Normal Factor Graphs[1] [2]

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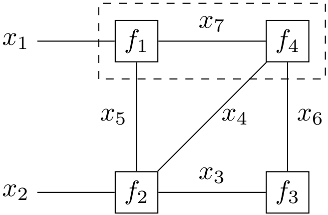
[1] Joint work with Y. Mao.

[2] Acknowledge P. O. Vontobel and D. Forney

Outline

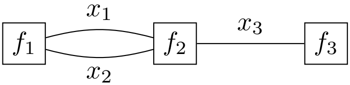
- Introduce normal factor graphs.
- Holographic algorithms and Duality theorem.
- Probabilistic models.
- Partition function estimation.
- Trace diagrams.

Normal factor graph (NFG)



$$Z_{\mathcal{G}}(x_1, x_2) := \sum_{x_3,\ldots,x_7} f_1(x_1, x_5, x_7) f_2(x_2, x_3, x_4, x_5) f_3(x_3, x_6) f_4(x_4, x_6, x_7)$$

Puzzle



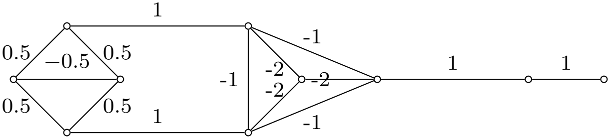
	00	01	10	11
f_1	0	1	1	0

	000	001	010	011	100	101	110	111
f_2	0	1	1	1	1	1	1	0

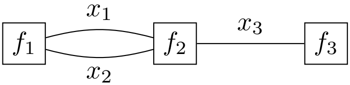
	0	1
f_3	1	1

x_1	x_2	x_3
0	1	0
0	1	1
1	0	0
1	0	1

, $Z_{\mathcal{G}} = 4$.



Puzzle



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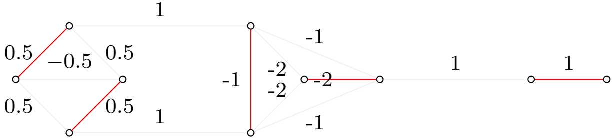
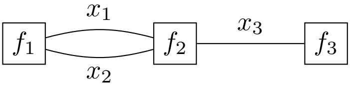


.5

Puzzle



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1	0	0
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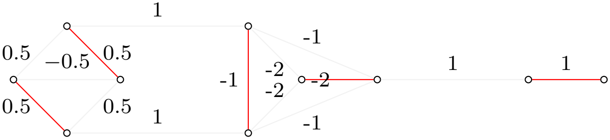
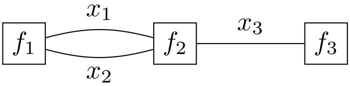


$.5 + .5$

Puzzle



	00	01	10	11
f_1	0	1	1	0

	000	001	010	011	100	101	110	111
f_2	0	1	1	1	1	1	1	0

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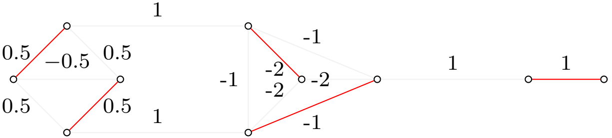
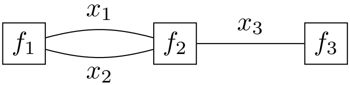


$.5 + .5 + .5$

Puzzle



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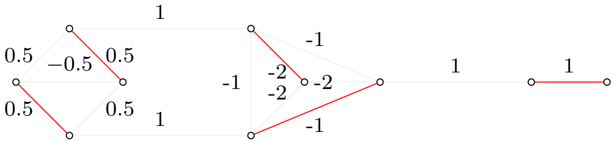
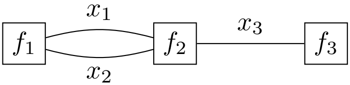


$.5 + .5 + .5 + .5$

Puzzle



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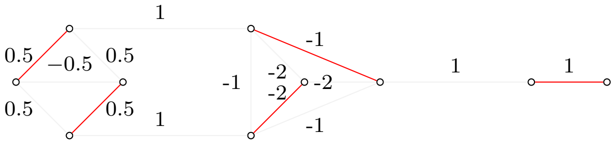
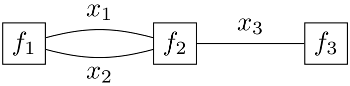


$$.5 + .5 + .5 + .5 + .5$$

Puzzle



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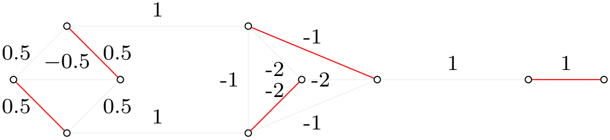
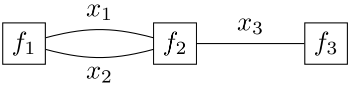


$$.5 + .5 + .5 + .5 + .5 + .5$$

Puzzle



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f_2	0	1	1	1	1	1	1	0

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1	0	0
1	0	1

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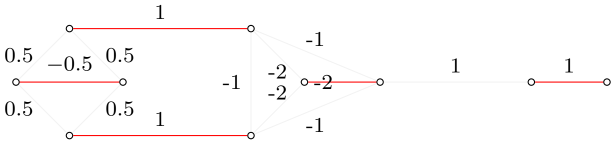
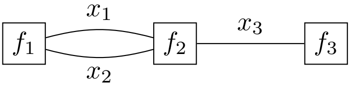


$$.5 + .5 + .5 + .5 + .5 + .5 + 1$$

Puzzle



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f_1	0	1	1	0

	000	001	010	011	100	101	110	111
f_2	0	1	1	1	1	1	1	0

	0	1
f_3	1	1

x_1	x_2	x_3
0	1	0
0	1	1
1	0	0
1	0	1

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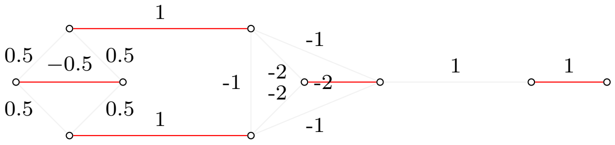
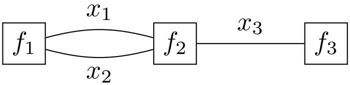


$$.5 + .5 + .5 + .5 + .5 + .5 + 1 = 4$$

Holographic algorithms [Valiant 2004]



