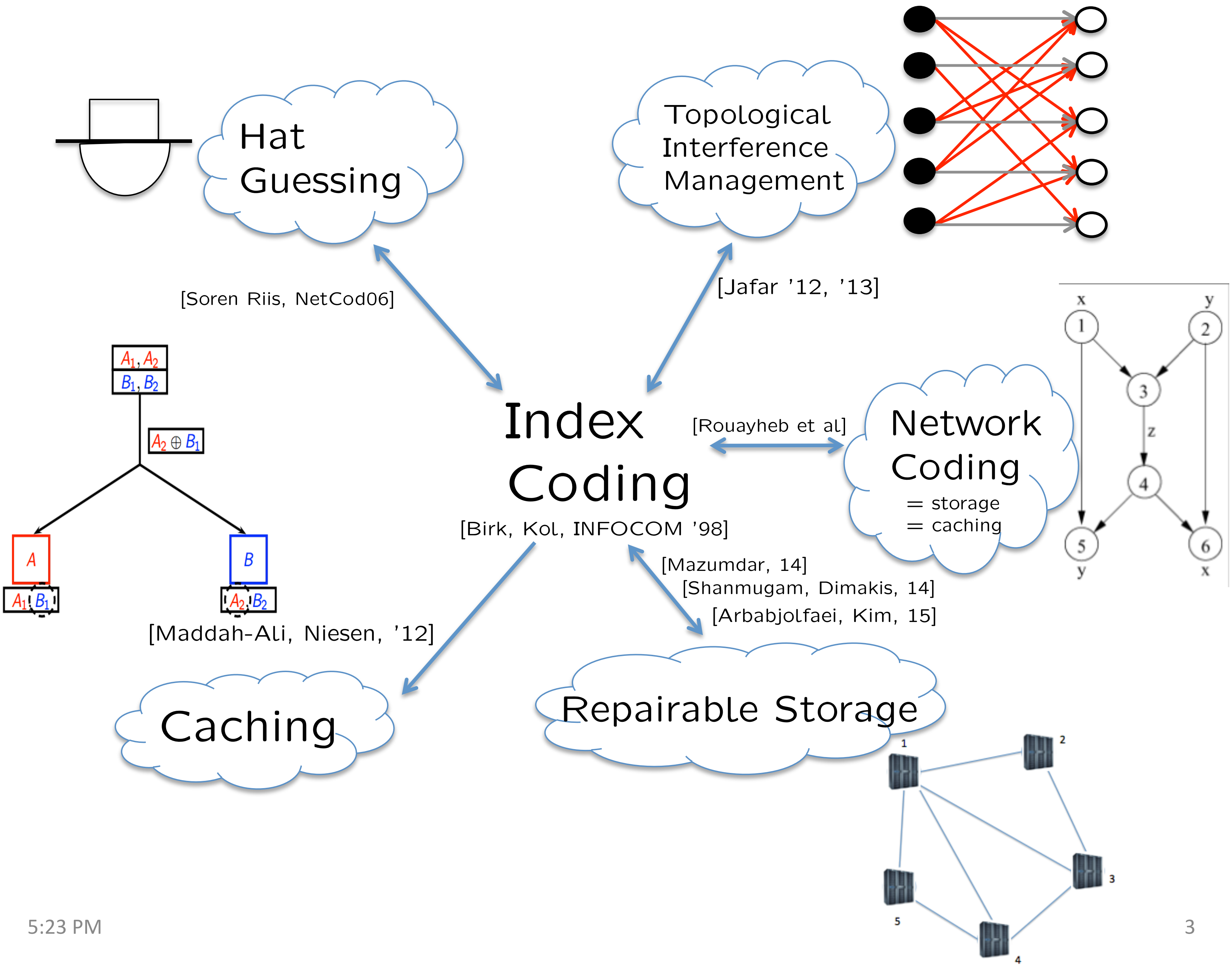
Joint work with Xinping Yi, Syed Jafar, David Gesbert

Index Coding

[Birk, Kol, INFOCOM98]

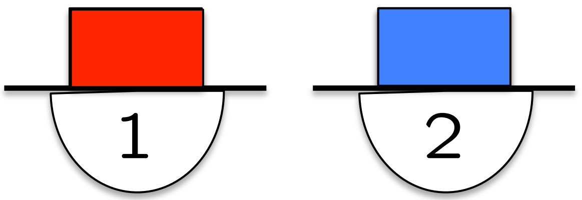


The "antidotes" simply mean that these undesired messages are known to the receivers, a-priori.


Hat Guessing
[Soren Riis, NetCod06]
Topological Interference Management
[Jafar '12, '13]
Index Coding
[Birk, Kol, INFOCOM '98]
[Rouayheb et al]
Network Coding
= storage
= caching
x
y
1
2
3
z
4
5
6
y
x
A_1, A_2
B_1, B_2
$A_2 \oplus B_1$
A
B
A_1 B_1
A_2 B_2
[Maddah-Ali, Niesen, '12]
Caching
[Mazumdar, 14]
[Shanmugam, Dimakis, 14]
[Arbabjolfaei, Kim, 15]
Repairable Storage
1
2
3
4
5

Hat Guessing Game

[Soren Riis, NetCod06]



2 players, each has a hat

The hat can be one of 2 colors

Each player sees the other's hat, but not his own

Guess the color of their own *simultaneously*

Can agree on a strategy before the hats are drawn
No communication allowed later on

Maximize the probability everybody guesses correctly

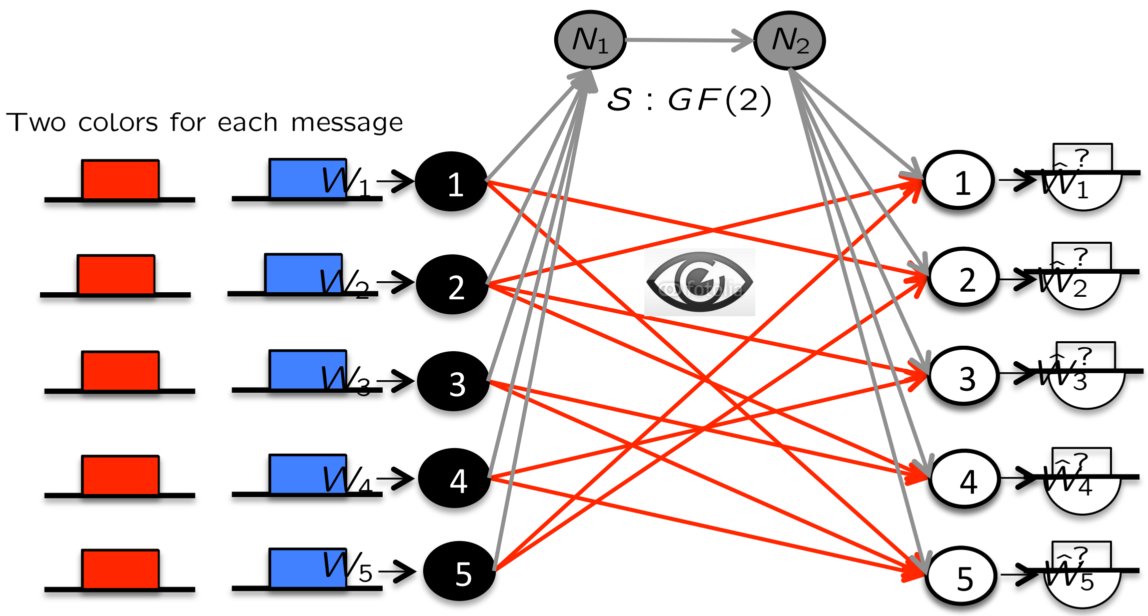
Is seeing *independent* information helpful?
Can we do better than $(\frac{1}{2})^2$?

Answer: $\frac{1}{2}$

Strategy: Common belief that both hats have the same color

Index coding view

A more challenging example



Independent Guesses: P(success) $= \left(\frac{1}{2}\right)^5$

Interference Alignment Scheme: P(success) $= \left(\frac{1}{2}\right)^2$

Improvement from $\frac{1}{32}$ to $\frac{1}{4}$

Solution:

Hat colors $x_i \in \{0, 1\}$

All players assume
$$x_1 + x_2 + x_5 = 0$$
$$x_2 + x_3 + x_4 = 0$$

Prob(assumed correct) $= 1/4$

If assumption is correct,
then everyone guesses their hat color correctly.

Approaches

- Graph Theoretic:

  coloring [Birk, Kol, 98]

  fractional coloring [Blasiak, Kleinberg, Lubetzky, 10]

  local fractional coloring [Shanmugam, Dimakis, Langberg, 13]

  (vector) minrank [Bar-Yossef et al, 11] [Lubetzky, Stav, 09] [Jafar, 13]

  acyclic outer bound [Bar-Yossef et al, 11] [Tehrani, Dimakis Neely, 12]

  graph product [Alon et al, 08] [Blasiak, Kleinberg, Lubetzky, 11] [Arbabjolfaei, Kim, 15]

  graph homomorphism [Ebrahimi, Siavoshani, 14]

- Information Theoretic:

  network coding:

  - matroid theory [Rouayheb, Sprintson, Georghiades, 10]
  - information inequalities [Blasiak, Kleinberg, Lubetzky, 10]
  - network equivalence [Effros, Rouayheb, Langberg, 14] and others

  random coding [Arbabjolfaei, Kim, et al, 14]

  rate distortion [Unal, Wagner, 14]

- Optimization:

  integer programming [Yu, Neely, 13]

  matrix completion [Jaganathan, Thramboulidis, Hassibi et al, 14]

- Interference Alignment

  [Hamed, Cadambe, Jafar, 11] [Jafar, 12, 13] [Sun, Jafar, 13]

Solved Classes of Index Coding Problems

- where sum capacity = 1 [Bar-Yossef et al, 11] [Tehrani, Dimakis Neely, 12]
- half-rate feasible instances [Blasiak, et al, 10] [Jafar, 13]
- alignment graph has no cycles or forks [Jafar, 13]
- alignment graph has no overlapping cycles [Sun, Jafar, 13]
- 5 or fewer messages, unicast [Arbabjolfaei, et al, 14]
- single uniprior instances [Ong, Ho, Lim, 14]
- each message not known at ≤ 2 Rx [Unal, Wagner, 14]

$\vdots$

When is the simplest coloring scheme optimal?

