PROBLEM 4.11.

Show that the solution to Eq. (4.71) gives a set of variables $\{x_i\}$ which guarantee that Eq. (4.72) is indeed the solution to Eq. (4.69).

SOLUTION

We proceed by showing that for Eq. (4.69) the left-hand side (LHS) equals the right-hand side (RHS) under the proposed solution Eq. (4.72) if the $\{x_i\}$ satisfy Eq. (4.71). Noting that $\delta_{k_i-1}\alpha_i(k_i) = \alpha_i(k_i)$ for each i, the LHS can be written as

$$\text{LHS} = p(k_1, \ldots, k_N) \sum_{i=1}^{N} \alpha_i(k_i)\mu_i .$$

The RHS is

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{k_j-1}\alpha_i(k_i+1)\mu_i r_{ij} p(k_1, \ldots, k_j-1, \ldots, k_i+1, \ldots, k_N) .$$

Using the proposed solution

$$p(k_1, \ldots, k_N) = \frac{1}{G(K)} \prod_{l=1}^{N} \frac{x_l^{k_l}}{\beta_l(k_l)}$$

we may write

$$p(k_1, \ldots, k_j-1, \ldots, k_i+1, \ldots, k_N) = \frac{1}{G(K)} \prod_{l=1}^{N} \frac{x_l^{k_l}}{\beta_l(k_l)} \cdot \frac{\dfrac{x_i}{x_j}}{\dfrac{\beta_i(k_i+1)}{\beta_i(k_i)} \cdot \dfrac{\beta_j(k_j-1)}{\beta_j(k_j)}}$$

$$= p(k_1, \ldots, k_N) \frac{x_i}{x_j} \cdot \frac{\beta_i(k_i)}{\beta_i(k_i+1)} \cdot \frac{\beta_j(k_j)}{\beta_j(k_j-1)}$$

Since $\beta_i(k_i+1) = \beta_i(k_i)\,\alpha_i(k_i+1)$ and $\beta_j(k_j) = \beta_j(k_j-1)\,\alpha_j(k_j)$ we have

$$p(k_1, \ldots, k_j-1, \ldots, k_i+1, \ldots, k_N) = p(k_1, \ldots, k_N) \frac{x_i}{x_j} \cdot \frac{\alpha_j(k_j)}{\alpha_i(k_i+1)}$$

Thus the RHS may be expressed as

$$\text{RHS} = \sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{k_j-1}\alpha_i(k_i+1)\mu_i r_{ij} p(k_1, \ldots, k_N) \frac{x_i}{x_j} \cdot \frac{\alpha_j(k_j)}{\alpha_i(k_i+1)}$$

$$= p(k_1, \ldots, k_N) \sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{k_j-1}\alpha_j(k_j)\mu_i r_{ij} \frac{x_i}{x_j}$$

But since $\delta_{k_j-1}\alpha_j(k_j) = \alpha_j(k_j)$,

$$\text{RHS} = p(k_1, \ldots, k_N) \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_j(k_j)\mu_i r_{ij} \frac{x_i}{x_j}$$

Thus the LHS equals the RHS if

$$\sum_{i=1}^{N} \alpha_i(k_i)\mu_i = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_j(k_j)\mu_i r_{ij} \frac{x_i}{x_j} .$$

We prove this last equality by applying Eq. (4.71). That is,

$$\mu_i = \sum_{j=1}^{N} \mu_j r_{ji} \frac{x_j}{x_i} \qquad i = 1, 2, \ldots, N .$$

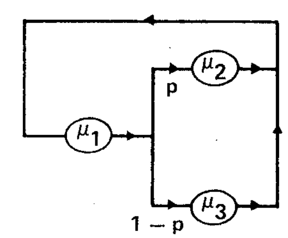
Multiplying each of these N equations by $\alpha_i(k_i)$, summing on i, and then interchanging the indices i and j gives

$$\sum_{i=1}^{N} \alpha_i(k_i)\mu_i = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_j(k_j)\mu_i r_{ij} \frac{x_i}{x_j}$$

which was to be shown. This completes the proof.

