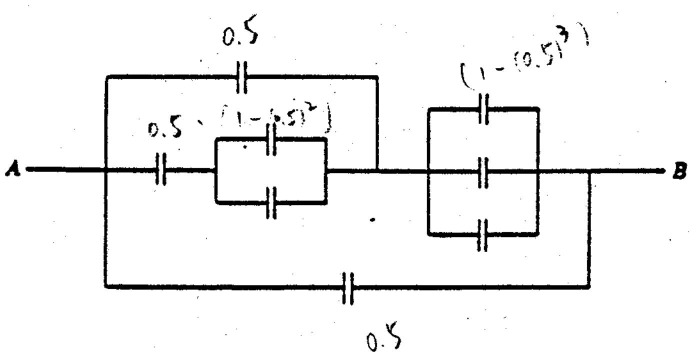
1. Five cars start out on a cross-country race. The probability that a car breaks down and drops out of the race is 0.2 . What is the probability that at least 3 cars finish the race?

2. We have two dice, A and B. Die A has 4 Red and 2 White faces and Die B has 2 Red and 4 White faces. We will play a game by first choosing a die; die A is selected with probability p. The chosen die is tossed until a White face appears, at which time the game is ended.

After playing the game a great many times, it is observed that the probability that a game ends in exactly 3 tosses of the selected die is 7/81. Determine the value of p.

3. A biased coin has a probability p of coming up Heads when tossed. It is tossed 6 times by your friend who tells you that Heads turned up in more than half of the tosses. Given that information, what is the probability that Heads appeared in all 6 of the tosses?

4. In the figure below, the notation —||— represents a communication link. Link failures are independent, and each link has a probability of 0.5 of being out of service. Towns A and B can communicate as long as they are connected by at least one communication path which contains only in-service links. In an efficient manner, determine the probability that A and B can communicate.



5. A pair of four-sided dice is thrown once. Each die has faces labeled 1,2,3,and 4. The discrete random variable X is defined to be the product of the down-face values. Determine the conditional variance of X^2 given that the sum of the down-face values is greater than the product of the down-face values.

6. A computer will fail in its kth month of use with probability

$$p_k = \frac{1}{5}\left(\frac{4}{5}\right)^{k-1} \qquad k=1,2,3,\dots$$

Four computers are life-tested simultaneously. Find the probability that:

a) None of the four computers fails during its first month of use.

b) Exactly two computers have failed by the end of the third month.

c) Exactly one computer fails during each of the first three months.

d) Exactly one computer has failed by the end of the second month, and exactly two computers are still working at the start of the fifth month.

7. a) A wheel of fortune is spun three times. What is the probability that none of the resulting spins is within 30 degrees of any other spin?

b) What is the smallest number of spins for which the probability that at least one other reading is within plus or minus 30 degrees of the first reading is at least 0.9 ?

8. The probability that a store will have exactly k customers on any given day is

$$p_k = \frac{1}{5}\left(\frac{4}{5}\right)^{k} \qquad k=0,1,2,\dots$$

On each day when the store has had at least one customer, one of the sales slips is selected at random and a door prize is mailed to the corresponding customer. (Each sales slip corresponds to a unique customer, and each customer buys exactly one item).

a) What is the probability that a customer selected randomly from the population of all customers will win a door prize?

b) Given a customer who has won a door prize, what is the probability that he was in the store on a day when it had exactly K customers?

1. $P = 0.2$ (probability that a car break down)

$q = 1 - P = 0.8$

The probability that at least 3 cars finish the race is

$$\binom{5}{3} 0.8^3 0.2^2 + \binom{5}{4} 0.8^4 0.2^1 + \binom{5}{5} 0.8^5 0.2^0 = 0.94208$$

2 When die A is selected probability of white face is $\frac{1}{3}$

probability of red face is $\frac{2}{3}$

When die B is selected probability of white face is $\frac{2}{3}$

probability of red face is $\frac{1}{3}$

$$\Rightarrow \quad \frac{7}{81} = P \cdot \left(\frac{2}{3}\right)^2 \frac{1}{3} + (1-P) \left(\frac{1}{3}\right)^2 \frac{2}{3} \qquad P = \frac{1}{6}$$

