where we write $\rho = \lambda/m\mu$. For stability we require $\lambda < m\mu$, or $\rho < 1$. We note that $D(z)$ is the negative of the polynomial in Eq. (1.84) (with m replacing r), and so we may use the results of Problem 4.10. Thus $D(z)$ has one root at $z = 1$, $m - 1$ roots in the range $|z| < 1$, and one root (say, z_m) in $|z| > 1$. By analyticity of $Q(z)$ for $|z| \leq 1$, the m roots of $D(z)$ satisfying $|z| \leq 1$ cancel with the m numerator roots leaving

$$Q(z) = K\frac{1}{z_m - z}$$

Now $Q(1) = 1$ implies $K = z_m - 1$. Thus

$$Q(z) = \frac{z_m - 1}{z_m - z}$$ □

PROBLEM 5.14

Consider an M/G/1 system with bulk service. Whenever the server becomes free, he accepts *two* customers from the queue into service simultaneously, or, if only one is on queue, he accepts that one; in either case, the service time for the group (of size 1 or 2) is taken from $B(x)$. Let q_n be the number of customers remaining after the nth service instant. Let v_n be the number of arrivals during the nth service. Define $B^*(s)$, $Q(z)$, and $V(z)$ as transforms associated with the random variables $\tilde{x}$, $\tilde{q}$, and $\tilde{v}$ as usual. Let $\rho = \lambda\bar{x}/2$.

(a) Using the method of imbedded Markov chains, find

$$E(\tilde{q}) = \lim_{n\to\infty} E(q_n)$$

in terms of ρ, σ_b^2, and $P[\tilde{q} = 0] \triangleq p_0$.

(b) Find $Q(z)$ in terms of $B^*(\cdot)$, p_0, and $p_1 \triangleq P[\tilde{q} = 1]$.

(c) Express p_1 in terms of p_0.

SOLUTION

(a) Clearly we may write

$$q_{n+1} = \begin{cases} q_n - 2 + v_{n+1} & q_n \geq 2 \\ q_n - 1 + v_{n+1} & q_n = 1 \\ v_{n+1} & q_n = 0 \end{cases}$$

Introducing the function

$$\Delta_{2,k} \triangleq \begin{cases} 2 & k \geq 2 \\ k & 0 \leq k \leq 2 \end{cases}$$

we have $q_{n+1} = q_n - \Delta_{2,q_n} + v_{n+1}$. Letting $n \to \infty$ and taking expectations

$$\bar{q} = \bar{q} - E\left[\Delta_{2,\tilde{q}}\right] + \bar{v}$$

But

$$E\left[\Delta_{2,\tilde{q}}\right] = \sum_{k=0}^{\infty} \Delta_{2,k} P[\tilde{q} = k]$$
$$= P[\tilde{q} = 1] + \sum_{k=2}^{\infty} 2P[\tilde{q} = k]$$
$$= p_1 + 2(1 - p_0 - p_1)$$

So

$$\bar{v} = E\left[\Delta_{2,\tilde{q}}\right] = 2 - 2p_0 - p_1$$

Recall that

$$q_{n+1} = q_n - \Delta_{2,q_n} + v_{n+1}$$

Squaring this equation gives

$$q_{n+1}^2 = q_n^2 - 2q_n\Delta_{2,q_n} + \Delta_{2,q_n}^2 + 2v_{n+1}(q_n - \Delta_{2,q_n}) + v_{n+1}^2$$

Let $n \to \infty$ and take expectations

$$\overline{q^2} = \overline{q^2} - 2E\left[\tilde{q}\Delta_{2,\tilde{q}}\right] + E\left[\Delta_{2,\tilde{q}}^2\right] + 2\bar{v}E\left[\tilde{q} - \Delta_{2,\tilde{q}}\right] + \overline{v^2}$$

since $\tilde{v}$ and $\tilde{q}$ are independent. So

$$2E\left[\tilde{q}\Delta_{2,\tilde{q}}\right] = E\left[\Delta_{2,\tilde{q}}^2\right] + \overline{v^2} + 2\bar{v}\left(\bar{q} - E\left[\Delta_{2,\tilde{q}}\right]\right)$$