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f_1	0	1	1	0

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f_2	0	1	1	1	1	1	1	0

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1	0	1

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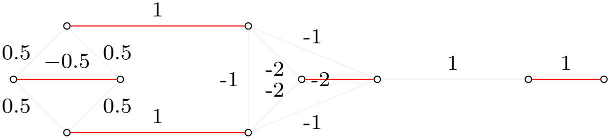
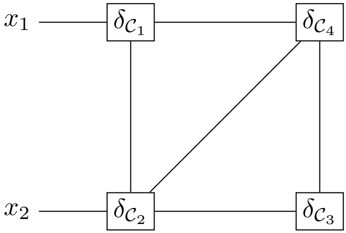


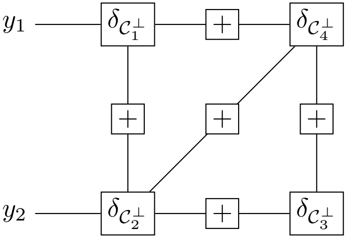
$$.5 + .5 + .5 + .5 + .5 + .5 + 1 = 4$$

- Of great importance.

Duality theorem [Forney 2001]



(a) $\mathcal{G}$



(b) $\mathcal{G}'$

- $Z_{\mathcal{G}} = \delta_{\mathcal{C}}$
- $Z_{\mathcal{G}'} = \delta_{\mathcal{C}^\perp}$
- Easy.
- General.
- Many applications.

Exterior-Function-Preserving Procedures

- Vertex grouping/splitting— closing/opening the box.

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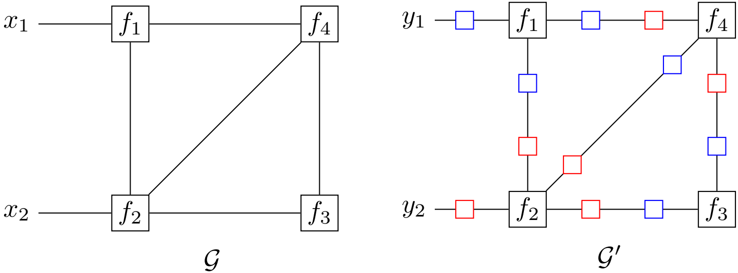
Exterior-Function-Preserving Procedures

- Vertex grouping/splitting— closing/opening the box.
- Equality insertion/deletion.

- Inverse-pair insertion/deletion.

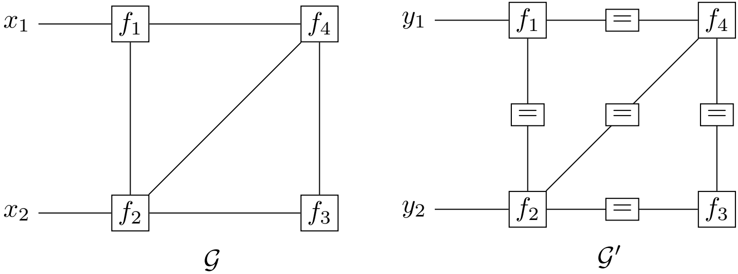
$$\sum_y A(x,y)B(x',y) = \delta_=(x,x').$$

In general



- $Z_{\mathcal{G}}$ is only altered be external transformations.
- Different kernels.

In general



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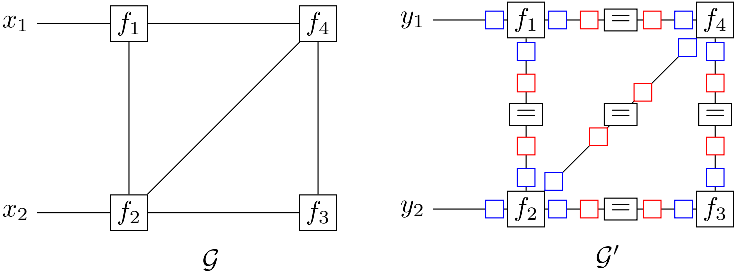
In general



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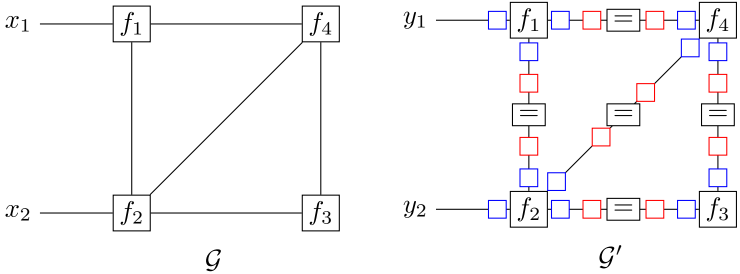
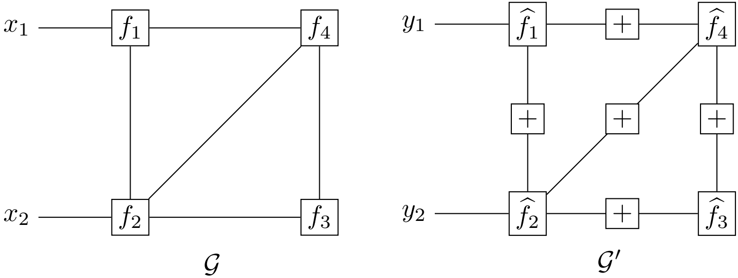


- $Z_{\mathcal{G}}$ is only altered be external transformations.
- Choose Fourier, then $Z_{\mathcal{G}'} = \hat{Z}_{\mathcal{G}}$