PROBLEM 4.12.

(a) Draw the state-transition-rate diagram showing local balance for the case ($N = 3, K = 5$) with the following structure:



(b) Solve for $p(k_1, k_2, k_3)$.

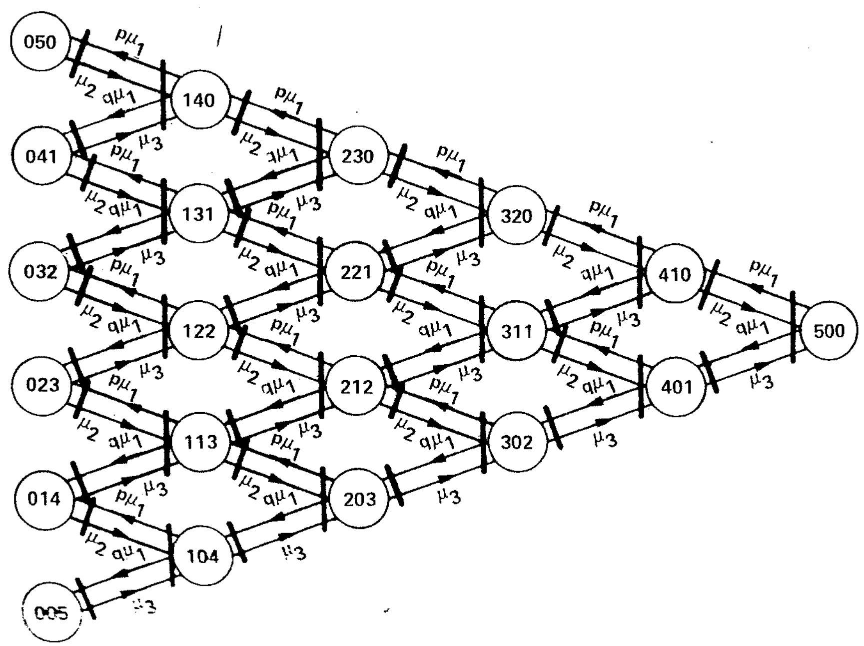
SOLUTION

The state of the system is represented by the circled triplet

$$k_1, k_2, k_3$$

where k_i = the number in the ith node ($i = 1, 2, 3$) and where the k_i must satisfy $k_1 + k_2 + k_3 = K = 5$. Let $q = 1 - p$. Recall that a local balance equation (with respect to a given network state and a network node i) equates the rate of flow out of that network state due to the departure of a customer from node i to the rate of flow into that network state due to the arrival of a customer to node i. In the state-transition-rate diagram below, a set of flows to be balanced in a local balance equation is joined by a heavy black line.

(a)



(b) The local balance equations among states for which $k_3 = 0$ are:

$$\mu_2 p(050) = \mu_1 p\, p(140)$$
$$\mu_2 p(140) = \mu_1 p\, p(230)$$
$$\mu_2 p(230) = \mu_1 p\, p(320)$$
$$\mu_2 p(320) = \mu_1 p\, p(410)$$
$$\mu_2 p(410) = \mu_1 p\, p(500)$$

which give

$$p(5-k, k, 0) = \left(\frac{\mu_1 p}{\mu_2}\right)^k p(500) \quad \text{for} \quad k = 1, 2, 3, 4, 5$$

The equations when $k_2 = 0$ are:

$$\mu_3 p(005) = \mu_1 q\, p(104)$$
$$\mu_3 p(104) = \mu_1 q\, p(203)$$
$$\mu_3 p(203) = \mu_1 q\, p(302)$$
$$\mu_3 p(302) = \mu_1 q\, p(401)$$
$$\mu_3 p(401) = \mu_1 q\, p(500).$$

which give

$$p(5-k, 0, k) = \left(\frac{\mu_1 q}{\mu_3}\right)^k p(500) \quad \text{for} \quad k = 1, 2, 3, 4, 5$$