Difficulty

Needs non-linear coding schemes [Rouayheb et al, 10] [Maleki et al, 12]

Needs non-shannon information inequalities

(computer search) [Riis '07, '13]

(matroids) [Blasiak, Kleinberg, Lubetzky, '10, '11]

(by hand, alignment perspective) [Sun, Jafar '13]

How far can we go with only Shannon Inequalities?

Outline

1. Optimality of the Simplest Coloring Scheme

2. Minimal Necessity of Non-Shannon Inequalities

3. Remaining Challenges

Outline

1. Optimality of the Simplest Coloring Scheme

    1a. Main Result

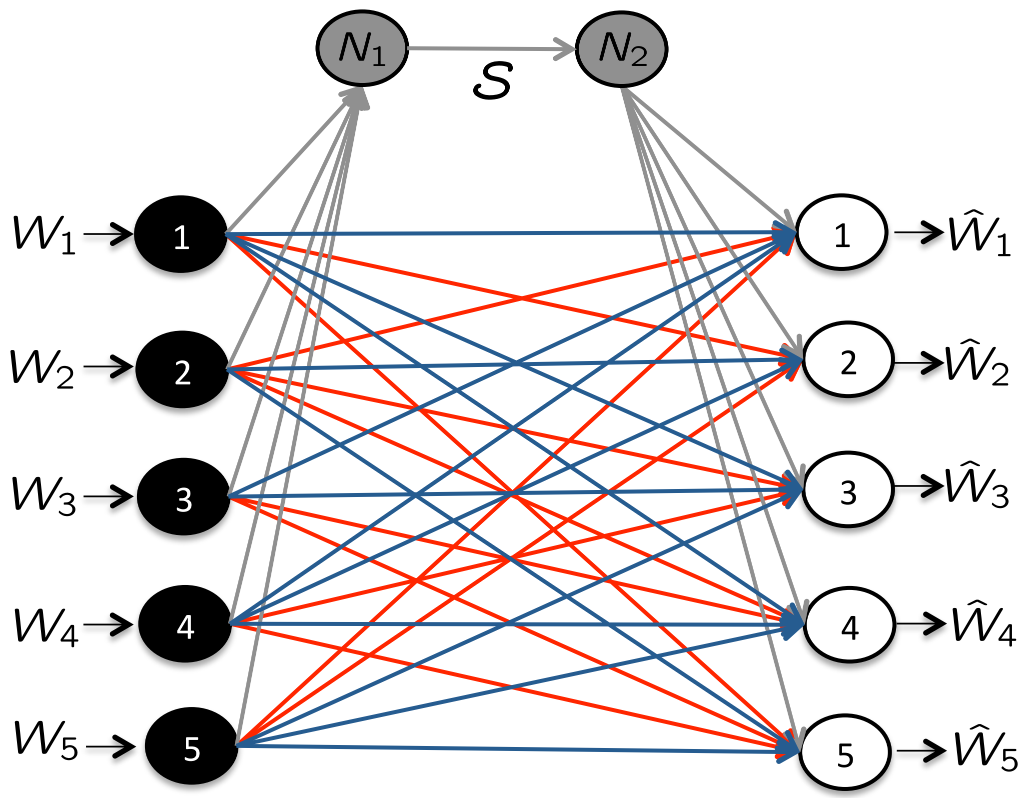
    1b. Special Case: Convex networks

    1c. Proof

2. Minimal Necessity of Non-Shannon Inequalities

3. Remaining Challenges

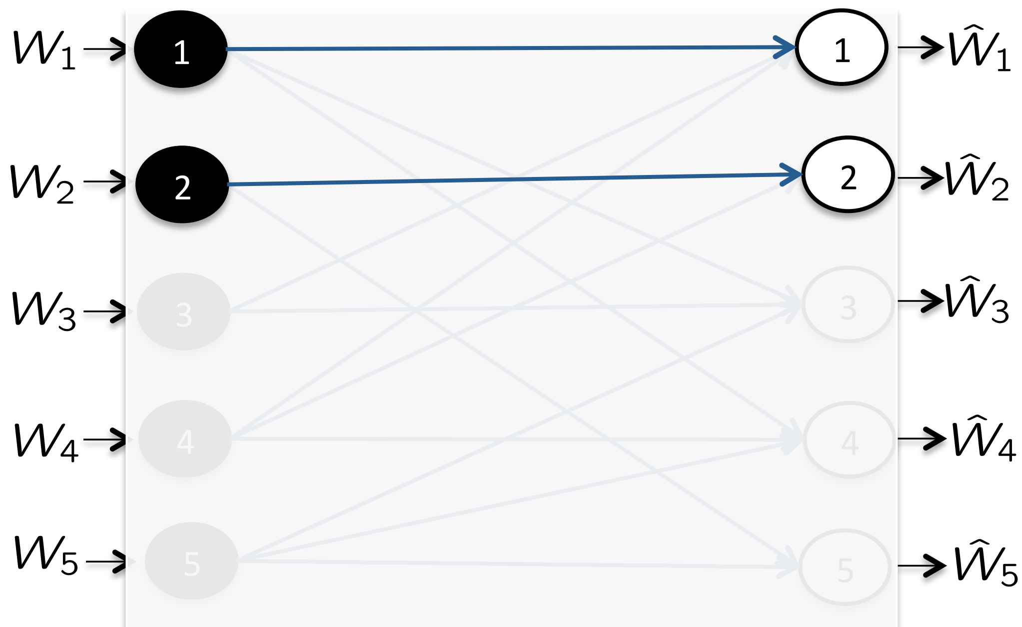
Fractional Coloring achieves All-unicast Capacity Region if and only if Network Topology is Chordal



Fractional Coloring achieves All-unicast Capacity Region if and only if Network Topology is Chordal

Coloring: (clique cover, TDMA, scheduling, orthogonal access)
Schedule messages that are non-interfering/orthogonal.

Fractional: Allows time sharing.



Network Topology: Complement of Antidote Graph

Fractional Coloring achieves All-unicast Capacity Region if and only if Network Topology is Chordal

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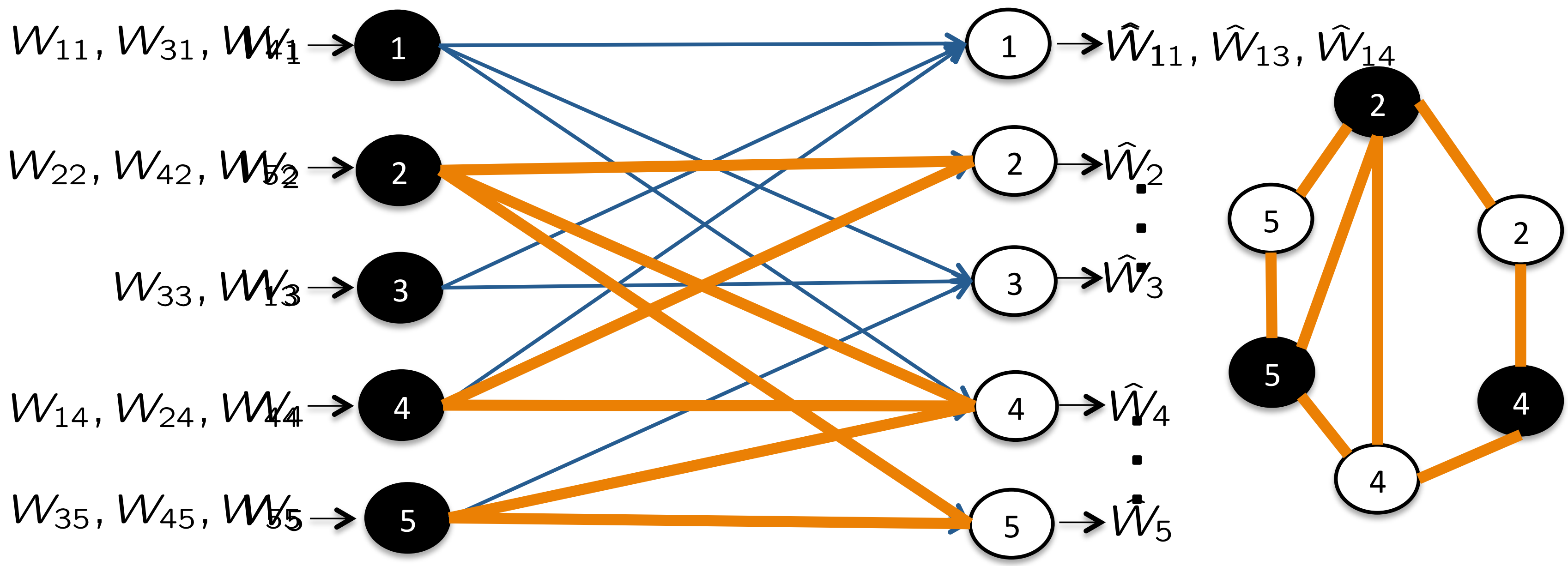