In general



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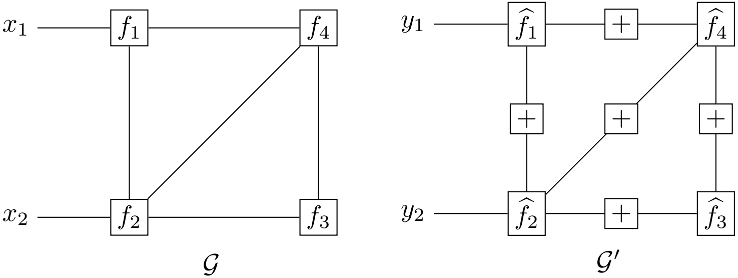
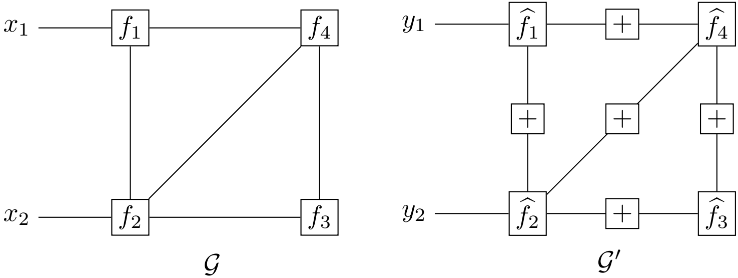
In general



- $Z_{\mathcal{G}}$ is only altered be external transformations.
- Choose Fourier, then $Z_{\mathcal{G}'} = \hat{Z}_{\mathcal{G}} = \delta_{\mathcal{C}^\perp}$
- $f_i := \delta_{\mathcal{C}_i} \Rightarrow \hat{f}_i = \delta_{\mathcal{C}_i^\perp}$

In general



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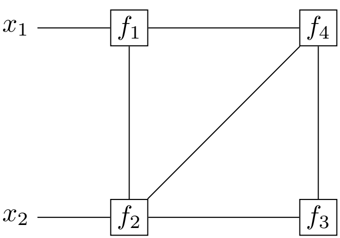
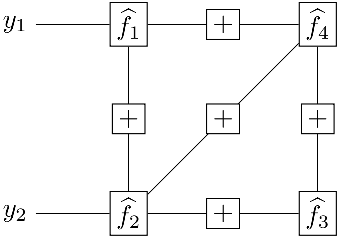
- Choose Fourier, then $Z_{\mathcal{G}'} = \hat{Z}_{\mathcal{G}} = \delta_{\mathcal{C}^\perp}$
- $f_i := \delta_{\mathcal{C}_i} \Rightarrow \hat{f}_i = \delta_{\mathcal{C}_i^\perp}$
- Forney's duality [A, Mao 2011], [Forney 2011]
- Applications?

In general

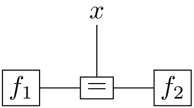


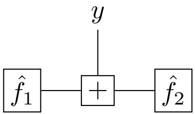
$\mathcal{G}$



$\mathcal{G}'$

- Choose Fourier, then $Z_{\mathcal{G}'} = \hat{Z}_{\mathcal{G}}$
- Convolutional FGs [Mao, Kschischang 2005]

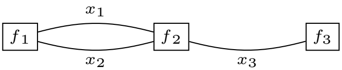




$$Z_{\mathcal{G}} = f_1(x)f_2(x) := (f_1 \cdot f_2)(x)$$

$$Z_{\mathcal{G}'} = \textstyle\sum_x \hat{f}_1(y-x)\hat{f}_2(x) := (\hat{f}_1 * \hat{f}_2)(y)$$

Holographic algorithms [Valiant 2004]

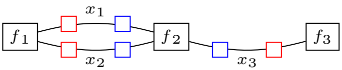


	00	01	10	11
f_1	0	1	1	0

	000	001	010	011	100	101	110	111
f_2	0	1	1	1	1	1	1	0

	0	1
f_3	1	1

Holographic algorithms [Valiant 2004]



	00	01	10	11
f_1	0	1	1	0

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f_2	0	1	1	1	1	1	1	0

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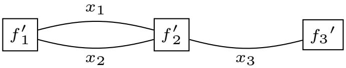
f_3	1	1

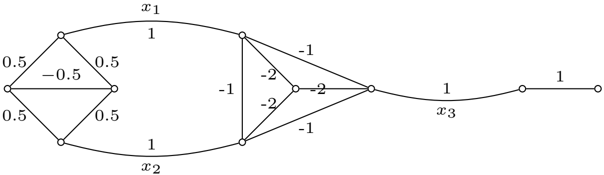
Holographic algorithms [Valiant 2004]



	00	01	10	11
f'_1	.5	0	0	-.5

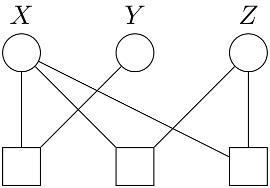
	000	001	010	011	100	101	110	111
f'_2	6	0	0	-2	0	-2	-2	0

	0	1
f'_3	1	0



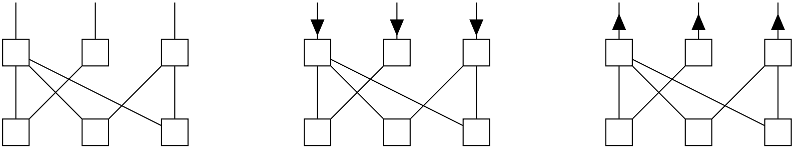
Probabilistic models

Some existing models



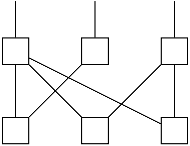
- FGs [Kschischang, Frey, Loeliger 2001].
- CFGs [Mao, Kschischang 2005]. CFGs as probabilistic model [Mao, Kschischang 2004].
- LCM [Bickson, Guestrin 2010], inference with heavy tail distributions.
- CDN [Huang, Frey 2011], ranking problems.
- Independence.

NFGs as probabilistic models



NFG model
Interface
Latent

NFGs as probabilistic models



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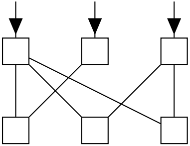
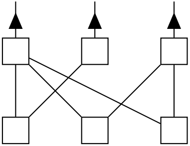


constrained
split



generative
conditional

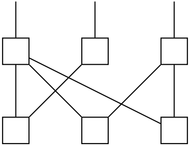
- Split function.

$$f(x_1, x_2, x_3) := f_1(x_1, x_2) f_2(x_1, x_3)$$

- Conditional function.

$$\sum_{x_1} g(x_1, x_2, x_3) = \text{constant}$$

NFGs as probabilistic models



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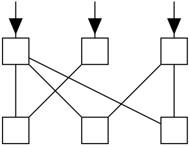
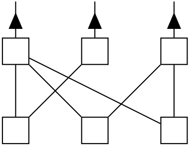


constrained
split
“shaping”



generative
conditional
“independent sources”

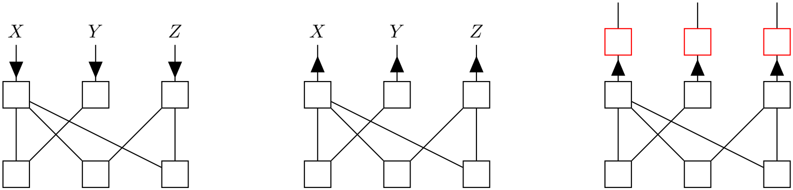
- Split function.

$$f(x_1, x_2, x_3) := f_1(x_1, x_2) f_2(x_1, x_3)$$

- Conditional function.