Similarly, when $k_3 = 1$ we obtain

$$p(4-k, k, 1) = \left(\frac{\mu_1 p}{\mu_2}\right)^k \left(\frac{\mu_1 q}{\mu_3}\right) p(500) \quad \text{for} \quad k = 1, 2, 3, 4$$

and when $k_2 = 1$ we find

$$p(4-k, 1, k) = \left(\frac{\mu_1 p}{\mu_2}\right)\left(\frac{\mu_1 q}{\mu_3}\right)^k p(500) \quad \text{for} \quad k = 1, 2, 3, 4$$

Finally, we get

$$p(k_1, k_2, k_3) = \left(\frac{\mu_1 p}{\mu_2}\right)^{k_2}\left(\frac{\mu_1 q}{\mu_3}\right)^{k_3} p(500).$$

To find $p(500)$, we use $\sum_{k_1+k_2+k_3=5} p(k_1, k_2, k_3) = 1$. We may eliminate k_1 by observing that for any $0 \leqslant k_2 \leqslant 5$, $0 \leqslant k_3 \leqslant 5 - k_2$ we must have $k_1 = 5 - k_3 - k_2$. Thus

$$\sum_{k_1+k_2+k_3=5} p(k_1, k_2, k_3) = \sum_{k_2=0}^{5} \sum_{k_3=0}^{5-k_2} p(5-k_2-k_3, k_2, k_3)$$

and so

$$p(500) = \frac{1}{\displaystyle\sum_{k_2=0}^{5} \sum_{k_3=0}^{5-k_2} \left(\frac{\mu_1 p}{\mu_2}\right)^{k_2} \left(\frac{\mu_1 q}{\mu_3}\right)^{k_3}}$$

Hence

$$p(k_1, k_2, k_3) = \frac{\left(\frac{\mu_1 p}{\mu_2}\right)^{k_2} \left(\frac{\mu_1 q}{\mu_3}\right)^{k_3}}{\displaystyle\sum_{k_2=0}^{5} \sum_{k_3=0}^{5-k_2} \left(\frac{\mu_1 p}{\mu_2}\right)^{k_2} \left(\frac{\mu_1 q}{\mu_3}\right)^{k_3}}$$ ■

where $k_1+k_2+k_3 = 5$.

PROBLEM 4.13.

Consider a two-node Markovian queueing network (of the more general type considered by Jackson) for which $N = 2$, $m_1 = m_2 = 1$, $\mu_{k_i} = \mu_i$ (constant service rate), and which has transition probabilities (r_{ij}) as described in the following matrix:

$r_{ij} =$

i \ j	0	1	2	3
0	0	1	0	0
1	0	0	$1-\alpha$	α
2	0	1	0	0

where $0 < \alpha < 1$ and nodes 0 and $N+1$ are the "source" and "sink" nodes, respectively. We also have (for some integer K)

$$\gamma(S(k_1, k_2)) = \begin{cases} \infty & k_1+k_2 \neq K \\ 0 & k_1+k_2 = K \end{cases}$$

and assume the system initially contains K customers.

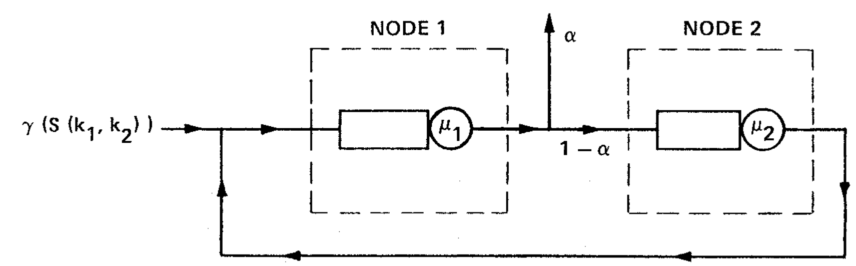
(a) Find e_i ($i = 1, 2$) as given in Eq. (4.75).

(b) Since $N = 2$, let us denote $p(k_1, k_2) = p(k_1, K-k_1)$ by p_{k_1}. Find the balance equations for p_{k_1}.

(c) Solve these equations for p_{k_1} explicitly.

(d) By considering the fraction of time the first node is busy, find the time between customer departures from the network (via node 1, of course).