Network Topology: Complement of Antidote Graph

Chordal: All Cycles have Chord

All Unicast: Each Tx has a message for each Rx

Include arbitrary subset of the 17 messages

Capacity Region: Includes symmetric/sum capacity

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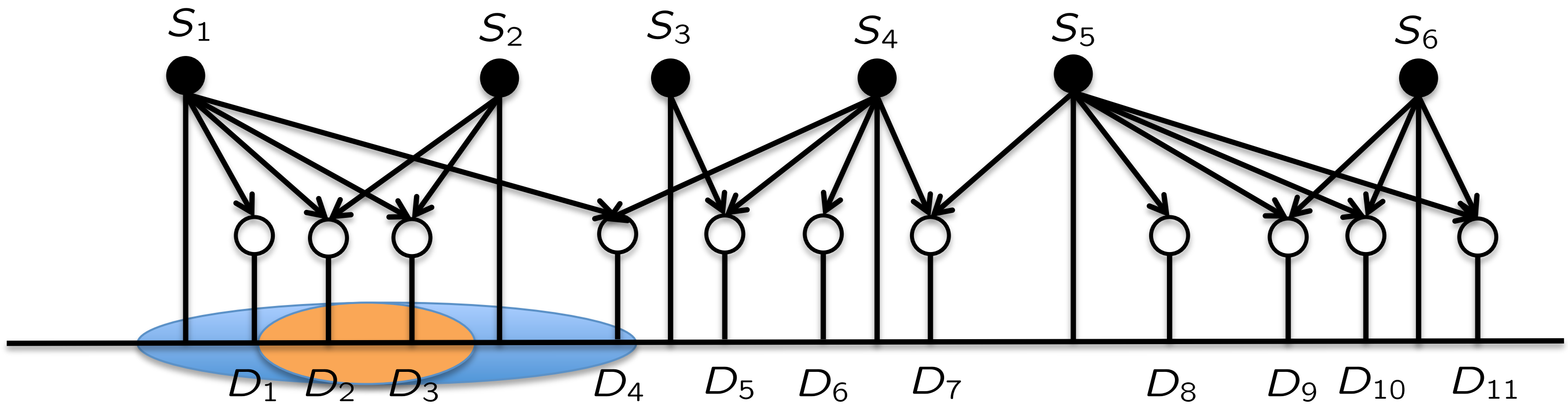
One-dimensional Convex Cellular Topologies

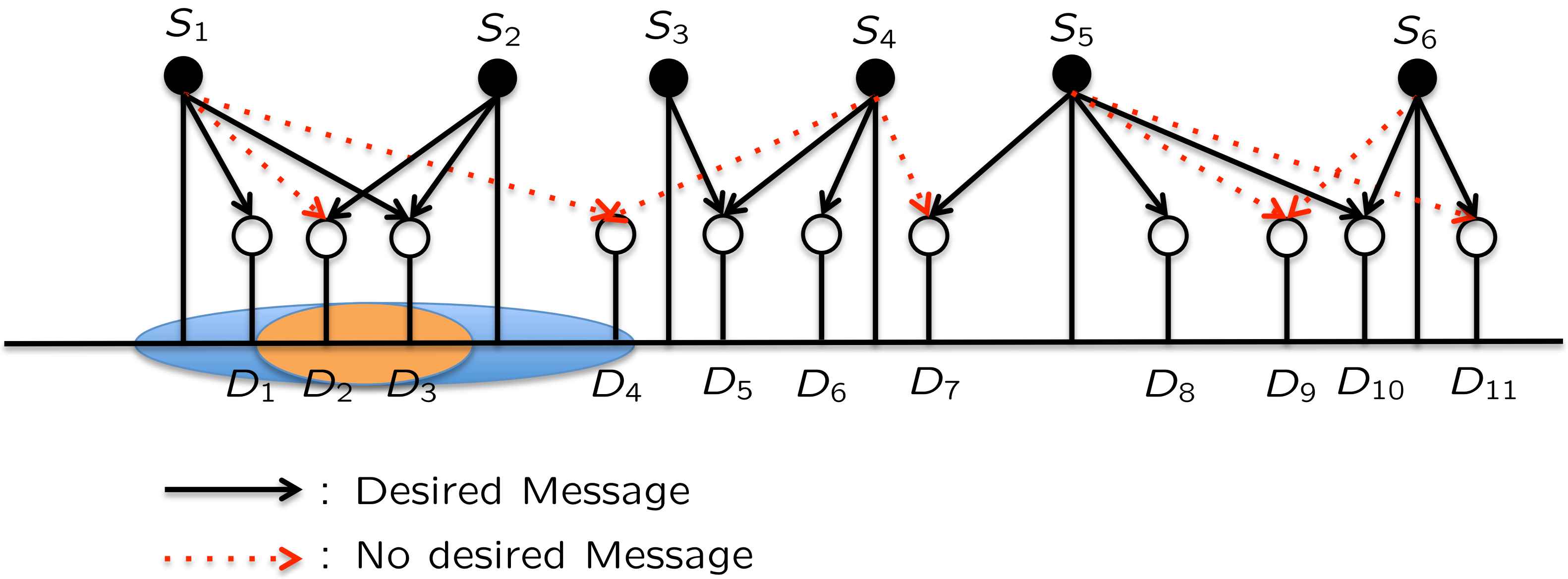
(index coding problem)

[Maleki, Jafar '13]



One-dimensional Convex Cellular Topologies

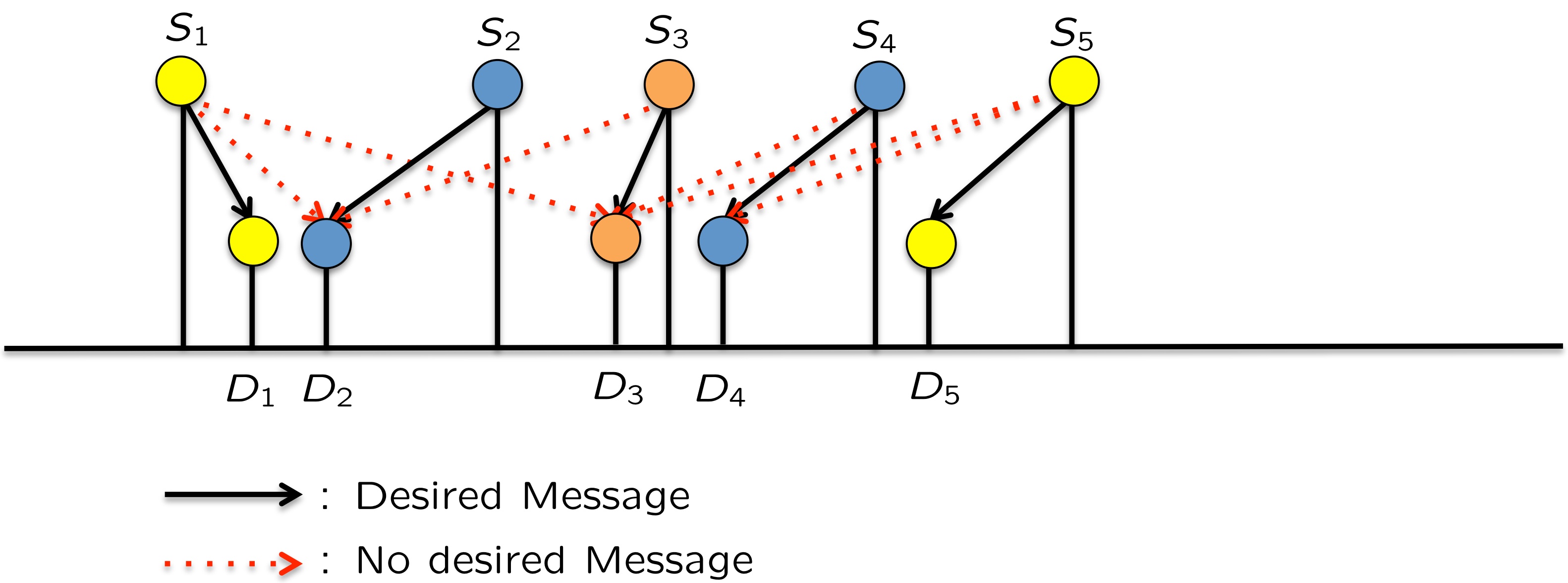
(index coding problem)



Each link can carry an *independent* desired message

Scheduling is IT optimal for symmetric capacity, sum capacity and capacity region

Illustrating Example



Sum capacity = 2

$(D_2, W_2) \geq D_3$

$(D_2, W_2, W_3) \geq D_1$

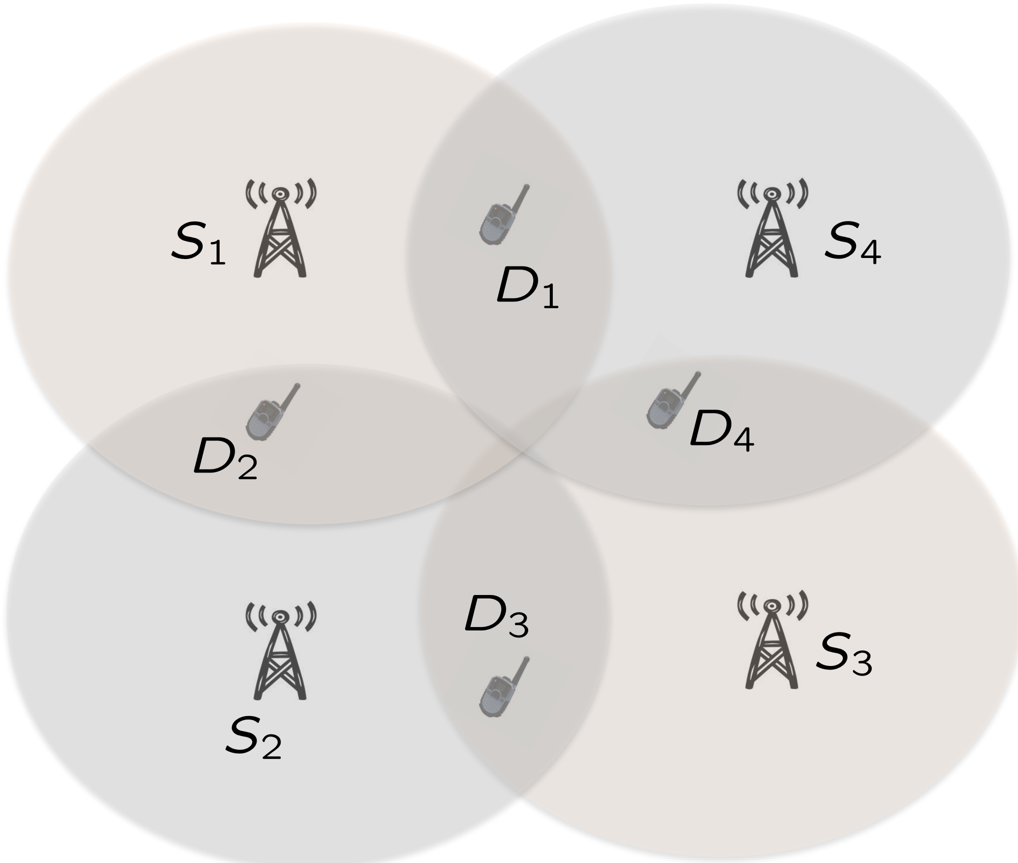
Acyclic outer bound $\begin{cases} R_1 + R_2 + R_3 \leq 1 \\ R_4 + R_5 \leq 1 \end{cases}$