3. Probability of Heads appeared 4 times is $\binom{6}{4} p^4 (1-p)^2$

5 times is $\binom{6}{5} p^5 (1-p)^1$

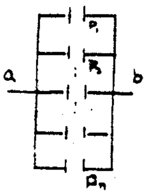
6 times is $\binom{6}{6} p^6 (1-p)^0$

P(Heads appeared 6 times | Heads appered > 3 times)

$$= \frac{p^6}{\binom{6}{4} p^4 (1-p)^2 + \binom{6}{5} p^5 (1-p) + \binom{6}{6} p^6 (1-p)^0}$$

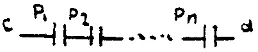
$$= \frac{p^2}{10p^2 - 24p + 15}$$

4. Let P_i be the probability of out of service



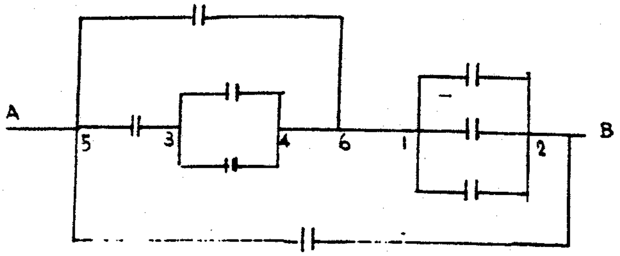
The probability that a and b can communicate is

$$p(a,b) = 1 - P_1 P_2 \quad P_n$$



The probability that c and d can communicate is

$$p(c.d) = (1-P_1)(1-P_2)\cdots(1-P_n)$$



$$P(1,2) = 1 - (\tfrac{1}{2})^3 = \frac{7}{8}$$

$$P(3.4) = 1 - (\tfrac{1}{2})^2 = \frac{3}{4}$$

$$P(5.4) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

$$P(5.6) = 1 - \frac{1}{2}\left(1 - \frac{3}{8}\right) = \frac{11}{16}$$

$$P(5,2) = \frac{7}{8} \cdot \frac{11}{16} = \frac{77}{128}$$

$$P(A.B) = 1 - \frac{1}{2}\left(1 - \frac{77}{128}\right) = \frac{205}{256}$$

X_1 \ X_2	1	2	3	4
1	2/1	3/2	4/3	5/4
2	3/2	4/4	5/6	6/8
3	4/3	5/6	6/9	7/12
4	5/4	6/8	7/12	8/16

Sum / product

Let X_1 and X_2 be two random variables for this pair of dice and

$$X = X_1 X_2$$

$$Y = X_1 + X_2 \geq X_1 X_2$$

σ^2 : the conditional variance of X^2 given Y

$$\Rightarrow \quad \begin{aligned} P[\,X=1 \mid Y\,] &= 1/7 \\ P[\,X=2 \mid Y\,] &= 2/7 \\ P[\,X=3 \mid Y\,] &= 2/7 \\ P[\,X=4 \mid Y\,] &= 2/7 \end{aligned}$$

$$\Rightarrow \quad E[X^2 \mid Y] = 1^2\, P[X=1 \mid Y] + 2^2\, P[X=2 \mid Y] + 3^2\, P[X=3 \mid Y] + 4^2\, P[X=4 \mid Y]$$

$$= \frac{1}{7} + \frac{8}{7} + \frac{18}{7} + \frac{32}{7} = \frac{59}{7}$$

$$E[X^4 \mid Y] = 1^4 \cdot \frac{1}{7} + 2^4 \cdot \frac{2}{7} + 3^4\, \frac{2}{7} + 4^4\, \frac{2}{7} = 101$$

$$\sigma^2 = E[X^4 \mid Y] - E^2[X^2 \mid Y]$$

$$= 101 - \frac{59}{7} = \frac{1468}{49} \approx 29.959$$

6. A computer will fail in its k^{th} month of use with probability

$$P_k = \frac{1}{5}\left(\frac{4}{5}\right)^{k-1} \qquad k = 1, 2, 3, \ldots$$

Let E_{ij} represent the event that the i^{th} computer will fail at the j^{th} month

where $i = 1, 2, 3, 4$

$j \geq 1$

(a)

$$P(E_{i1}) = \frac{1}{5}\left(\frac{4}{5}\right)^0 = \frac{1}{5} \qquad i = 1, 2, 3, 4$$