Now $E\left[\Delta_{2,\tilde{q}}\right] = \bar{v}$ and

$$E\left[\Delta_{2,\tilde{q}}^2\right] = \sum_{k=1}^{\infty} \Delta_{2,k}^2 P[\tilde{q} = k]$$
$$= P[\tilde{q} = 1] + 4\sum_{k=2}^{\infty} P[\tilde{q} = k]$$
$$= p_1 + 4(1 - p_0 - p_1)$$

Also

$$E\left[\tilde{q}\Delta_{2,\tilde{q}}\right] = \sum_{k=1}^{\infty} k\Delta_{2,k} P[\tilde{q} = k]$$
$$= P[\tilde{q} = 1] + \sum_{k=2}^{\infty} 2kP[\tilde{q} = k]$$
$$= p_1 + 2\sum_{k=1}^{\infty} kP[\tilde{q} = k] - 2p_1$$
$$= 2\bar{q} - p_1$$

Thus

$$2(2\bar{q} - p_1) = 4 - 4p_0 - 3p_1 + \overline{v^2} + 2\bar{v}(\bar{q} - \bar{v})$$

Therefore, using $p_1 = 2 - 2p_0 - \bar{v}$, we find

$$\bar{q} = \frac{2 - 2p_0 + \bar{v} + \overline{v^2} - 2(\bar{v})^2}{4 - 2\bar{v}}$$

From Problem 5.11, we know that $V(z) = B^*(\lambda - \lambda z)$. Then, by differentiation, we have

$$\bar{v} = V^{(1)}(1) = \lambda\bar{x} = 2\rho$$

and

$$\overline{v^2} - \bar{v} = V^{(2)}(1) = \lambda^2\overline{x^2}$$

Thus

$$\bar{q} = \frac{2(1 - p_0) + 2\rho + \lambda^2\overline{x^2} + 2\rho - 2(4\rho^2)}{4 - 4\rho}$$
$$\bar{q} = \rho + \frac{2(1 - p_0) + \lambda^2\overline{x^2} - 4\rho^2}{4(1 - \rho)}$$
$$\bar{q} = \rho + \frac{2(1 - p_0) + \lambda^2\sigma_b^2}{4(1 - \rho)}$$

(b) We have

$$Q(z) = E\left[z^{\tilde{q}}\right] = E\left[z^{\tilde{q} - \Delta_{2,\tilde{q}} + \tilde{v}}\right]$$
$$= E\left[z^{\tilde{v}}\right]E\left[z^{\tilde{q} - \Delta_{2,\tilde{q}}}\right]$$
$$= V(z)E\left[z^{\tilde{q} - \Delta_{2,\tilde{q}}}\right]$$

But

$$E[z^{\tilde{q}-\Delta_{2,\tilde{q}}}] = \sum_{k=0}^{\infty} z^{k-\Delta_{2,k}} P[\tilde{q} = k]$$

$$= P[\tilde{q} = 0] + P[\tilde{q} = 1] + \sum_{k=2}^{\infty} z^{k-2} P[\tilde{q} = k]$$

$$= p_0 + p_1 + \frac{1}{z^2}[Q(z) - p_0 - p_1 z]$$

Thus

$$Q(z) = V(z)\left[p_0 + p_1 + \frac{1}{z^2}[Q(z) - p_0 - p_1 z]\right]$$

$$Q(z) = V(z)\frac{p_0(1-z^2) + p_1 z(1-z)}{V(z) - z^2}$$

Finally

$$Q(z) = B^*(\lambda - \lambda z)\frac{p_0(1-z^2) + p_1 z(1-z)}{B^*(\lambda - \lambda z) - z^2}$$