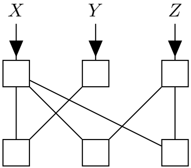
$$\sum_{x_1} g(x_1, x_2, x_3) = \text{constant}$$

NFGs probabilistic models



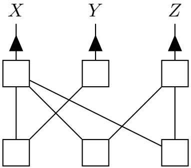
- constrained.
- generative.
- transformed.

NFGs probabilistic models



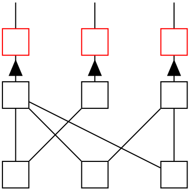
- constrained.
- conditional indep.

$$Y \perp\!\!\!\perp Z | X$$



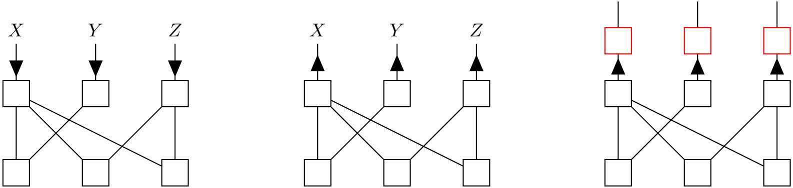
- generative.
- marginal indep.

$$Y \perp\!\!\!\perp Z$$



- transformed.
- respects indep.

NFGs probabilistic models

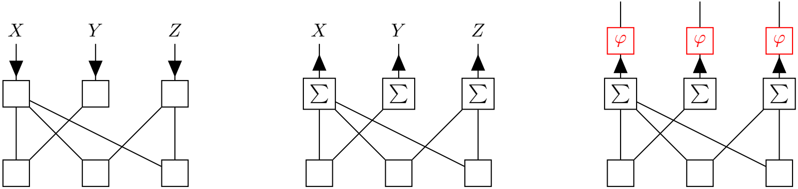


- constrained.
- conditional indep.
- FGs. [KFL2001]

- generative.
- marginal indep.

- transformed.
- respects indep.

NFGs probabilistic models

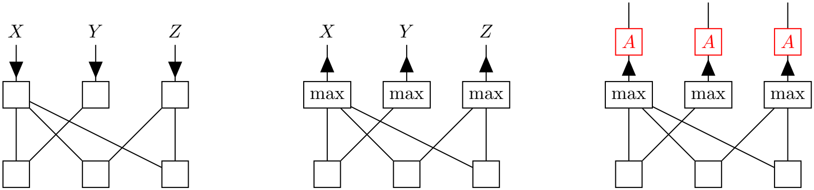


- constrained.
- conditional indep.
- FGs. [KFL2001]

- generative.
- marginal indep.
- CFGs. [MK2005]

- transformed.
- respects indep.
- LCM. [BG2010]

NFGs probabilistic models



- constrained.
- conditional indep.
- FGs. [KFL2001]

- generative.
- marginal indep.
- CFGs. [MK2005]
- SMM.

- transformed.
- respects indep.
- LCM. [BG2010]
- CDN. [HF2011]

SMM-CDN

- $\mathcal{X}$ ordered set.

$$A_{\mathcal{X}} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, D_{\mathcal{X}} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

- $\mathcal{X} := \mathcal{X}_1 \times \cdots \mathcal{X}_n$,

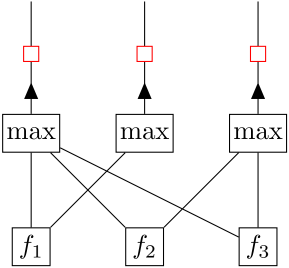
$$A_{\mathcal{X}}(x,y) := \prod_i A_{\mathcal{X}_i}(x_i,y_i), D_{\mathcal{X}}(x,y) := \prod_i D_{\mathcal{X}_i}(x_i,y_i)$$

- If f a probability distribution on $\mathcal{X}$, then $A \cdot f$ is its corresponding cumulative distribution. Conversely, ...
- Remark: $\mathcal{X} = \{0,1\}^n$, then $\mathcal{X} \leftrightarrow 2^N$, where $N = \{1,\ldots,n\}$. Mobius inversion

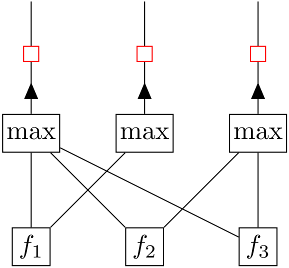
$$F = A \cdot f \Leftrightarrow f = D \cdot F$$

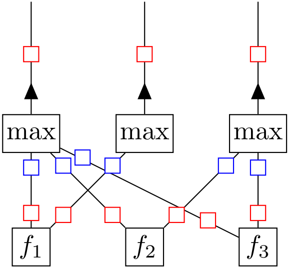
$$F(J) = \sum_{K \subseteq J} f(K), \forall J \subseteq N \Leftrightarrow f(K) = \sum_{J \subseteq K} (-1)^{|K \setminus J|} F(J), \forall K \subseteq N$$

SMM-CDN



SMM-CDN





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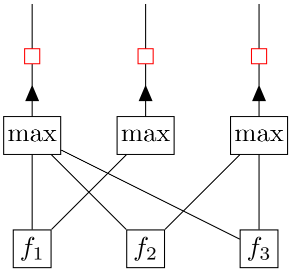
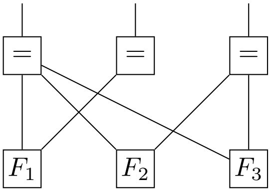
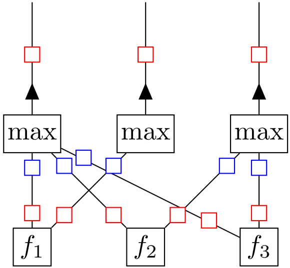
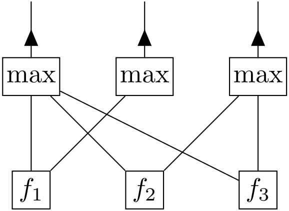






