$M/G/1$ queues and Busy Cycle Analysis

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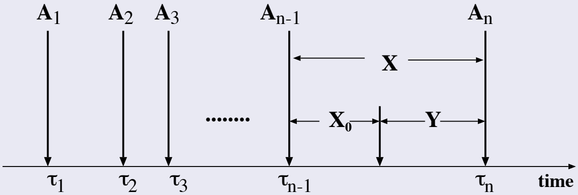
Outline

Another notation for transform

- Given a discrete r.v $\tilde{G}$ with $g_i = Prob[\tilde{g} = i]$
  1. $G(Z) = \sum_{i=0}^{\infty} g_i Z^i$
  2. $E[Z^{\tilde{G}}] = \sum_{i=0}^{\infty} Z^i Prob[\tilde{g} = i] = \sum_{i=0}^{\infty} Z^i g_i$ Therefore, $E[Z^{\tilde{G}}] = G(Z)$
- Given a continuous r.v $\tilde{X}$ with $f_{\tilde{X}}(x)$
  1. $F^*_{\tilde{X}}(s) = \int_{x=0}^{\infty} f_{\tilde{X}}(x) e^{-sx} dx$
  2. $E[e^{-s\tilde{X}}] = \int_{x=0}^{\infty} e^{-sx} f_{\tilde{X}}(x) dx$

Residual Life



- Interarrival time of bus is exponential w/ rate λ while hippie arrives at an arbitrary instant in time
- Question: How long must the hippie wait, on the average, till the bus comes along?

Answer

- Answer 1 : Because the average interarrival time is $\frac{1}{\lambda}$, therefore $\frac{1}{2\lambda}$
- Answer 2 : Because of memoryless, it has to wait $\frac{1}{\lambda}$
- General Result:

$$\begin{aligned} f_X(x)dx &= kxf(x)dx = \frac{xf(x)}{\int_0^\infty xf(x)dx} \\ f_Y(y) &= \frac{1-F(y)}{\int_0^\infty xf(x)dx} \\ F^*(s) &= \frac{1-F^*(s)}{m_1} \\ r_n &= \frac{m_{n+1}}{(n+1)m_1} \end{aligned}$$

Particularly, $r_1 = \frac{\bar{x^2}}{2\bar{x}}$

Derivation

$$\begin{aligned} P[x < X \leq x + dx] &= f_X(x)dx = kxf(x)dx \\ \int_{x=0}^{\infty} f_X(x)dx &= k \int_{x=0}^{\infty} xf(x)dx \Rightarrow 1 = km_1 \end{aligned}$$

Therefore,

$$f_X(x) = \frac{1}{m_1} xf(x)$$

$f_Y(y)$ = ?

$$\begin{aligned} P[Y \leq y | X = x] &= \frac{y}{x} \\ P[y < Y \leq y + dy, x < X \leq x + dx] &= \left(\frac{dy}{x}\right)\left(\frac{xf(x)}{m_1}\right) dx \end{aligned}$$

continue:

$$
\begin{aligned}
f_Y(y)dy &= \int_{x=y}^{\infty} P[y < Y \leq y + dy, x < X \leq x + dx] \\
&= \int_{x=y}^{\infty} (\frac{dy}{x})(\frac{xf(x)}{m_1})dx = \frac{1 - F(y)}{m_1} dy \\
f_Y(y) &= \frac{1 - F(y)}{m_1} \quad \text{since} \quad f(y) = \frac{dF(y)}{dy} \\
&= \frac{1 - F^*(s)}{sm_1}
\end{aligned}
$$

$M/G/1$

$$
\begin{aligned}
A(t) &= 1 - e^{-\lambda t} & t \geq 0 \\
a(t) &= \lambda e^{-\lambda t} & t \geq 0 \\
b(t) &= \text{general}
\end{aligned}
$$

- Describe the state

  $$[N(t), X_o(t)]$$

  $N(t)$: The no. of customers present at time t
  $X_o(t)$: Service time already received by the customer in service at time t (or remaining service time).
- Rather than using this approach, we use the method of the *imbedded Markov Chain*