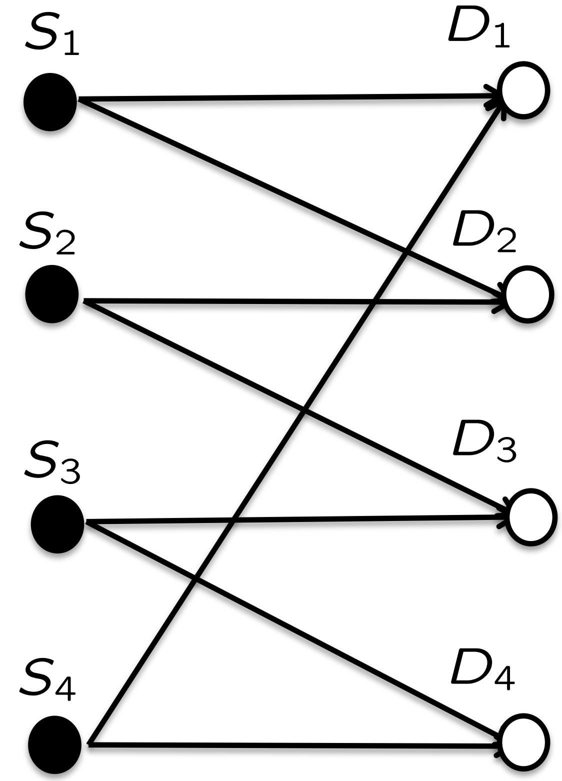
Symmetric capacity = 1/3

Capacity region:
set of all acyclic outer bounds = convex hull of all TDMA points

Two-dimensional Convex Cellular Topologies

Fact: 2-dim convex networks are not chordal.





Fact: Scheduling is not optimal.

Conjecture: Scheduling is still close to optimal.

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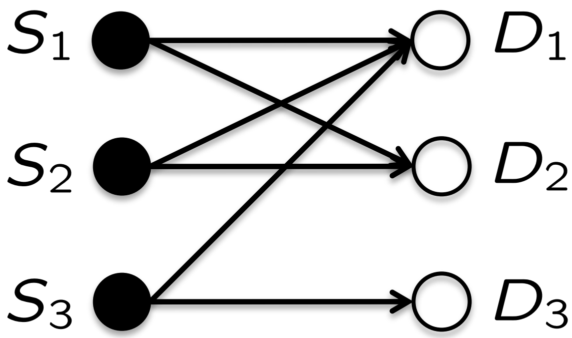
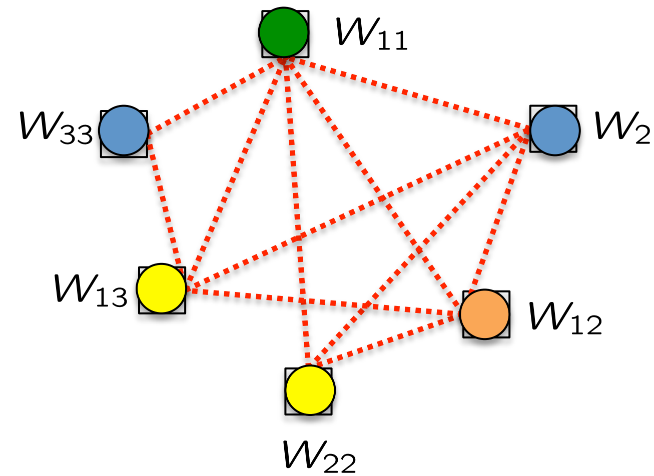
Sufficiency: Chordal $\rightarrow$ Coloring is optimal

$G\,(\text{chordal}) \xrightarrow{*} (\text{perfect})\, G_e^2$ = Message Conflict Graph





$R_{11} + R_{21} + R_{12} \leq 1$

For a clique C:

$$\bigcup_C \sum_{W_{ji} \in C} R_{ji} \leq 1$$

(G is chordal)

Acyclic demand graph

Clique Polytope

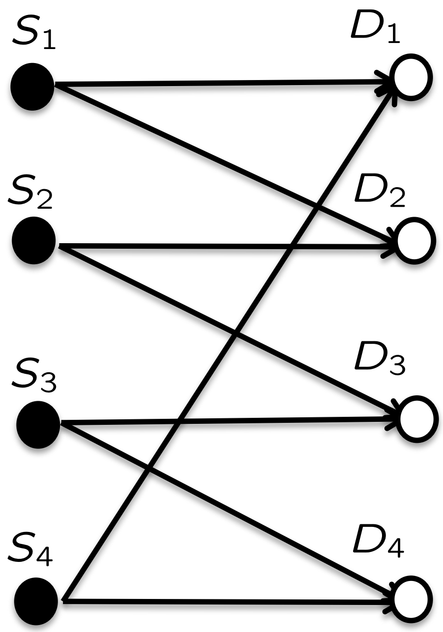
* (G_e^2 is perfect)

all vertices are integral

Capacity region:
set of all acyclic outer bounds = convex hull of all TDMA points

Fractional Coloring achieves All-unicast Capacity Region if and only if Network Topology is Chordal

Necessity: Not chordal → Coloring is sub-optimal



For the cyclic sub-network,

∃ a rate tuple that is not achievable by coloring.

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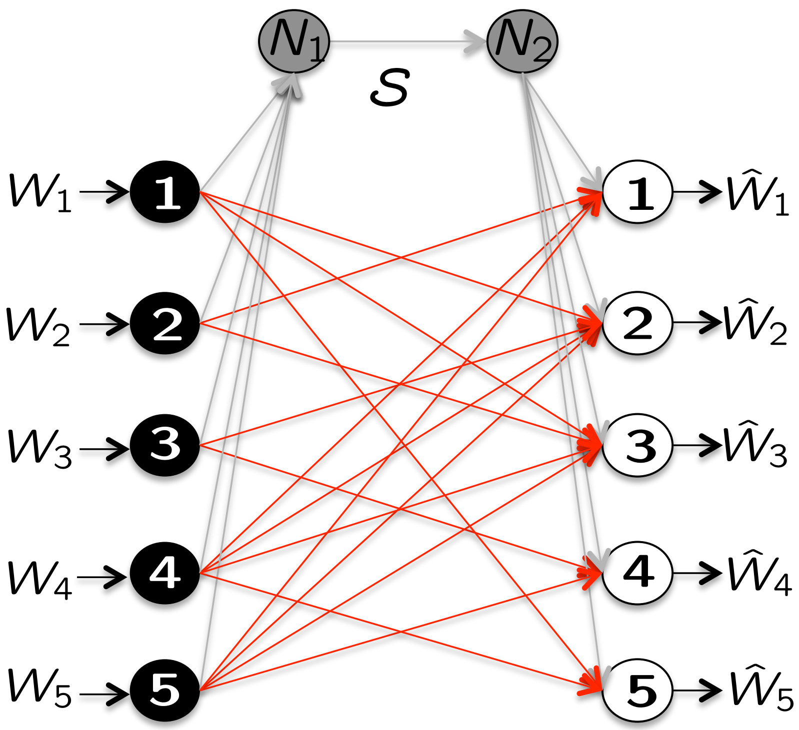
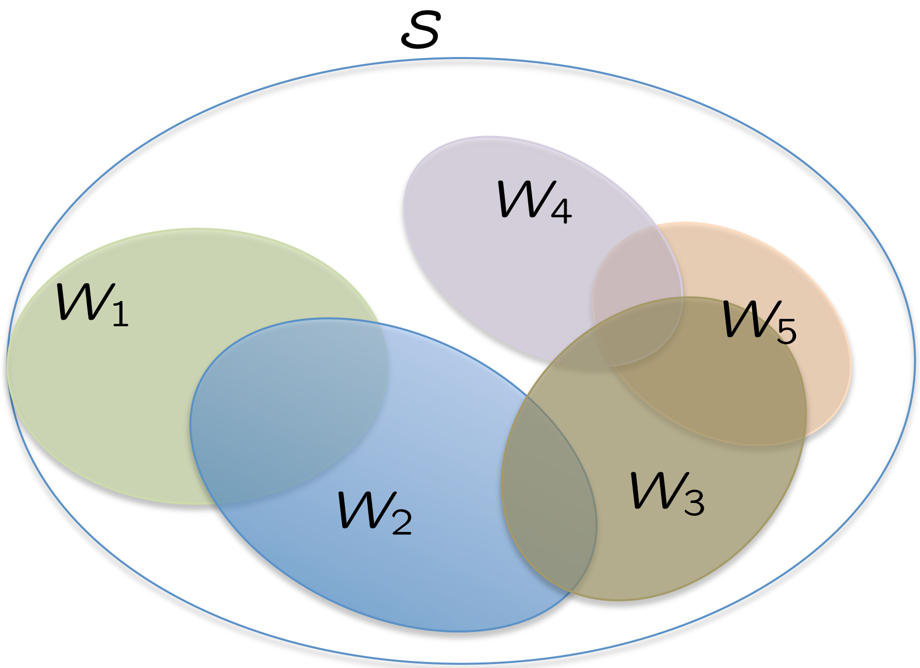
    2a. An Interference Alignment Perspective