SMM-Group testing (GT)



- Specify to binary alphabets
- max becomes `OR`

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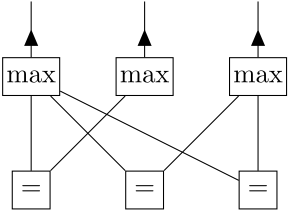

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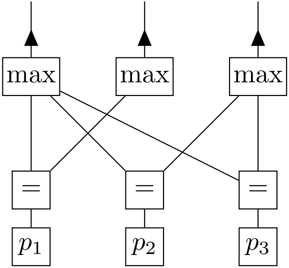

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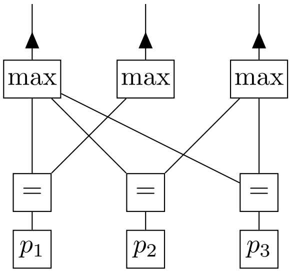

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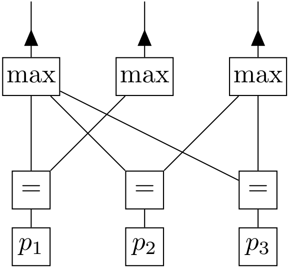

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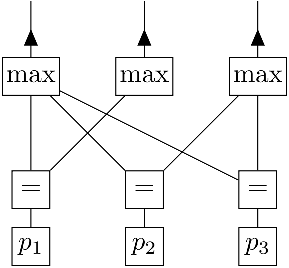

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- SMM reduces to GT: Given observed tests, infer the samples
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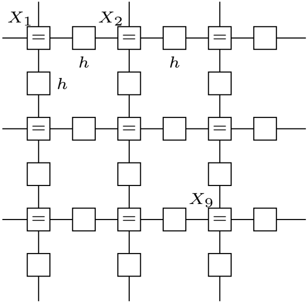
- Connections to CDNs
- [Wadayama, Izumi 2014] used an NFG approach to GT— Reduced complexity

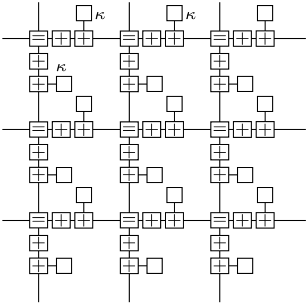
Stochastic estimation

Partition function estimation



- Stat. model.
- 2D-Nearest neighbour.
- $Z_{\mathcal{G}} := \sum_{\underline{x} \in \mathcal{X}^N} f_{\mathcal{G}}(\underline{x})$
- $p_{\mathcal{G}} := f_{\mathcal{G}} / Z_{\mathcal{G}}$

Partition function estimation



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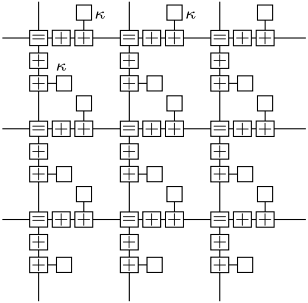
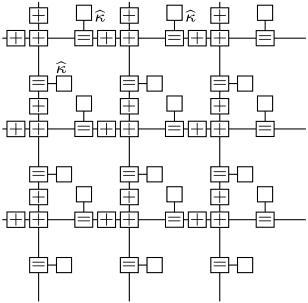
- Potts model
  $h(x, x') := \kappa(x - x')$,
  $\forall x, x' \in \mathbb{Z}_q$

  $$\kappa(x) := \begin{cases} e^{\beta}, & x = 0 \\ e^{-\beta}, & x \neq 0. \end{cases}$$
- $\beta := 1/kT$ inverse temperature.

Partition function estimation





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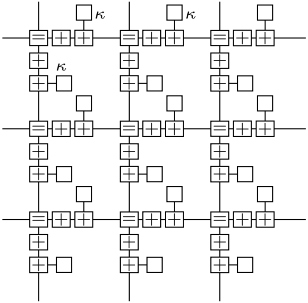
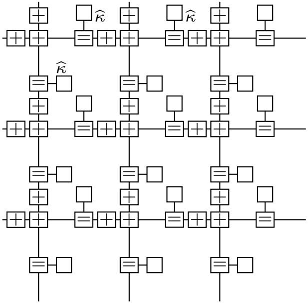
  $$\widehat{\kappa}(x) = \begin{cases} e^{\beta} + (q-1)e^{-\beta}, & x = 0 \\ e^{\beta} - e^{-\beta}, & x \neq 0. \end{cases}$$

Partition function estimation





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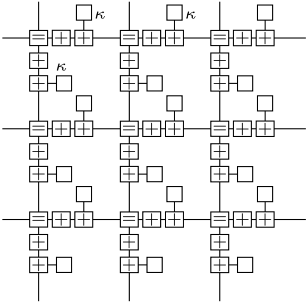
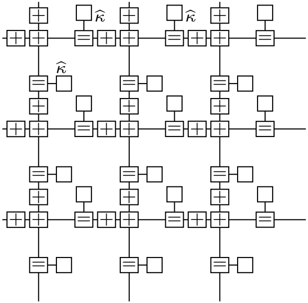
Partition function estimation





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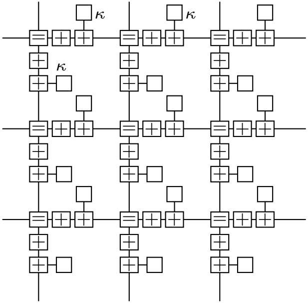
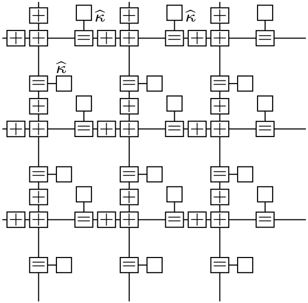
Partition function estimation