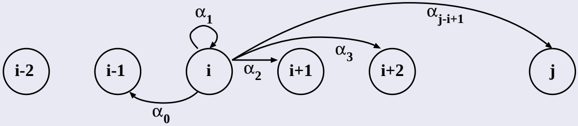
Imbedded Markov Chain $[N(t), X_o(t)]$

- Select the "departure" points, we therefore eliminate $X_o(t)$
- Now $N(t)$ is the no. of customers left behind by a departure customer.
  1. For Poisson arrival: $P_k(t) = R_k(t)$ for all time t. Therefore, $p_k = r_k$.
  2. If in any system (even it's non-Markovian) where $N(t)$ makes discontinuous changes in size(plus or minus)one, then

$$r_k = d_k = \text{Prob[departure leaves } k \text{ customers behind]}$$

- Therefore, for $M/G/1$,

$$p_k = d_k = r_k$$



α_k = Prob[k arrivals during the service of a customer]

$$P = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots \\ 0 & 0 & 0 & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

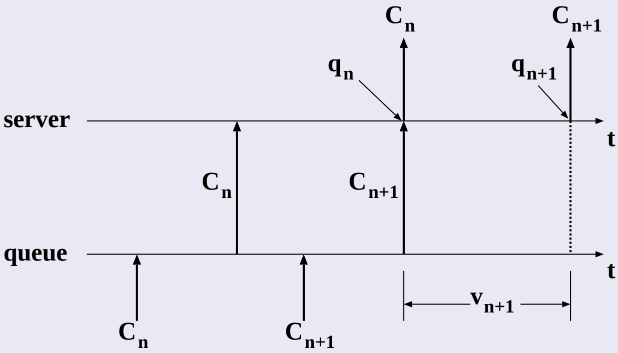
$$\alpha_k = P[\tilde{v} = k] = \int_0^\infty P[\tilde{v} = k | \tilde{x} = x] b(x) dx = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx$$

$\pi = \pi P$ and $\sum \pi_i = 1$. **Question:** why not $\pi Q = 0, \sum \pi_i = 1$?

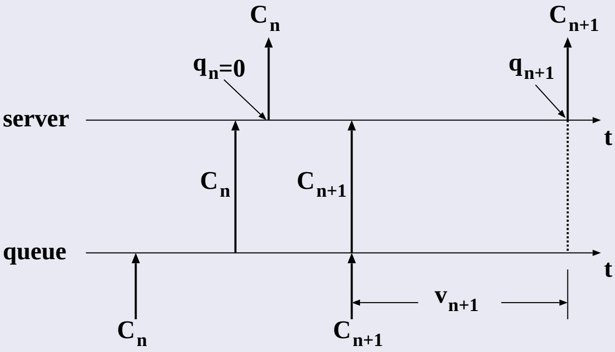
The mean queue length

We have two cases.

Case 1: $q_{n+1} = q_n - 1 + v_{n+1} \quad \text{for} \quad q_n > 0$



Case 2: $q_{n+1} = v_{n+1}$ for $q_n = 0$



Let $\Delta_k = \begin{cases} 1 & \text{for } k = 1, 2, \ldots \\ 0 & \text{for } k = 0 \end{cases}$

$$q_{n+1} = q_n - \Delta_{q_n} + v_{n+1}$$

$E[q_{n+1}] = E[q_n] - E[\Delta_{q_n}] + E[v_{n+1}]$

- Take the limit as $n \to \infty$, $E[\tilde{q}] = E[\tilde{q}] - E[\Delta_{\tilde{q}}] + E[\tilde{v}]$
- We get,

$$E[\Delta_{\tilde{q}}] = E[\tilde{v}] = \text{average no. of arrivals in a service time}$$

- On the other hand,

$$\begin{aligned} E[\Delta_{\tilde{q}}] &= \sum_{k=0}^{\infty} \Delta_k P[\tilde{q} = k] \\ &= \Delta_0 P[\tilde{q} = 0] + \Delta_1 P[\tilde{q} = 1] + \cdots \\ &= P[\tilde{q} > 0] \end{aligned}$$