    2b. The Simplest Hard Problem

3. Remaining Challenges

Index Coding – Interference Alignment Perspective





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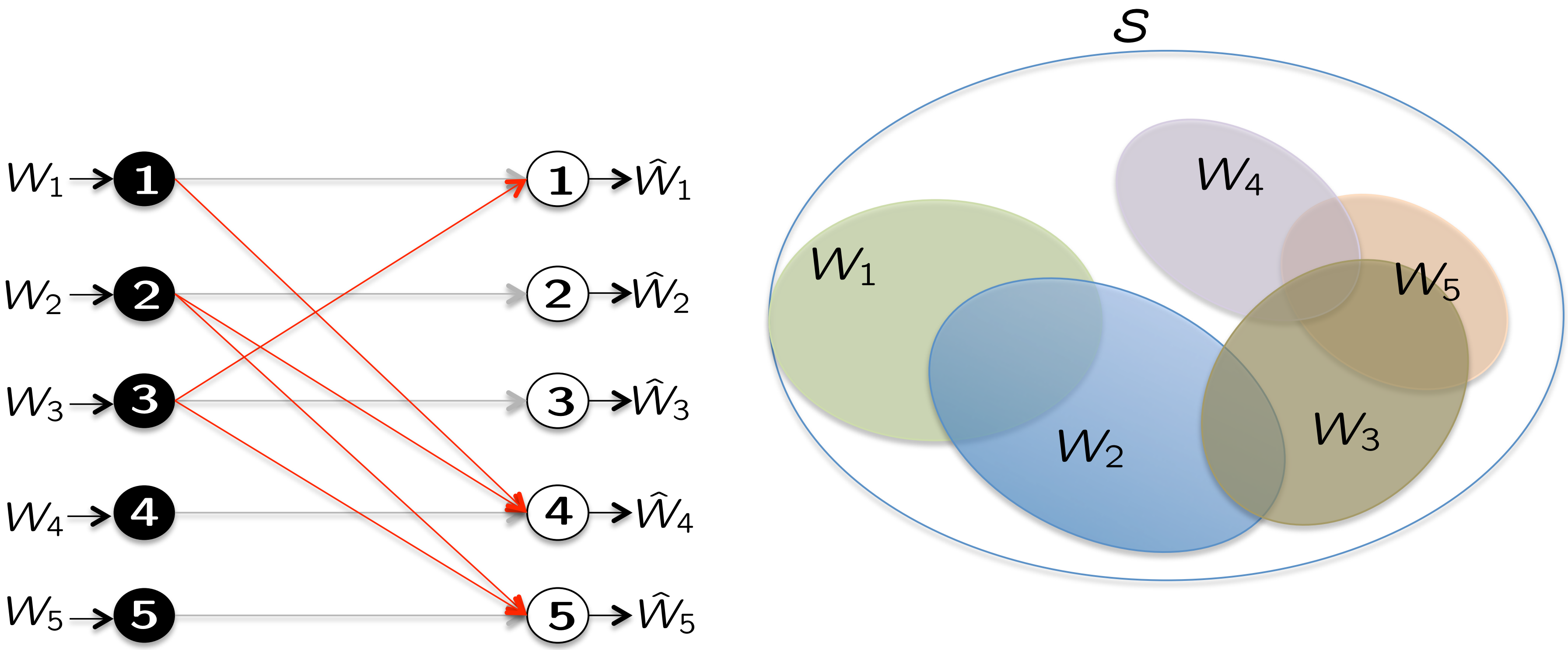

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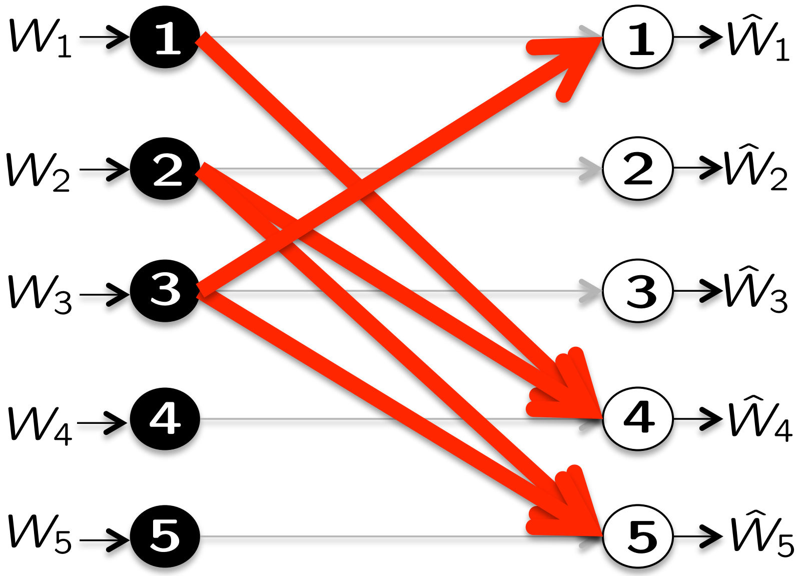
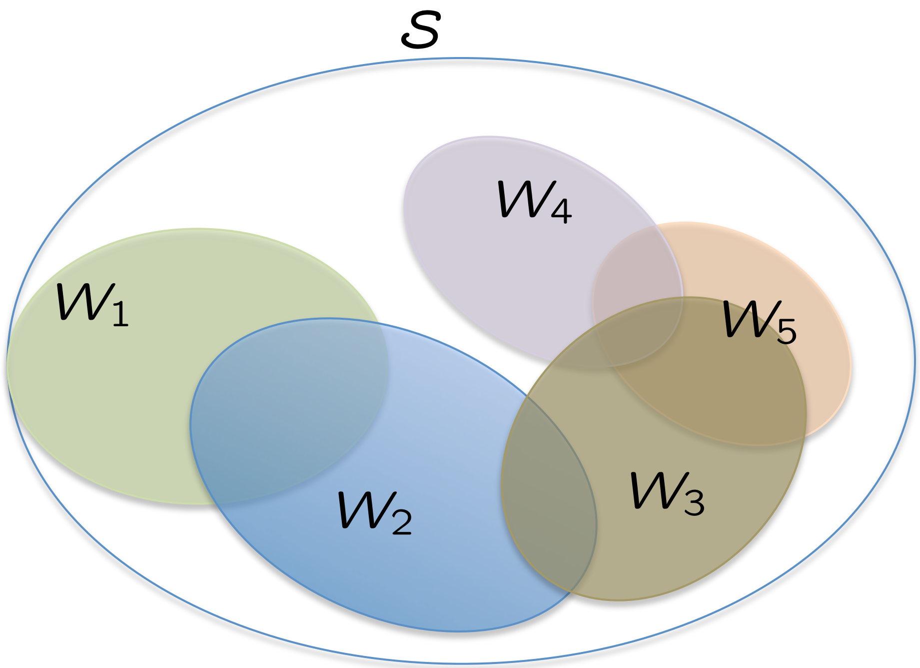
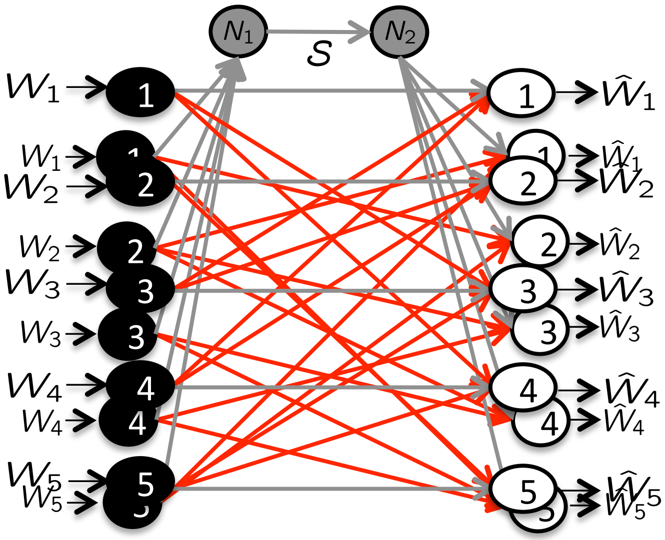




Interference Alignment Perspective

[Jafar '12, '13]



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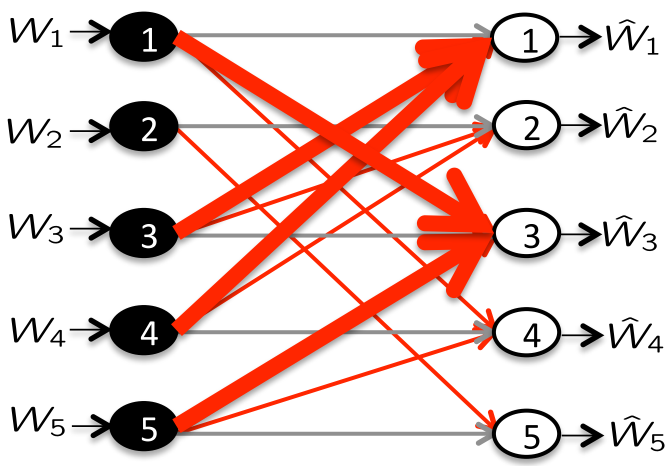
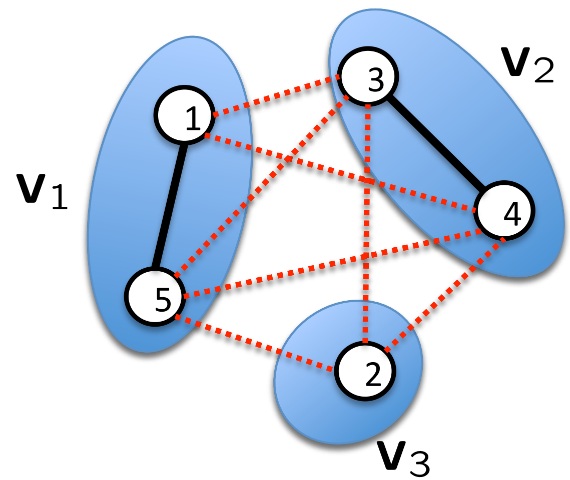




$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: 2×1 vectors

Interference Alignment Conditions:

1) Interferers should align as much as possible — **Alignment graph** (solid black edges)

2) Desired signal must not align with interference — **Conflict graph** (dashed red edges)

Connected components of alignment graph are **alignment sets**

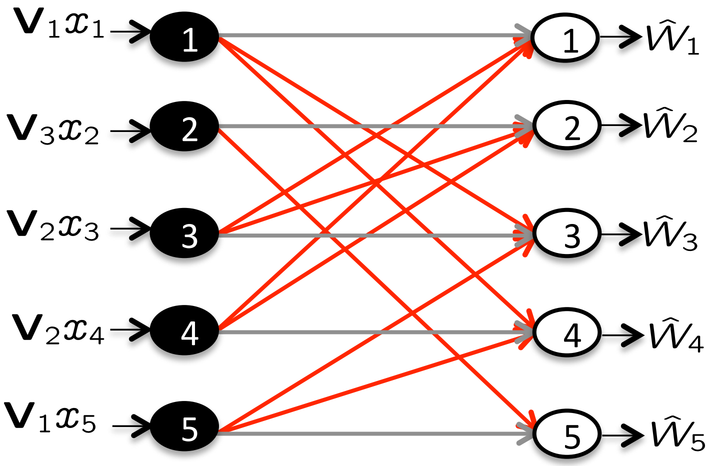
There is no internal conflict.