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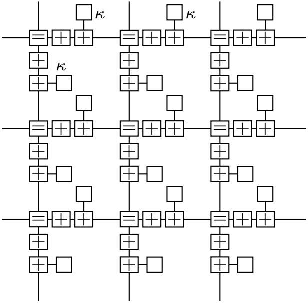
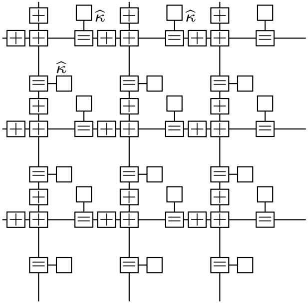
Partition function estimation





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- $\widehat{\kappa} > 0$.
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- $Z_{\mathcal{G}'} = Z_{\mathcal{G}}/q^N$
- [Molkaraie, Loeliger 2013]

Partition function estimation— Analysis

- Uniform and Ogata-Tanemura (OT) estimators.

$$Z_{\mathcal{G}}^{\text{unif}}(M) := \frac{|\mathcal{X}_{\mathcal{G}}|}{M} \sum_{i=1}^{M} f_{\mathcal{G}}(Y_i), \qquad Z_{\mathcal{G}}^{\text{OT}}(M) := \frac{|\mathcal{X}_{\mathcal{G}}|}{\frac{1}{M} \sum_{i=1}^{M} \frac{1}{f_{\mathcal{G}}(Y_i)}}$$

- Lower and upper bounds on $\lim_{M\to\infty} M \mathrm{Var}[\log(\tilde{Z}(M))]$
- Primal
  - At high temp (small β), UB's $\to 0$
  - At low temp (large β), LB's exponential in N
- Dual
  - At high temp (small β), LB's exponential in N
  - At low temp (large β), UB's $\to 0$
- Remark
  - Prim: There exists T_0 below which unif is better than OT[3]
  - Dual: There exists T_0' above which unif is better than OT

[3] [Potamianos and Goutsias, IT1997]

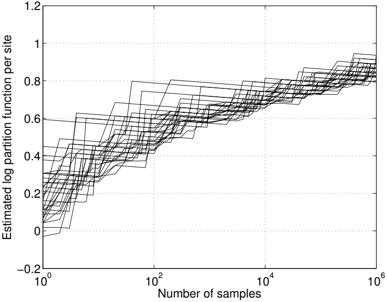
Partition function estimation— Experiments

"If it disagrees with experiment, its WRONG."[4]

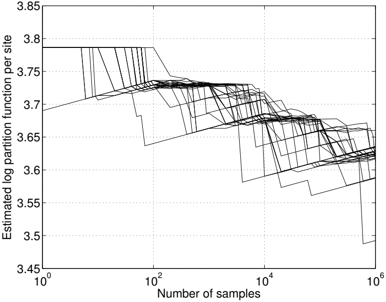
[4] R. Feynman

Partition function estimation— Experiments

Low temperature $\beta = 1.2$, $q = 4, N = 10 \times 10$



(c) Uniform

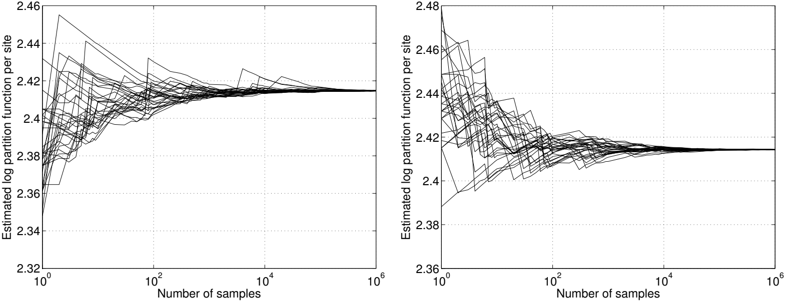


(d) OT

Primal

Partition function estimation— Experiments

Low temperature $\beta = 1.2$, $q = 4, N = 10 \times 10$

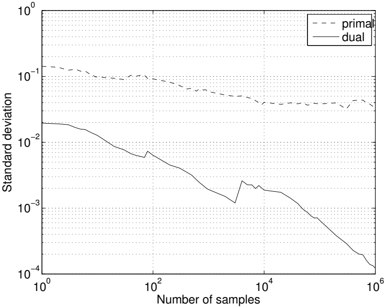


(a) Uniform (b) OT

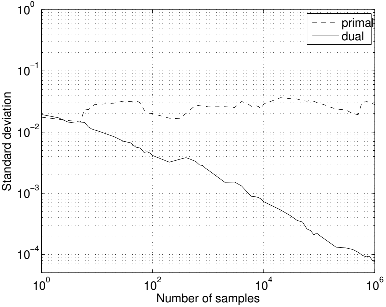
Dual

Partition function estimation— Experiments

Low temperature $\beta = 1.2$, $q = 4, N = 10 \times 10$



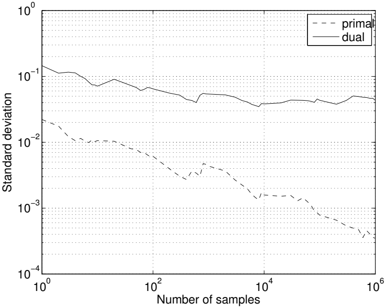
(a) Uniform



(b) OT

Partition function estimation— Experiments

High temperature $\beta = 0.18$, $q = 4, N = 10 \times 10$



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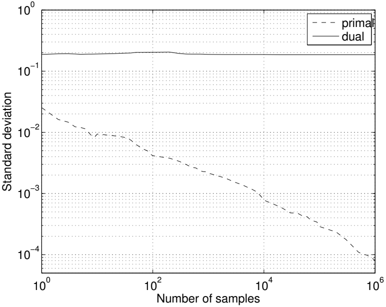
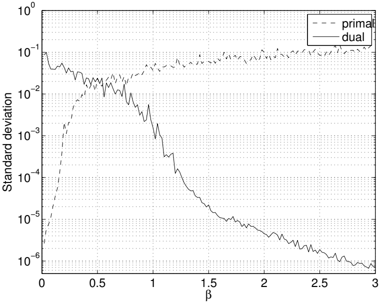