- Therefore $E[\Delta_{\tilde{q}}] = P[\tilde{q} > 0]$. Since we are dealing with single server, it is also equal to P[busy system]=$\rho = \lambda/mu = \lambda\bar{x}$. Therefore,

$$E[\tilde{v}] = \rho$$

Since we have

$$
\begin{aligned}
q_{n+1} &= q_n - \Delta_{q_n} + v_{n+1} \\
q_{n+1}^2 &= q_n^2 + \Delta_{q_n}^2 + v_{n+1}^2 - 2q_n\Delta_{q_n} + 2q_n v_{n+1} - 2\Delta_{q_n} v_{n+1}
\end{aligned}
$$

Note that : $(\Delta_{q_n})^2 = \Delta_{q_n}$ and $q_n\Delta_{q_n} = q_n$

$$
\begin{aligned}
\lim_{n\to\infty} E[q_{n+1}^2] &= \lim_{n\to\infty} \{E[q_n^2] + E[\Delta_{q_n}^2] + E[v_{n+1}^2] - \\
&\qquad 2E[q_n] + 2E[q_n v_{n+1}] - 2E[\Delta_{q_n} v_{n+1}]\} \\
0 &= E[\Delta_{\tilde{q}}] + E[\tilde{v}^2] - 2E[\tilde{q}] + 2E[\tilde{q}]E[\tilde{v}] - 2E[\Delta_{\tilde{q}}]E[\tilde{v}] \\
E[\tilde{q}] &= \rho + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1-\rho)}
\end{aligned}
$$

Now the remaining question is how to find $E[\tilde{v}^2]$.

- Let $V(Z) = \sum_{k=0}^{\infty} P[\tilde{v} = k] Z^k$

$$
\begin{aligned}
V(Z) &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx Z^k \\
&= \int_0^{\infty} e^{-\lambda x} \left(\sum_{k=0}^{\infty} \frac{(\lambda x Z)^k}{k!} \right) b(x) dx \\
&= \int_0^{\infty} e^{-\lambda x} e^{\lambda x Z} b(x) dx \\
&= \int_0^{\infty} e^{-(\lambda - \lambda Z) x} b(x) dx
\end{aligned}
$$

- Look at $B^*(s) = \int_0^{\infty} e^{-sx} b(x) dx$. Therefore,

$$
V(Z) = B^*(\lambda - \lambda Z)
$$

- From this, we can get $E[\tilde{v}], E[\tilde{v}^2], \ldots$

$$
\begin{aligned}
\frac{dV(Z)}{dZ} = \frac{dB^*(\lambda - \lambda Z)}{dZ} &= \frac{dB^*(\lambda - \lambda Z)}{d(\lambda - \lambda Z)} \bullet \frac{d(\lambda - \lambda Z)}{dZ} \\
&= -\lambda \frac{dB^*(y)}{dy} \\
\frac{dV(Z)}{dZ} \bigg|_{Z=1} &= -\lambda \frac{dB^*(y)}{dy} \bigg|_{y=0} = +\lambda \bar{x} = \rho
\end{aligned}
$$

$\frac{d^2V(Z)}{dZ^2} = \bar{v^2} - \bar{v}$, since $V(Z) = B^*(\lambda - \lambda Z)$

$$\begin{aligned}
\frac{d^2V(Z)}{dZ^2} &= \frac{d}{dZ}[-\lambda \frac{dB^*(y)}{dy}] = -\lambda \frac{d^2B^*(y)}{dy^2}\frac{dy}{dZ} \\
\frac{d^2V(Z)}{dZ^2}|_{Z=1} &= \lambda^2 \frac{dB^{2*}(y)}{dy^2}|_{y=0} = \lambda^2 B^{*(2)}(0)
\end{aligned}$$

$$\bar{v^2} - \bar{v} = \lambda^2 \bar{x^2} \quad \Rightarrow \quad \bar{v^2} = \bar{v} + \lambda^2 \bar{x^2}$$

Go back, since

$$\begin{aligned}
E[\tilde{q}] &= \rho + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1-\rho)} \\
E[\tilde{q}] &= \rho + \frac{\lambda^2 \bar{x^2}}{2(1-\rho)} = \rho + \rho^2 \frac{(1 + C_b^2)}{2(1-\rho)}
\end{aligned}$$