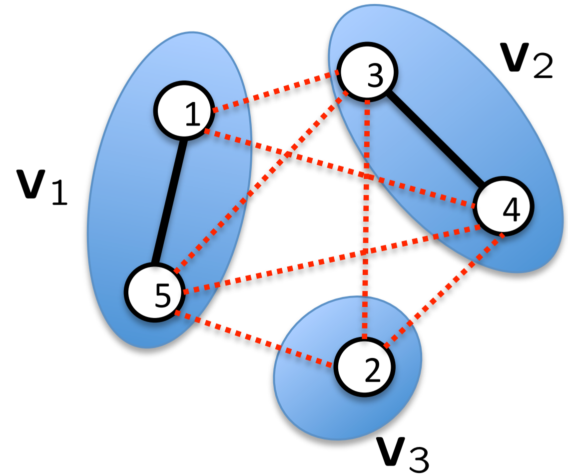
Assign a 2×1 vector to each alignment set

Achieved Rate $= 1/2$ per user

Interference Alignment Perspective

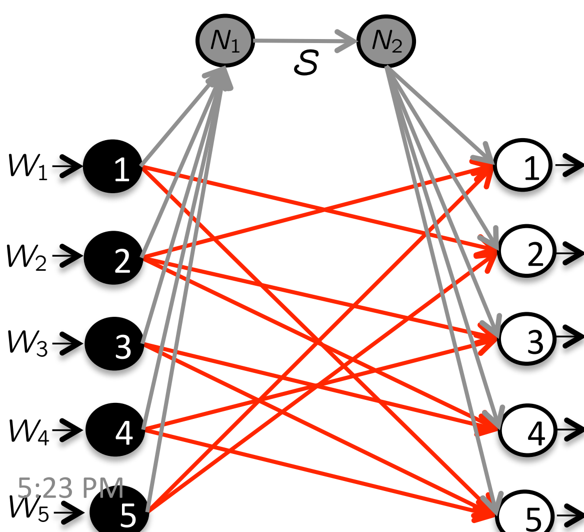
[Jafar '12, '13]





$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: 2×1 vectors

$$S = \mathbf{v}_1 x_1 + \mathbf{v}_3 x_2 + \mathbf{v}_2 x_3 + \mathbf{v}_2 x_4 + \mathbf{v}_1 x_5$$



$$Y_1 = \mathbf{v}_1 x_1 + \mathbf{v}_2(x_3 + x_4)$$
$$Y_2 = \mathbf{v}_3 x_2 + \mathbf{v}_2(x_3 + x_4)$$
$$Y_3 = \mathbf{v}_2 x_3 + \mathbf{v}_1(x_1 + x_5)$$
$$Y_4 = \mathbf{v}_2 x_4 + \mathbf{v}_1(x_1 + x_5)$$
$$Y_5 = \mathbf{v}_1 x_5 + \mathbf{v}_3 x_2$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S(1) = x_1 + x_2 + x_5$$
$$S(2) = x_2 + x_3 + x_4$$

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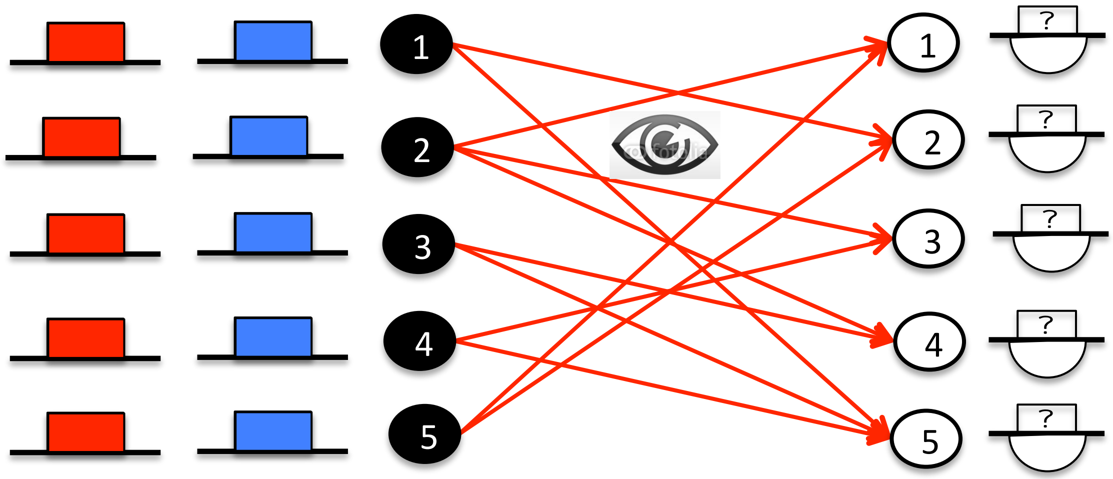
Hat Guessing View

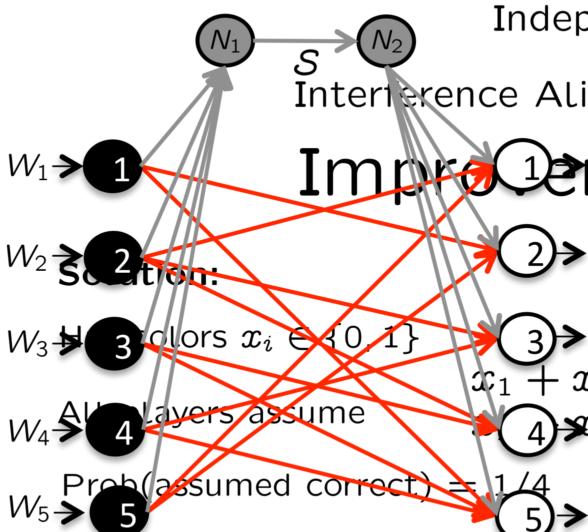
Two colors for each message



Independent Guesses: $P(\text{success}) = \left(\frac{1}{2}\right)^5$

Interference Alignment Scheme: $P(\text{success}) = \left(\frac{1}{2}\right)^2$

Improvement from $\frac{1}{32}$ to $\frac{1}{4}$



Solution:

Hat colors $x_i \in \{0, 1\}$

All players assume

$x_1 + x_2 + x_5 = 0$

$x_3 + x_4 = 0$

Prob(assumed correct) = 1/4

If assumption is correct, then everyone guesses their hat color correctly.

$S(1) = x_1 + x_2 + x_5$

$S(2) = x_2 + x_3 + x_4$

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3. Remaining Challenges

Non-Shannon Inequalities

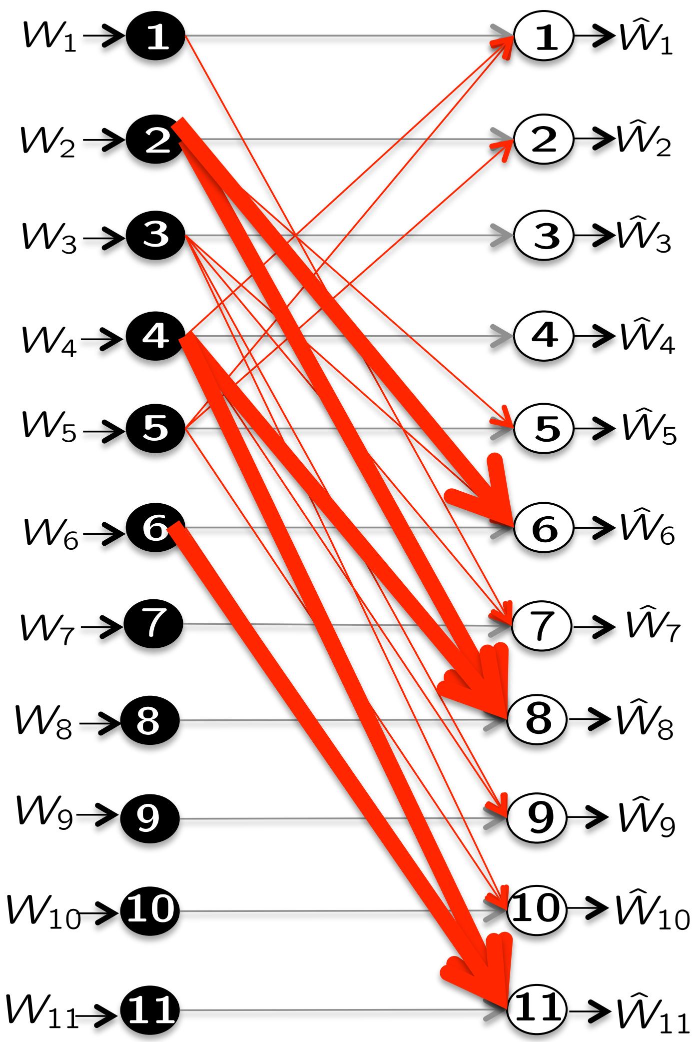
	IT Capacity		Linear Capacity
	Entropy Space (random variables)		Vector Space (subspaces)
	Information Inequalities	$\subset$	Linear Rank Inequalities
≤ 3	Polymatroidal Axioms (non-negativeness of Shannon information measurements)		Polymatroidal Axioms (non-negativeness of Shannon information measurements)
$= 4$	Non-Shannon-type Information Inequality [Zhang,Yeung,TIT98] Infinite many [Matus, ISIT2007] Open		Ingleton Inequalities [Ingleton, 71]
$= 5$	Open		24 more inequalities [Dougherty,Freiling,Zeger, 10]
> 5	Open		Open