$$P(\bar{E}_{i1}) = 1 - \frac{1}{5} = \frac{4}{5} \qquad i = 1, 2, 3, 4$$

The probability of none of the 4^{th} computers fails during first month of use is

$$P(\bar{E}_{11} \cap \bar{E}_{21} \cap \bar{E}_{31} \cap \bar{E}_{41}) = P(\bar{E}_{11})\, P(\bar{E}_{21})\, P(\bar{E}_{31})\, P(\bar{E}_{41}) \qquad (\because E_{i1} \text{ are independent})$$

$$= \boxed{\left(\frac{4}{5}\right)^4 = 0.4096}$$

(b) The probability of a computer has failed by the end of the third month is

$$P(E_{i1} \cup E_{i2} \cup E_{i3}) = P(E_{i1}) + P(E_{i2}) + P(E_{i3})$$

$$= \frac{1}{5} + \frac{4}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5} = \frac{61}{125} = 0.488$$

The probability of exactly two computers have failed by the end of the third month is

$$P = \binom{4}{2}\left(\frac{61}{125}\right)^2\left(1 - \frac{61}{125}\right)^2 = 6\left(\frac{61}{125}\right)^2\left(\frac{64}{125}\right)^2 \approx 0.3746$$

(c) Let F_i be the event that exactly one computer fails in the i^{th} month

The probability of exactly one computer fails during each of the first three month is

$$P(F_1 \cap F_2 \cap F_3) = P(F_3 \mid F_2 \cap F_1)\, P(F_2 \mid F_1)\, P(F_1)$$

$$= \binom{4}{1}\binom{3}{1}\binom{2}{1}\, \frac{1}{5}\left(\frac{4}{5}\right)^3 \frac{4}{25}\left(\frac{21}{25}\right)^2 \frac{16}{125}\, \frac{109}{125} \approx 0.031$$

where $P(F_1) = \binom{4}{1}\frac{1}{5}\left(\frac{4}{5}\right)^3$

$$P(F_2 \mid F_1) = \binom{3}{1} P(E_{i2})\,[1 - P(E_{i2})]^2 = \binom{3}{1}\left(\frac{4}{5}\,\frac{1}{5}\right)\left(\frac{21}{25}\right)^2$$

$$P(F_3 \mid F_2 \cap F_1) = \binom{2}{1} P(E_{i3})\,[1 - P(E_{i3})] = \binom{2}{1}\left(\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}\right)\left(\frac{109}{125}\right)$$

(d) Let E be the event that exactly one computer has failed by the end of the second month and F be the event that exactly two computers are still working at the start of the fifth month

$$P(E \cap F) = P(E \mid F)\, P(F)$$

$$= \binom{3}{1}\binom{4}{1} \frac{9}{25}\left(\frac{16}{25}\right)^3 \left(\frac{144}{625}\right)\left(\frac{481}{625}\right)^2$$

where $E \mid F$ means exactly one computer has failed during 3^{rd} and 4^{th} month

$$P(E \mid F) = \binom{3}{1}\,[P(E_{i3}) + P(E_{i4})]\,[1 - (P(E_{i3}) + P(E_{i4}))]^2$$

$$= \binom{3}{1}\left[\left(\frac{4}{5}\right)^2\frac{1}{5} + \left(\frac{4}{5}\right)^3\frac{1}{5}\right]\left[1 - \left(\left(\frac{4}{5}\right)^2\frac{1}{5} + \left(\frac{4}{5}\right)^3\frac{1}{5}\right)\right]^2$$

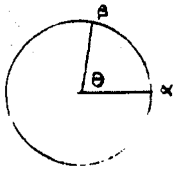
$$= \binom{3}{1}\left(\frac{144}{625}\right)\left(\frac{481}{625}\right)^2$$

$$P(F) = \binom{4}{1}\,[P(E_{i1}) + P(E_{i2})]\,[1 - (P(E_{i1}) + P(E_{i2}))]^3$$

$$= \binom{4}{1}\left[\frac{1}{5} + \frac{4}{5}\,\frac{1}{5}\right]\left[1 - \left(\frac{1}{5} + \frac{4}{5}\,\frac{1}{5}\right)\right]^3 = \binom{4}{1}\frac{9}{25}\left(\frac{16}{25}\right)^3$$

7 (a) Let S_1, S_2 and S_3 be the three positions of the spin, and let $S_1 = 0$

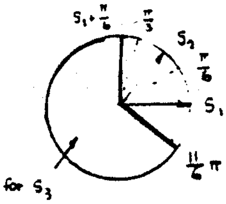
The probability for a spin to be between α and β is



$$P(\alpha < \theta < \beta) = \int_\alpha^\beta \frac{d\theta}{2\pi} = \frac{\beta - \alpha}{2\pi}$$

Now, we have three cases to consider.