This is the famous **Pollaczek - Khinchin Mean Value Formula.**

- For M/M/1, $b(x) = \mu e^{-\mu x}, \bar{x} = \frac{1}{\mu}; \bar{x^2} = \frac{2}{\mu^2}$

$$\begin{aligned}\bar{q} &= \rho + \frac{\lambda^2 \bar{x^2}}{2(1-\rho)} = \rho + \frac{2\frac{\lambda^2}{\mu^2}}{2(1-\rho)} = \rho + \rho^2 \frac{2}{2(1-\rho)} \\ \bar{q} &= \frac{\rho}{1-\rho} = \bar{N} \text{ in } M/M/1\end{aligned}$$

- For $M/D/1, \bar{x} = x; \bar{x^2} = x^2$

$$\bar{q} = \rho + \rho^2 \frac{1}{2(1-\rho)} = \frac{\rho}{1-\rho} - \frac{\rho^2}{2(1-\rho)}$$

$\rightarrow$ It's less than $M/M/1$!

- For $M/H_2/1$, let $b(x) = \frac{1}{4}\lambda e^{-\lambda x} + \frac{3}{4}(2\lambda)e^{-2\lambda x}; \bar{x} = \frac{5}{8\lambda}; \bar{x^2} = \frac{56}{64\lambda^2}$

$$\bar{q} = \rho + \frac{\frac{56}{64}}{2(1-\rho)} \text{ where } \rho = \lambda \bar{x} = \frac{5}{8}; \bar{q} = 1.79$$

Distribution of Number in the System

$$
\begin{aligned}
q_{n+1} &= q_n - \Delta_{q_n} + v_{n+1} \\
Z^{q_{n+1}} &= Z^{q_n - \Delta_{q_n} + v_{n+1}} \\
E[Z^{q_{n+1}}] &= E[Z^{q_n - \Delta_{q_n} + v_{n+1}}] = E[Z^{q_n - \Delta_{q_n}} \cdot Z^{v_{n+1}}]
\end{aligned}
$$

Taking limit as $n \to \infty$

$$
\begin{aligned}
Q(Z) &= E[Z^{q-\Delta_q}] \cdot E[Z^v] = E[Z^{q-\Delta_q}] V(Z) \to (1) \\
E[Z^{q-\Delta_q}] &= Z^{0-0}\text{Prob}[q=0] + \sum_{k=1}^{\infty} Z^{k-1}\text{Prob}[q=k] \\
&= Z^0\text{Prob}[q=0] + \frac{1}{Z}[Q(Z) - P[q=0]] \\
&= \text{Prob}[q=0] + \frac{1}{Z}[Q(Z) - P[q=0]]
\end{aligned}
$$

Continue

Putting them together, we have:

$$Q(Z) \quad = \quad V(Z)(\text{Prob}[q=0] + \frac{1}{Z}[Q(Z) - P[q=0]])$$

But $P[q=0] = 1 - \rho$, we have:

$$Q(Z) = V(Z)[\frac{(1-\rho)(1-\frac{1}{Z})}{1 - \frac{V(Z)}{Z}}] = B^*(\lambda - \lambda Z)[\frac{(1-\rho)(1-Z)}{B^*(\lambda - \lambda Z) - Z}]$$

This is the famous **P-K Transform equation.**

Example

$$
\begin{aligned}
Q(Z) &= B^*(\lambda - \lambda Z)\frac{(1-\rho)(1-Z)}{B^*(\lambda-\lambda Z) - Z} \\
\text{For } M/M/1 &: \quad B^*(s) = \frac{\mu}{s+\mu} \\
Q(Z) &= \left(\frac{\mu}{\lambda - \lambda Z + \mu}\right)\frac{(1-\rho)(1-Z)}{\left[\frac{\mu}{(\lambda-\lambda Z+\mu)}\right] - Z} = \frac{1-\rho}{1-\rho Z} = \frac{(1-\rho)}{1-\rho Z}
\end{aligned}
$$