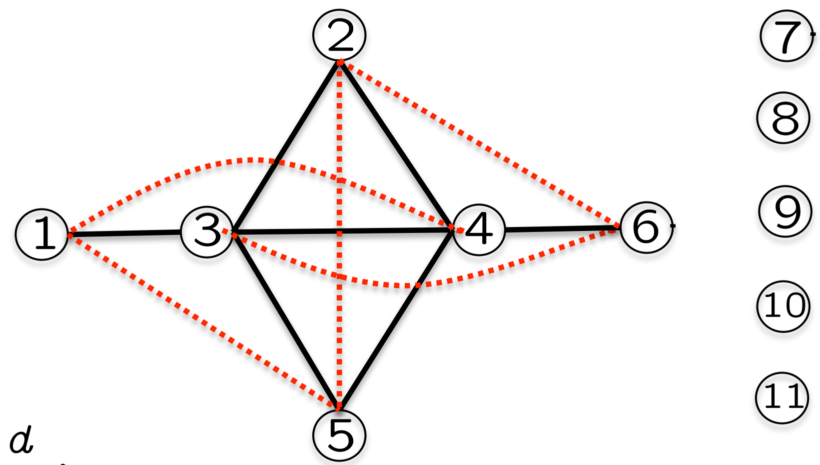
Simplest Example (for non-Shannon needed)

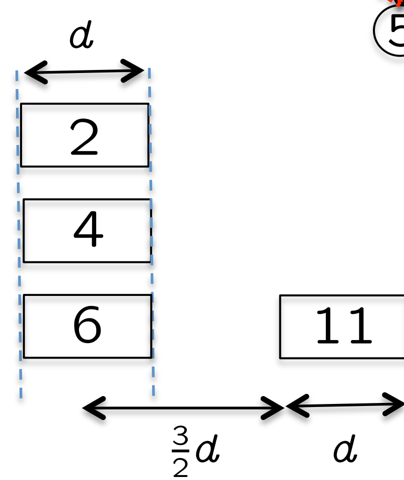
(from alignment perspective)

by hand, from scratch

[Sun, Jafar, 13]







For receiver 11

$$5d/2 \leq 1$$

$$d \leq 2/5$$

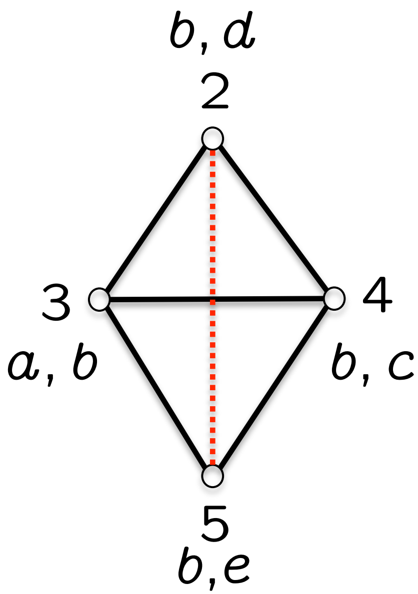
2/5 is the best bound possible through Shannon inequalities

Can be tightened to 11/28 with Zhang-Yeung non-Shannon inequality

Simplest Example

[Sun, Jafar, 13]

Vector Space Interpretation



a, b, c, d, e:
1/5-size generic space

○ $= 2/5$: rate constraint

—— $= 3/5$: interference constraint

△ $= 4/5$: submodularity constraint

Above three satisfy all polymatroidal constraints.

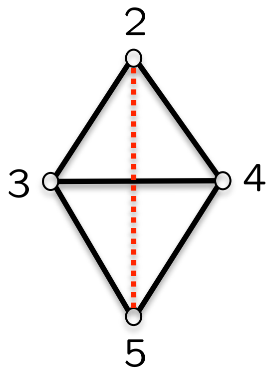
For example, $(2, 3) + (3, 4) \geq (2, 3, 4) + 3$?

$\Leftarrow$ |——| + |——| $\geq$ |△| + |○|

$\Leftarrow \; 3/5 + 3/5 \geq 4/5 + 2/5$

Overlap of 3 and 4 = |○| + |○| − |——|

$= 1/5$

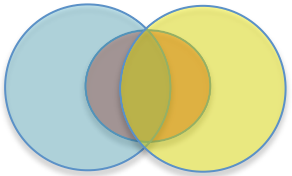


Simplest Example

[Sun, Jafar, 13]

Vector Space Interpretation

Vector Space used by W_i : $\mathbf{V}_i$



$$\begin{aligned}\dim(\mathbf{V}_2 \cap \mathbf{V}_5) &\geq \dim(\mathbf{V}_2 \cap (\mathbf{V}_3 \cap \mathbf{V}_4)) + \dim(\mathbf{V}_5 \cap (\mathbf{V}_3 \cap \mathbf{V}_4)) - \dim(\mathbf{V}_3 \cap \mathbf{V}_4) \\ &\geq \dim(\mathbf{V}_2 \cap \mathbf{V}_3) + \dim(\mathbf{V}_2 \cap \mathbf{V}_4) - \dim(\mathbf{V}_2 \cap (\mathbf{V}_3, \mathbf{V}_4)) \\ &\quad +\dim(\mathbf{V}_5 \cap \mathbf{V}_3) + \dim(\mathbf{V}_5 \cap \mathbf{V}_4) - \dim(\mathbf{V}_5 \cap (\mathbf{V}_3, \mathbf{V}_4)) - \dim(\mathbf{V}_3, \mathbf{V}_4)\end{aligned}$$

$$\Rightarrow \quad \dim(\mathbf{V}_2, \mathbf{V}_3) + \dim(\mathbf{V}_2, \mathbf{V}_4) + \dim(\mathbf{V}_3, \mathbf{V}_4) + \dim(\mathbf{V}_3, \mathbf{V}_5) + \dim(\mathbf{V}_4, \mathbf{V}_5)$$
$$\geq \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) + \dim(\mathbf{V}_2, \mathbf{V}_5) + \dim(\mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4) + \dim(\mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5)$$

Ingleton inequality

$$\Rightarrow 5 \times \frac{3}{5} \geq 4 \times \frac{2}{5} + 2 \times \frac{4}{5}$$

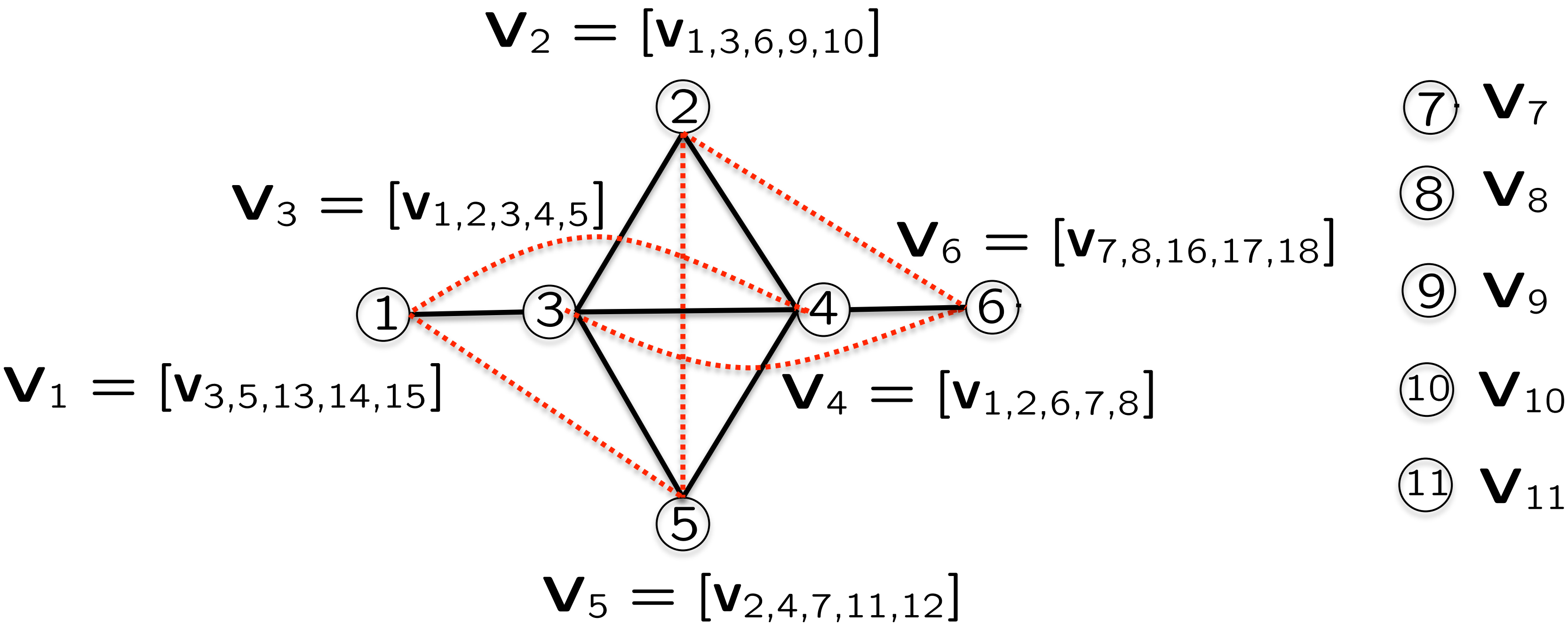
$$\Rightarrow \frac{15}{5} \geq \frac{16}{5}, \text{ contradiction!}$$

Simplest Example

[Sun, Jafar, 13]

Linear Capacity (= 5/13)