(b) OT

Partition function estimation— Experiments

10^6 uniform samples, $q = 4, N = 10 \times 10$



Partition function estimation— Experiments

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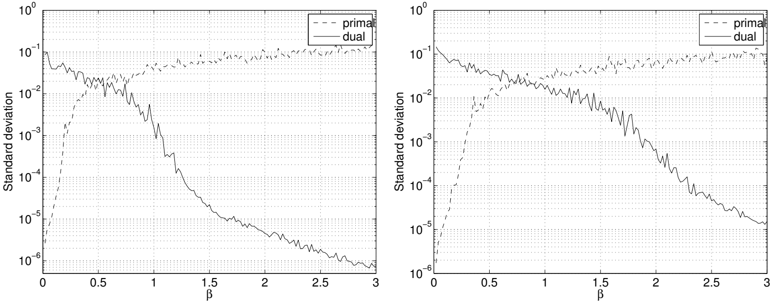


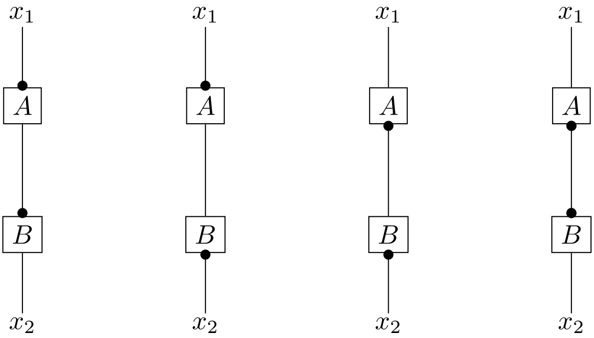
(a) Potts model (b) Clock model

$$\kappa_{\text{clock}}(x) = e^{\beta \cos(2\pi x/q)}, \forall x \in \mathbb{Z}_q.$$

Trace diagrams Skip.

NFGs-Trace diagrams.

- [Cvitanovic 2008], [Peterson 2009], [Morse, Peterson 2010]

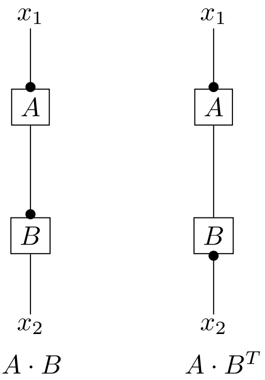


$A \cdot B$

$$\sum_t A(x_1, t)B(t, x_2)$$

NFGs-Trace diagrams.

- [Cvitanovic 2008], [Peterson 2009], [Morse, Peterson 2010]



$$\sum_t A(x_1, t)B(x_2, t)$$

NFGs-Trace diagrams.

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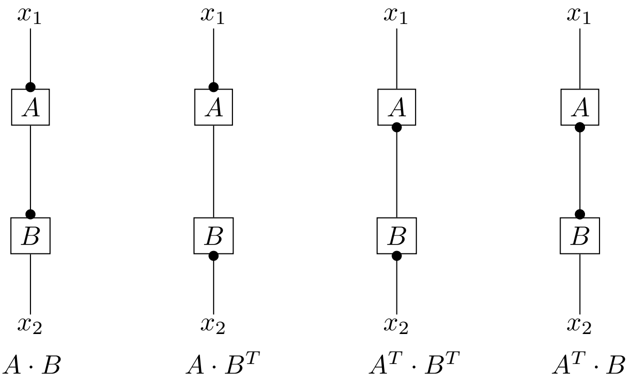
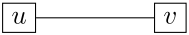


Product of vectors

Product of vectors

- Dot product.

$$Z = \sum_t u(t)v(t).$$



$$Z = u \cdot v.$$

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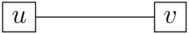


$$Z = u \cdot v.$$

- Levi-Civita

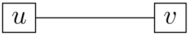
$$\varepsilon(x_1, \ldots, x_n) = \begin{cases} \operatorname{sgn} \begin{pmatrix} 1 & \cdots & n \\ x_1 & \cdots & x_n \end{pmatrix}, & \begin{pmatrix} 1 & \cdots & n \\ x_1 & \cdots & x_n \end{pmatrix} \in S_n \\ 0, & \text{otherwise} \end{cases}$$

$\forall \underline{x} \in \{1, \ldots, n\}^n.$

Product of vectors

- Dot product.

$$Z = \sum_t u(t)v(t).$$

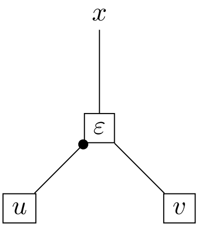


$$Z = u \cdot v.$$

- Cross Product. $u, v \in \mathbb{C}^3$

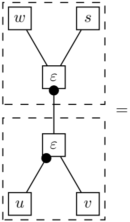
$$Z(x) = \sum_{t_1,t_2} u(t_1)v(t_2)\varepsilon(t_1,t_2,x).$$

$$Z(1) = u(2)v(3) - u(3)v(2)$$
$$Z(2) = u(3)v(1) - u(1)v(3)$$
$$Z(3) = u(1)v(2) - u(2)v(1)$$



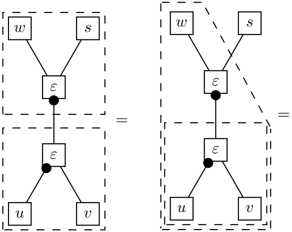
$$Z = u \times v.$$

Cross Product– Example



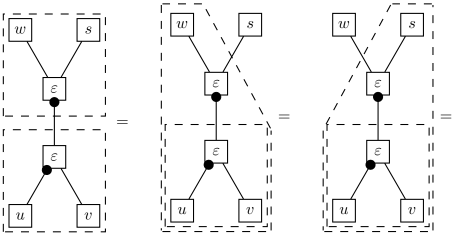
$$(u \times v) \cdot (s \times w)$$

Cross Product– Example



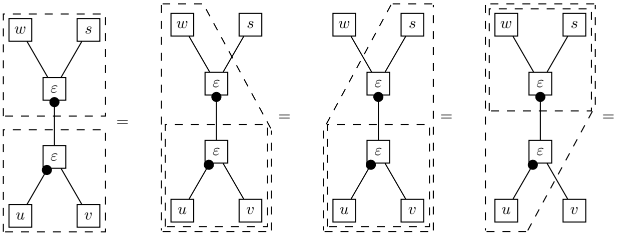
$$(u \times v) \cdot (s \times w) = (w \times (u \times v)) \cdot s$$