Therefore,

$$P[\bar{q} = k] = (1-\rho)\rho^k \qquad k \geq 0$$

This is the same as

$$P[\tilde{N} = k] = (1-\rho)\rho^k$$

Continue

$$
\begin{aligned}
Q(Z) &= B^*(\lambda - \lambda Z)\frac{(1-\rho)(1-Z)}{B^*(\lambda-\lambda Z) - Z} \\
\text{For } \; M/H_2/1 \quad &: \quad B^*(s) = \frac{1}{4}\frac{\lambda}{s+\lambda} + \frac{3}{4}\frac{2\lambda}{s+2\lambda} = \frac{7\lambda s + 8\lambda^2}{4(s+\lambda)(s+2\lambda)} \\
Q(Z) &= \frac{(1-\rho)(1-z)[8+7(1-z)]}{8+7(1-z)-4z(2-z)(3-z)} \\
&= \frac{(1-\rho)[1-\frac{7}{15}z]}{[1-\frac{2}{5}z][1-\frac{2}{3}z]} = (1-\rho)\left[\frac{\frac{1}{4}}{1-\frac{2}{5}z} + \frac{\frac{3}{4}}{1-\frac{2}{3}z}\right]
\end{aligned}
$$

Where $\rho = \lambda\bar{x} = \frac{5}{8}$

$$
P_k = Prob[\tilde{q} = k] = \frac{3}{32}\left(\frac{2}{5}\right)^k + \frac{9}{32}\left(\frac{2}{3}\right)^k \qquad k = 0, 1, 2, \cdots
$$

We know $q_{n+1} = q_n - \Delta_{q_n} + v_{n+1}$. From the r.v. equation, we derived:

$$\bar{q} \quad = \quad \rho + \frac{\lambda^2 \overline{x^2}}{2(1-\rho)} = \rho + \rho^2 \frac{(1+C_b^2)}{2(1-\rho)}, \tag{1}$$

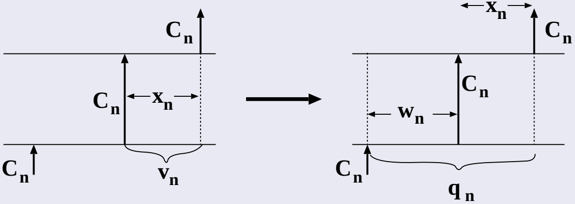
where $C_b^2 = \frac{\sigma_b^2}{\bar{x}^2}$

$$\begin{aligned} Q(Z) &= V(Z)\frac{(1-\rho)(1-\frac{1}{Z})}{1-\frac{V(Z)}{Z}} \\ &= \frac{B^*(\lambda-\lambda Z)(1-\rho)(1-Z)}{B^*(\lambda-\lambda Z)-Z} \qquad (2) \\ \text{because} \quad V(Z) &= B^*(\lambda-\lambda Z) \end{aligned}$$

$$Q(Z) \quad = \quad S^*(\lambda-\lambda Z) = B^*(\lambda-\lambda Z)\frac{(1-\rho)(1-Z)}{B^*(\lambda-\lambda Z)-Z} \tag{3}$$

$$\rightarrow \quad W^*(s) = \frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)}$$

Distribution of Waiting Time



$$V^*(z) = B^*(\lambda - \lambda z) \qquad Q(z) = S^*(\lambda - \lambda z)$$

$$S^*(\lambda - \lambda Z) = B^*(\lambda - \lambda Z)\frac{(1-\rho)(1-Z)}{B^*(\lambda - \lambda Z) - Z}$$

Let $s = \lambda - \lambda z$, then $z = 1 - \frac{s}{\lambda}$

$$S^*(s) = B^*(s)\frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)}$$

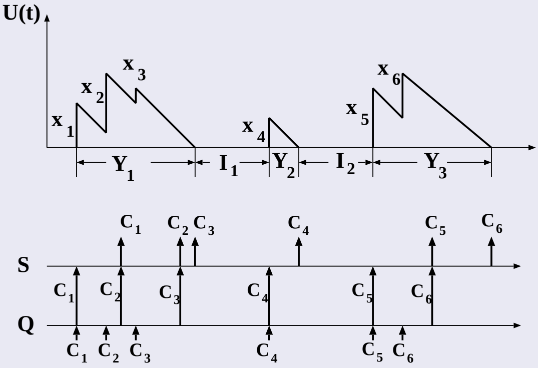
What is $W^*(s)$?