(c) From part (a),

$$\bar{v} = 2 - 2p_0 - p_1$$

But, from part (b),

$$\bar{v} = \lambda\bar{x} = 2\rho$$

Equating these two expressions gives

$$p_1 = 2(1 - p_0 - \rho)$$

or

$$p_1 = 2(1 - p_0) - \lambda\bar{x}$$ □

PROBLEM 5.15

Consider an M/G/1 queueing system with the following variation. The server refuses to serve any customers unless at least two customers are ready for service, at which time both are "taken into" service. These two customers are served individually and independently, one after the other. The instant at which the second of these two is finished is called a "critical" time and we shall use these critical times as the points in an imbedded Markov chain. Immediately following a critical time, if there are two more ready for service, they are both "taken into" service as above. If one or none is ready, then the server waits until a pair is ready, and so on. Let

q_n = number of customers left behind in the system immediately following the nth critical time

v_n = number of customers arriving during the combined service time of the nth *pair* of customers

(a) Derive a relationship between q_{n+1}, q_n, and v_{n+1}.

(b) Find

$$V(z) = \sum_{k=0}^{\infty} P[v_n = k]z^k$$

(c) Derive an expression for $Q(z) = \lim_{n\to\infty} Q_n(z)$ in terms of $p_0 = P[\tilde{q} = 0]$, where

$$Q_n(z) = \sum_{k=0}^{\infty} P[q_n = k]z^k$$

(d) How would you solve for p_0?

(e) Describe (do *not* calculate) two methods for finding $\bar{q}$.

SOLUTION

(a) Since the server refuses to serve any customers unless at least two are in the system, we see that

$$q_{n+1} = \begin{cases} q_n - 2 + v_{n+1} & q_n \geq 2 \\ v_{n+1} & q_n \leq 1 \end{cases}$$

Introducing the function

$$\Delta_{2,k} \triangleq \begin{cases} 2 & k \geq 2 \\ k & 0 \leq k \leq 2 \end{cases}$$

we have

$$q_{n+1} = q_n - \Delta_{2,q_n} + v_{n+1}$$

(b) $V(z)$ is the z-transform for the number of arrivals in an interval, which is the sum of two service times. The transform of the pdf for this interval is clearly $[B^*(s)]^2$. By analogy to the development in Problem 5.11, we see that

$$V(z) = [B^*(\lambda - \lambda z)]^2$$

(c) From part (a) we have

$$Q_{n+1}(z) = E\left[z^{q_{n+1}}\right] = E\left[z^{q_n-\Delta_{2,q_n}+v_{n+1}}\right] = E\left[z^{v_{n+1}}\right]E\left[z^{q_n-\Delta_{2,q_n}}\right]$$

by independence of q_n and v_{n+1}. Letting $n \to \infty$, we have

$$Q(z) = E\left[z^{\tilde{q}}\right] = V(z)E\left[z^{\tilde{q}-\Delta_{2,\tilde{q}}}\right]$$

But

$$E\left[z^{\tilde{q}-\Delta_{2,\tilde{q}}}\right] = \sum_{k=0}^{\infty} z^{k-\Delta_{2,k}}P[\tilde{q}=k]$$

$$= p_0 + p_1 + \sum_{k=2}^{\infty} z^{k-2}p_k$$

$$= p_0 + p_1 + \frac{1}{z^2}[Q(z) - p_0 - p_1 z]$$

Thus

$$Q(z) = \frac{V(z)}{z^2}\left[Q(z) + (z^2-1)p_0 + (z^2-z)p_1\right]$$

$$Q(z) = V(z)\frac{(1-z^2)p_0 + (z-z^2)p_1}{V(z)-z^2}$$

To eliminate p_1 we proceed as follows:

$$1 = \frac{Q(1)}{V(1)} = \lim_{z\to 1}\frac{(1-z^2)p_0 + (z-z^2)p_1}{V(z)-z^2}$$