SOLUTION



(a) Using Eq. (4.75) e_i is found as follows:

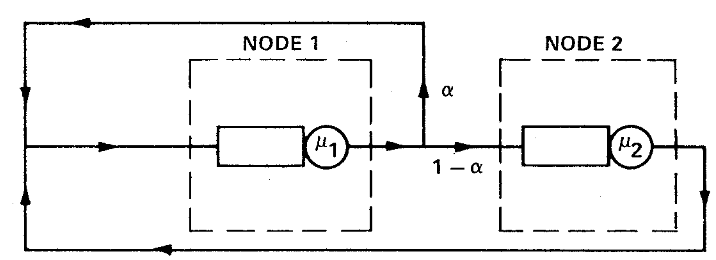
$$e_1 = r_{01} + \sum_{j=1}^{2} e_j r_{j1} \Rightarrow e_1 = 1 + e_1 \cdot 0 + e_2 \cdot 1 = 1 + e_2$$

$$e_2 = r_{02} + \sum_{j=1}^{2} e_j r_{j2} \Rightarrow e_2 = 0 + e_1 \cdot (1-\alpha) + e_2 \cdot 0 = e_1(1-\alpha)$$

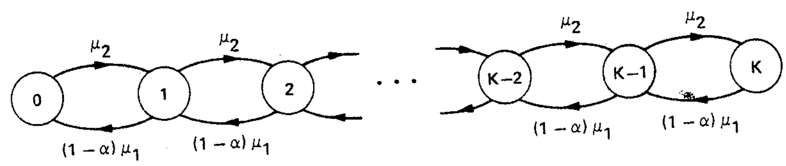
Solving gives

$$e_1 = \frac{1}{\alpha}, \quad e_2 = \frac{1-\alpha}{\alpha}$$ ■

(b) Since $\gamma(S(k_1, k_2)) = 0$ for $k_1+k_2 = K$, no-one enters the system if K customers are already there. But as soon as a departure takes place, another customer immediately enters the system (since $\gamma(S(k_1, k_2)) = \infty$ for $k_1+k_2 \neq K$). Thus we have a *closed* queueing network as follows:



For this network we have the following state diagram (labeling the states only by the number k_1 present at node 1):



The balance equations for p_{k_1} are:

$$\mu_2 p_0 = (1-\alpha)\mu_1 p_1 \quad k_1 = 0$$ ■

$$[\mu_2 + (1-\alpha)\mu_1] p_{k_1} = \mu_2 p_{k_1-1} + (1-\alpha)\mu_1 p_{k_1+1} \quad 0 < k_1 < K$$ ■

$$(1-\alpha)\mu_1 p_K = \mu_2 p_{K-1} \quad k_1 = K$$ ■

(c) Noting that this is the same state diagram as Fig. 3.8 for M/M/1/K (and also for Exercise 3.12), we immediately solve for p_{k_1} from Eq. (3.43).

$$p_{k_1} = \frac{1 - \dfrac{\mu_2}{(1-\alpha)\mu_1}}{1 - \left[\dfrac{\mu_2}{(1-\alpha)\mu_1}\right]^{K+1}} \left[\frac{\mu_2}{(1-\alpha)\mu_1}\right]^{k_1} \quad 0 \leqslant k_1 \leqslant K$$ ■

(d) The first node is busy a fraction $(1-p_0)$ of the time. For a very long time interval τ, the first node is busy for $(1-p_0)\tau$ seconds. While node 1 is busy, customers leave the system at rate $\alpha\mu_1$. Thus $\alpha\mu_1(1-p_0)\tau$ is the average number of departures during τ. Therefore, by renewal theory arguments, the average time between departures will be $\dfrac{1}{\alpha\mu_1(1-p_0)}$ or, upon substituting for p_0, the average interdeparture time is

$$\frac{1-\alpha}{\alpha\mu_2} \, \frac{1 - \left[\dfrac{\mu_2}{(1-\alpha)\mu_1}\right]^{K+1}}{1 - \left[\dfrac{\mu_2}{(1-\alpha)\mu_1}\right]^{K}}$$ ■

4.13.