For $M/M/1$,

$$
\begin{aligned}
S^*(s) &= B^*(s)\frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)} \qquad B^*(s) = \frac{\mu}{s+\mu} \\
&= [\frac{\mu}{s+\mu}][\frac{s(1-\rho)}{s-\lambda+\lambda\frac{\mu}{s+\mu}}] \\
&= [\mu][\frac{s(1-\rho)}{s^2+s\mu-s\lambda}] \\
&= \frac{s\mu(1-\rho)}{s[s+\mu-\lambda]} = \frac{\mu(1-\rho)}{s+\mu-\lambda} \\
&= \frac{\mu(1-\rho)}{s+\mu(1-\rho)} \\
s(y) &= \mu(1-\rho)e^{-\mu(1-\rho)y} \qquad y \geq 0 \\
S(y) &= 1-e^{-\mu(1-\rho)y} \qquad y \geq 0
\end{aligned}
$$

$$
\begin{aligned}
W^*(s) &= \frac{s(1-\rho)}{s-\lambda+\lambda\frac{\mu}{s+\mu}} = \frac{s(1-\rho)(s+\mu)}{s^2+s\mu-s\lambda} \\
&= \frac{s(1-\rho)(s+\mu)}{s[s+\mu-\lambda]} = \frac{(1-\rho)(s+\mu)}{s+\mu-\lambda} \\
&= (1-\rho)+\frac{\lambda(1-\rho)}{s+\mu-\lambda} = (1-\rho)+\frac{\lambda(1-\rho)}{s+\mu(1-\rho)} \\
w(y) &= (1-\rho)\mu_0(y)+\lambda(1-\rho)e^{-\mu(1-\rho)y} \qquad y \geq 0 \\
W(y) &= 1-\rho e^{-\mu(1-\rho)y} \qquad y \geq 0
\end{aligned}
$$

- Let $U(t)$ =the unfinished work in the system at time t



- Y_i are the i^{th} busy period; I_i is the i^{th} idle period.
- The function $U(t)$ is INDEPENDENT of the order of service!!! The only requirement to this statement hold is the server remains busy where there is job.

- For $M/G/1$

$$
\begin{aligned}
A(t) &= P[t_n \leq t] = 1 - e^{-\lambda t} \quad t \geq 0 \\
B^*(x) &\Leftrightarrow P[X_n \leq x]
\end{aligned}
$$

- Let

$$
\begin{aligned}
F(y) &= P[I_n \leq y] \\
&= \text{idle-period distribution} \\
G(y) &= P[Y_n \leq y] \\
&= \text{busy-period distribution}
\end{aligned}
$$

$$
F(y) = 1 - e^{-\lambda t} \qquad t \geq 0
$$

- $G(y)$ is not that trivial! Well, thanks to Takacs, he came to the rescue.

The Busy Period

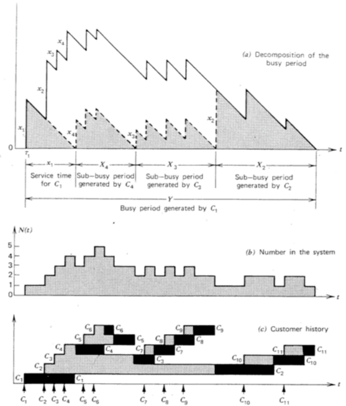


Figure 5.11 The busy period: last-come-first-served

- The busy period is independent of order of service
- Each sub-busy period behaves statistically in a fashion identical to the major busy period.