Cross Product– Example



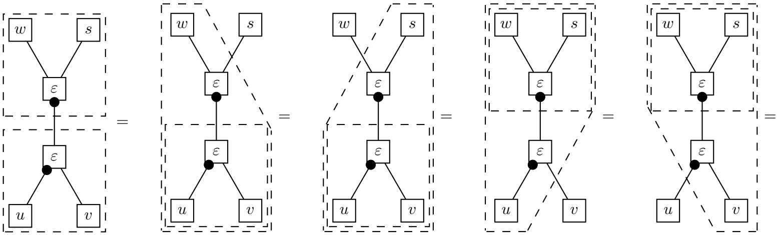
$$(u \times v) \cdot (s \times w) = (w \times (u \times v)) \cdot s = ((u \times v) \times s) \cdot w$$

Cross Product– Example



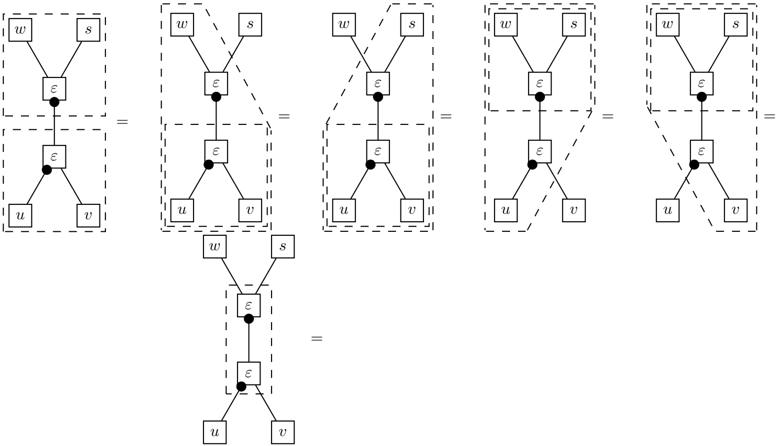
$$(u \times v) \cdot (s \times w) = (w \times (u \times v)) \cdot s = ((u \times v) \times s) \cdot w$$
$$= ((s \times w) \times u) \cdot v$$

Cross Product– Example



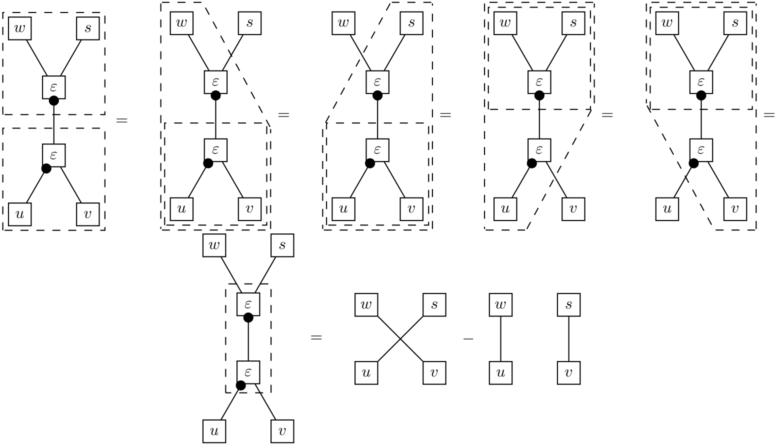
$$
\begin{aligned}
(u \times v) \cdot (s \times w) &= (w \times (u \times v)) \cdot s = ((u \times v) \times s) \cdot w \\
&= ((s \times w) \times u) \cdot v = (v \times (s \times w)) \cdot u
\end{aligned}
$$

Cross Product– Example



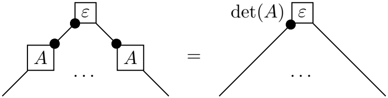
$$(u \times v) \cdot (s \times w) = (w \times (u \times v)) \cdot s = ((u \times v) \times s) \cdot w$$
$$= ((s \times w) \times u) \cdot v = (v \times (s \times w)) \cdot u$$

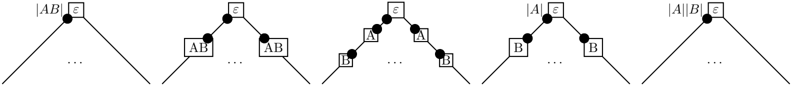
Cross Product– Example

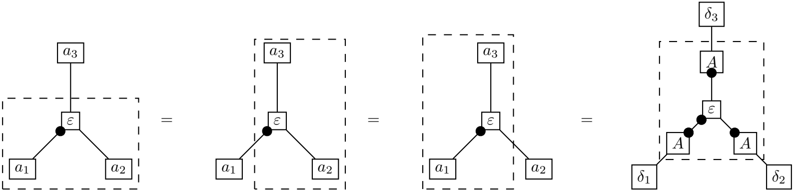


$$
\begin{aligned}
(u \times v) \cdot (s \times w) &= (w \times (u \times v)) \cdot s = ((u \times v) \times s) \cdot w \\
&= ((s \times w) \times u) \cdot v = (v \times (s \times w)) \cdot u \\
&= (u \cdot s)(v \cdot w) - (u \cdot w)(v \cdot s).
\end{aligned}
$$

Determinant







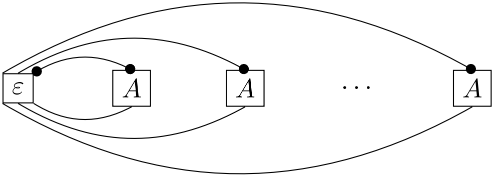
$$(a_1 \times a_2) \cdot a_3 = \quad (a_2 \times a_3) \cdot a_1 = \quad (a_3 \times a_1) \cdot a_2 = \quad \det(A)$$

Pfaffian

- A is a $2n \times 2n$ skew-symmetric, then
- Pfaffian of A, denoted $\mathrm{Pf}(A)$, is defined as

$$\mathrm{Pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A\big(\sigma(2i-1), \sigma(2i)\big).$$

- Then,

$\mathrm{Pf}(A) = \frac{1}{n!2^n}$ 

- Affirms a conjecture of [Peterson 2009]

Thank you

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