Using L'Hospital's rule and $V^{(1)}(1) = \overline{v}$, we find

$$\overline{v} = 2 - 2p_0 - p_1$$

(This could also be obtained from the equation $\overline{q} = \overline{q} + E\left[\Delta_{2,\tilde{q}}\right] + \overline{v}$.) But $V(z) = [B^*(\lambda - \lambda z)]^2$, so that

$$\overline{v} = V^{(1)}(1) = 2\lambda\overline{x}$$

Thus

$$2\lambda\overline{x} = 2 - 2p_0 - p_1$$

and

$$p_1 = 2 - 2p_0 - 2\lambda\overline{x}$$

Substituting this expression for p_1 into $Q(z)$, we have

$$Q(z) = [B^*(\lambda-\lambda z)]^2\frac{(1-z)[p_0(1-z) + 2z(1-\lambda\overline{x})]}{[B^*(\lambda-\lambda z)]^2 - z^2}$$

(d) Equate roots of the denominator of $Q(z)$ with that of the numerator for $|z| < 1$ using analyticity of $Q(z)$.

(e) **(1)** Use the relation $\overline{q} = Q^{(1)}(1)$.

(2) Square the equation in part (a), let $n \to \infty$, and take expectations. □

PROBLEM 5.16

Consider an M/G/1 queueing system in which service is given as follows. Upon entry into service, a coin is tossed, which has probability p of giving Heads. If the result is Heads, then the service time for that customer is zero seconds. If Tails, his service time is drawn from the following exponential distribution:

$$pe^{-px} \qquad x \geq 0$$

(a) Find the average service time $\overline{x}$.
(b) Find the variance of service time σ_b^2.
(c) Find the expected waiting time W.
(d) Find $W^*(s)$.
(e) From (d), find the expected waiting time W.
(f) From (d), find $W(t) = P[\text{waiting time} \leq t]$.

SOLUTION

The service time density $b(x)$ is given by

$$b(x) = pu_0(x) + (1-p)pe^{-px} \qquad x \geq 0$$

(a) The mean service time is

$$\overline{x} = 0 \cdot p + \frac{1}{p}\cdot(1-p) = \frac{1-p}{p}$$

[Thus, for stability, we require $p > \lambda/(\lambda+1)$.]

(b) The second moment of service time is

$$\overline{x^2} = 0\cdot p + \frac{2}{p^2}\cdot(1-p) = \frac{2(1-p)}{p^2}$$

Thus

$$\sigma_b^{\,2} = \overline{x^2} - (\overline{x})^2 = \frac{2(1-p)}{p^2} - \frac{(1-p)^2}{p^2}$$

$$\sigma_b^2 = \frac{1-p^2}{p^2}$$

(c) For M/G/1, $W = \lambda\overline{x^2}/2(1-\rho)$ by Eq. (1.101). Using $\rho = \lambda\bar{x} = \lambda(1-p)/p$, we have

$$W = \frac{\lambda \dfrac{2(1-p)}{p^2}}{2\left(1 - \dfrac{\lambda(1-p)}{p}\right)}$$

$$W = \frac{\rho}{p(1-\rho)}$$

(d) We first note that

$$B^*(s) = p + (1-p)\frac{p}{s+p} = \frac{p(s+1)}{s+p}$$

Thus, by Eq. (1.105),

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda \dfrac{p(s+1)}{s+p}}$$

$$W^*(s) = \frac{(1-\rho)(s+p)}{s+p(1-\rho)}$$

(e) Differentiating gives

$$W^{*(1)}(s) = (1-\rho)\left[\frac{[s+p(1-\rho)] - (s+p)}{[s+p(1-\rho)]^2}\right]$$

$$= (1-\rho)\frac{-p\rho}{[s+p(1-\rho)]^2}$$

Thus

$$W = -W^{*(1)}(0) = \frac{\rho}{p(1-\rho)}$$

[same as part (c)].

(f) We have

$$W^*(s) = \frac{(1-\rho)(s+p)}{s+p(1-\rho)}$$

$$= (1-\rho) + \frac{p\rho(1-\rho)}{s+p(1-\rho)}$$

Inverting we find the pdf as

$$w(y) = (1-\rho)u_0(y) + p\rho(1-\rho)e^{-p(1-\rho)y} \qquad y \geq 0$$

Thus the PDF is

$$W(y) = 1 - \rho e^{-p(1-\rho)y} \qquad y \geq 0$$

□