- The duration of busy period Y, is the sum of $1+\tilde{v}$ random variables where
$$Y = x_1 + X_{\tilde{v}} + \cdots + X_1$$
where x_1 is the service time of C_1, $X_{\tilde{v}}$ is the $(\tilde{v}+1)^{th}$ sub-busy period and $\tilde{v}$ is the r.v. of the number of arrival during the service of C_1.
- Let $G(y) = P[Y \leq y]$ and $G^*(s) = \int_0^\infty e^{-sy} dG(y) = E[e^{-sY}]$

$$\begin{aligned}
E[e^{-sY}|x_1 = x, \tilde{v} = k] &= E[e^{-s(x+X_{k+1}+X_k+\cdots+X_2)}] \\
&= E[e^{-sx}]E[e^{-sX_{k+1}}]E[e^{-sX_k}]\cdot\cdot E[e^{-sX_2}] \\
&= e^{-sx}[G^*(s)]^k \\
E[e^{-sY}|x_1 = x] &= \sum_{k=0}^{\infty} E[e^{-sY}|x_1 = x, \tilde{v} = k] P[\tilde{v} = k]
\end{aligned}$$

$$
\begin{aligned}
E[e^{-sY}|x_1 = x] &= \sum_{k=0}^{\infty} e^{-sx}[G^*(s)]^k \frac{(\lambda x)^k}{k!} e^{-\lambda x} \\
&= e^{-x[s+\lambda-\lambda G^*(s)]} \\
E[e^{-sY}] = G^*(s) &= \int_0^{\infty} E[e^{-sY}|x_1 = x] dB(x) \\
&= \int_0^{\infty} e^{-x[s+\lambda-\lambda G^*(s)]} dB(x)
\end{aligned}
$$

Therefore, we have

$$
G^*(s) = B^*[s + \lambda - \lambda G^*(s)]
$$

$G^*(s) = B^*[s+\lambda-\lambda G^*(s)]$
Since,

$$
\begin{aligned}
g_k &= E[Y^k] = (-1)^k G^{*(k)}(0) \quad \text{and} \quad \bar{x^k} = (-1)^k B^{*(k)}(0) \\
g_1 &= (-1)G^{*(1)}(0) = -B^{*(1)}(0)\frac{d}{ds}[s+\lambda-\lambda G^*(s)]|_{s=0} \\
&= -B^{*(1)}(0)[1-\lambda G^{*(1)}(0)] \\
g_1 &= \bar{x}(1+\lambda g_1)
\end{aligned}
$$

- Therefore $g_1 = \frac{\bar{x}}{1-p}$ where $\rho = \lambda\bar{x}$
- The average length of busy period for $M/G/1$ is equal to the average time a customer spends in an $M/M/1$ system

$$
g_2 = G^{*(2)}(s)|_{s=0} = \frac{d}{ds}[B^{*(1)}[s+\lambda-\lambda G^*(s)][1-\lambda G^{*(1)}(s)]|_{s=0}
$$

$$
= B^{*(2)}(0)[1-\lambda G^{*(1)}(0)]^2 + B^{*(1)}(0)[-\lambda G^{*(2)}(0)] = \frac{\bar{x^2}}{(1-\rho)^3}
$$

The number of customers served in a busy period

- Let N_{bp} = r.v. of no. of customers served in a busy period.

$$
\begin{aligned}
f_n &= Prob[N_{bp} = n] \\
F(Z) &= E[Z^{N_{bp}}] = \sum_{n=1}^{\infty} f_n Z^n
\end{aligned}
$$

$$
\begin{aligned}
E[Z^{N_{bp}} | \tilde{v} = k] &= E[Z^{1+M_k+M_{k-1}+\cdots+M_1}] \\
\text{(where } M_i &= \text{no. of customers served in the } i^{th} \text{ sub-busy period)} \\
E[Z^{N_{bp}} | \tilde{v} = k] &= E[Z]E[Z^{M_k}] \cdots E[Z^{M_1}] = E[Z](E[Z^{M_i}])^k \\
&= Z[F(Z)]^k
\end{aligned}
$$

Continue

$$
\begin{aligned}
F(Z) &= \sum_{k=0}^{\infty} E[Z^{N_{bp}} | \tilde{v} = k] P[\tilde{v} = k] \\
&= \sum_{k=0}^{\infty} Z[F(Z)]^k P[\tilde{v} = k] \\
&= ZV[F(Z)]
\end{aligned}
$$

$$
\Rightarrow F(Z) = ZB^*(\lambda - \lambda F(Z))
$$