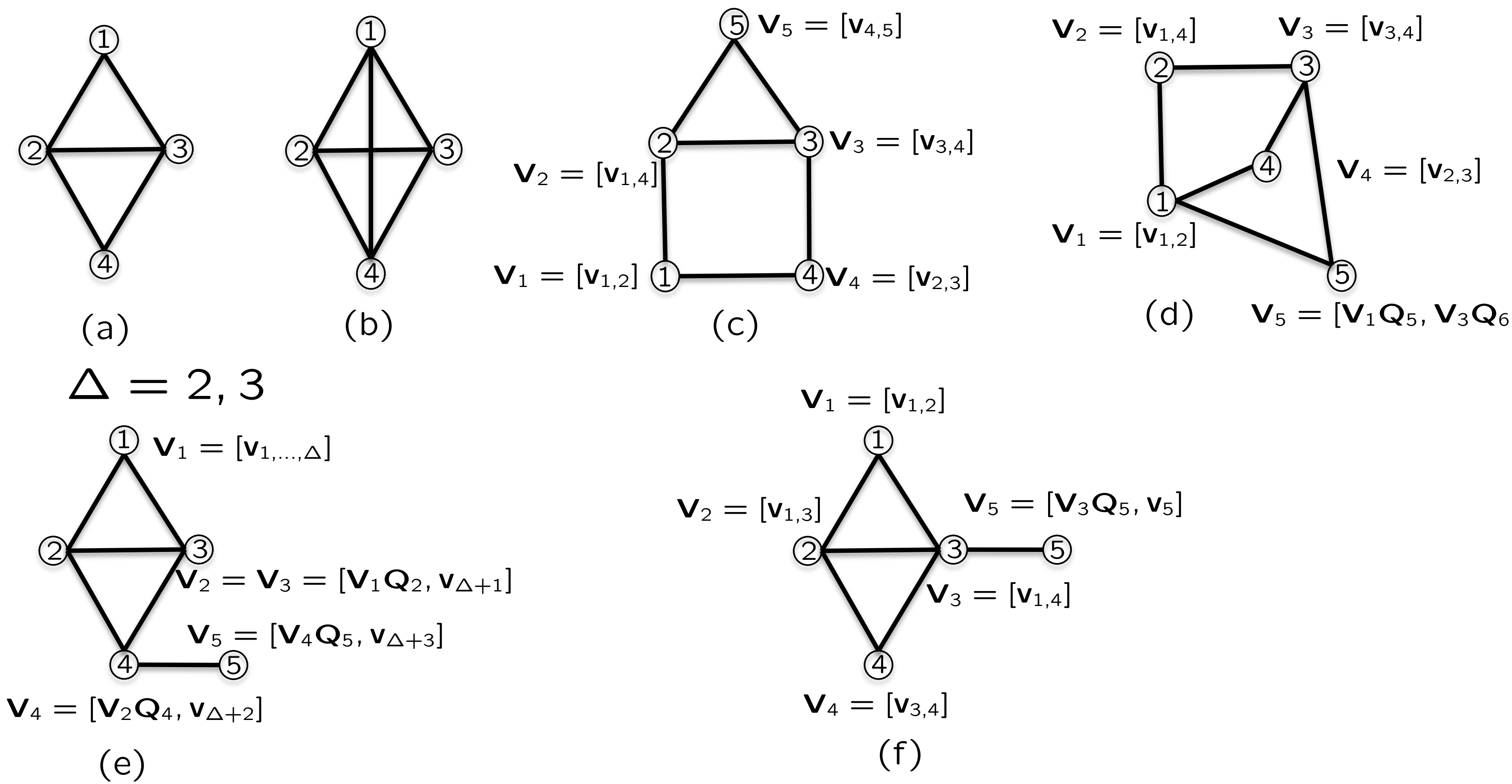
$\mathbf{v}_1, \ldots, \mathbf{v}_{18}$: Generic 13 × 1 random vectors
$\mathbf{V}_7, \ldots, \mathbf{V}_{11}$: Generic 13 × 5 random matrices



No Simpler Example

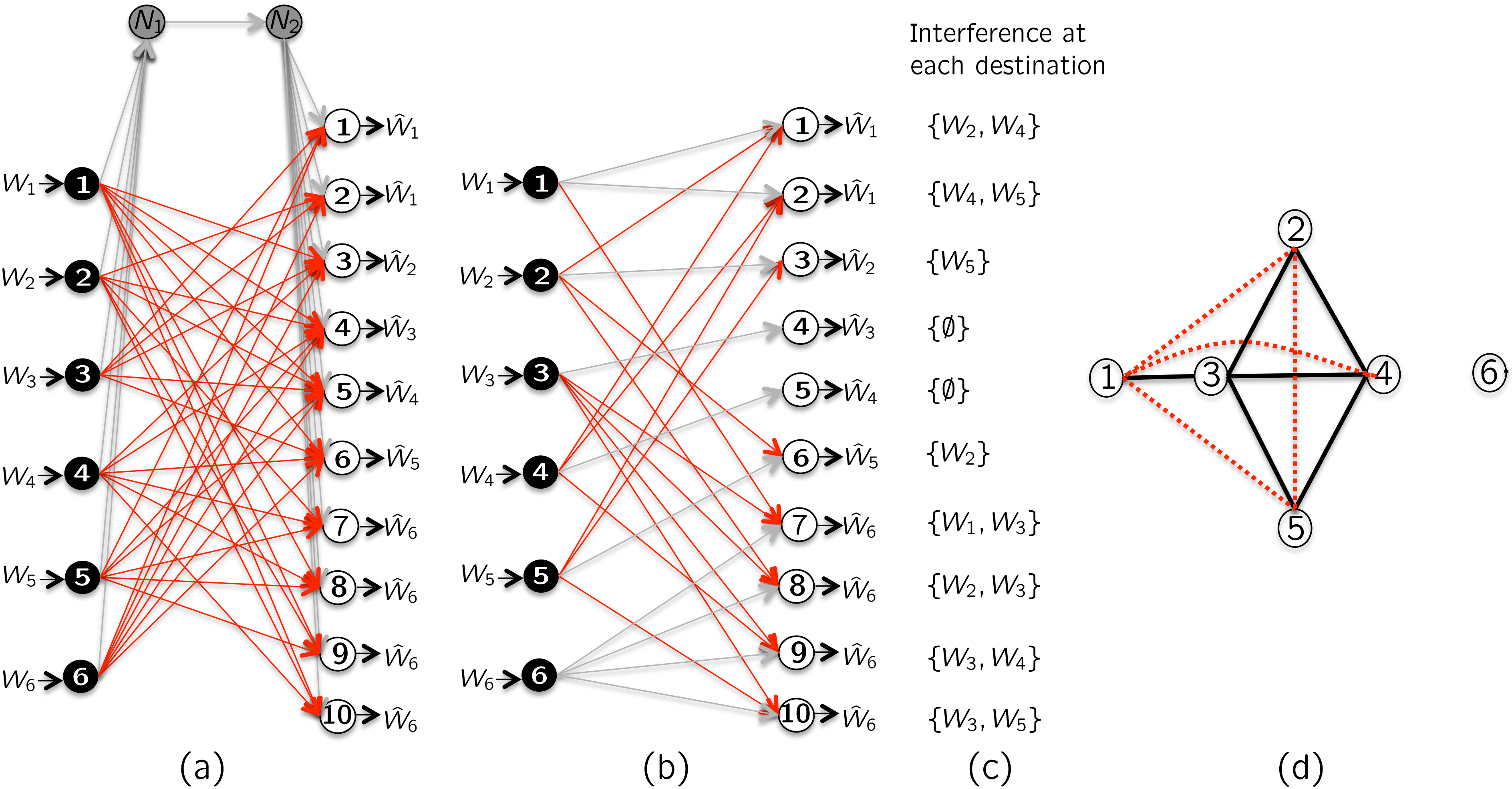
[Sun, Jafar, 13]

solved all cases with 6 or fewer edges in each alignment set



Simplest Example for groupcast

[Sun, Jafar, 13]



Open: Simplest unicast example with min number of messages

Outline

1. Optimality of the Simplest Coloring Scheme
   - 1a. Main Result
   - 1b. Special Case: Convex networks
   - 1c. Proof
2. Minimal Necessity of Non-Shannon Inequalities
   - 2a. An Interference Alignment Perspective
   - 2b. The Simplest Hard Problem
3. Remaining Challenges

Open Problem 1: Compute best linear rate?

Best linear rate = vector minrank

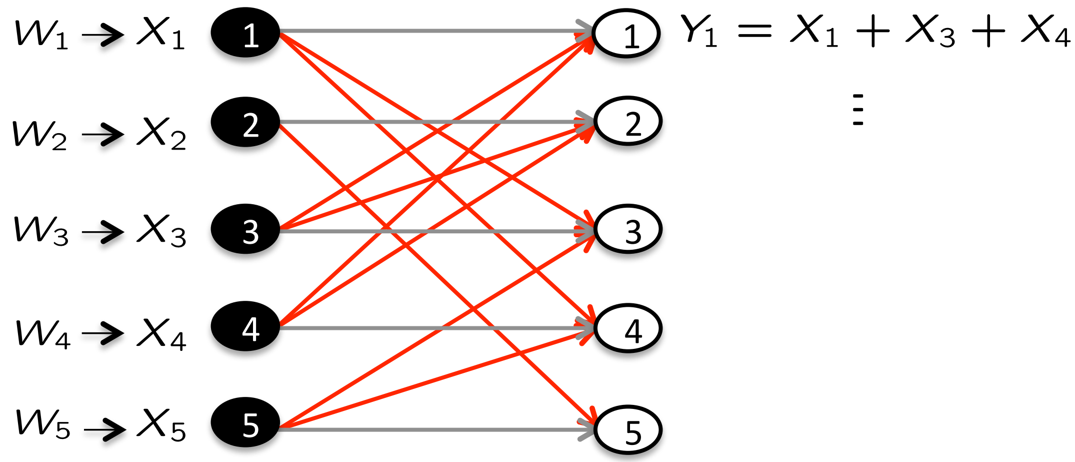
Can also be stated as alignment constraints.

Multi-letter in essence.

No bound on the number of symbol extension needed.

Open Problem 2: Separate encoding

$$S = X_1 + X_2 + X_3 + X_4 + X_5$$



Non-shannon needed?

Non-linear needed?

Relates to the DoF of Gaussian networks.

Thanks!

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