1. If $S_2 \in (\frac{\pi}{6}, \frac{\pi}{3})$ then $S_3 \in (S_2 + \frac{\pi}{6}, \frac{11}{6}\pi)$

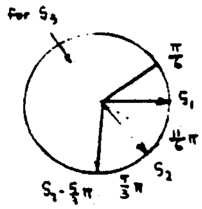


$$P_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{S_2 + \frac{\pi}{6}}^{\frac{11}{6}\pi} \frac{dS_3}{2\pi} \frac{dS_2}{2\pi}$$

$$= \frac{1}{4\pi^2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{5\pi}{3} - S_2\right) dS_2$$

$$= \frac{17}{288}$$

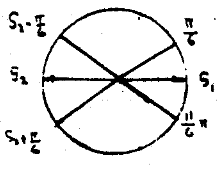
2. If $S_2 \in (\frac{5}{3}\pi\ \frac{11}{6}\pi)$ then $S_3 \in (\frac{\pi}{6}, \frac{5}{3}\pi - \frac{\pi}{6})$



$$P_2 = \int_{\frac{5}{3}\pi}^{\frac{11}{6}\pi} \int_{\frac{\pi}{6}}^{S_2 - \frac{5}{3}\pi} \frac{dS_3}{2\pi} \frac{dS_2}{2\pi}$$

$$= \frac{17}{288}$$

3. If $S_2 \in (\frac{\pi}{3}, \frac{5}{3}\pi)$ then $S_3 \in (\frac{\pi}{6}, S_2 - \frac{\pi}{6}) \cup (S_2 + \frac{\pi}{6}, \frac{11}{6}\pi)$



$$P_3 = \int_{\frac{\pi}{3}}^{\frac{5}{3}\pi} \left[\int_{\frac{\pi}{6}}^{S_2 - \frac{\pi}{6}} \frac{dS_3}{2\pi} + \int_{S_2 + \frac{\pi}{6}}^{\frac{11}{6}\pi} \frac{dS_3}{2\pi} \right] \frac{dS_2}{2\pi}$$

$$= \frac{4}{9}$$

$\Rightarrow$ $\therefore$ The probability that none of the resulting spins is within 30° of any other spin is

$$P = P_1 + P_2 + P_3 = \frac{17}{288} + \frac{17}{288} + \frac{4}{9} = \frac{9}{16}$$

(b) The probability that the 2nd spin is out of $\pm 30^\circ$ of the first on is $\frac{5}{6}$

- - - - - - - 2nd and 3rd - - - - - - - $(\frac{5}{6})^2$

$\vdots$ $\vdots$

- - - - - - - 2nd, 3rd.... n^{th} - - - - - - - $(\frac{5}{6})^{n-1}$

$\Rightarrow$ To find the smallest value of n such that $1 - (\frac{5}{6})^{n-1} > 0.9$

$$n \geq 13.629$$

$\Rightarrow$ The smallest value $n = 14$

8 The probability that a store will have exactly k customers on any given day is

$$P_k = \frac{1}{5}\left(\frac{4}{5}\right)^k \qquad k = 0, 1, 2, 3, \cdots$$

(a) Let X be the r.v. of the number of customers on any given day. and Y be the r.v. of a customer is selected at randomly from the population of all customers will win a door prize.

$$\begin{aligned}
P(Y) &= \sum P(Y \mid X)\, P(X) \\
&= \sum_{k=1}^{\infty} P(Y \mid X = k)\, P(X = k) \\
&= \sum_{k=1}^{\infty} \left[\frac{1}{5}\left(\frac{4}{5}\right)^k\right] \frac{1}{k} \\
&= \frac{1}{5} \sum_{k=1}^{\infty} \frac{\left(\frac{4}{5}\right)^k}{k} \\
&= \frac{1}{5}\left[-\ln\left(1 - \frac{4}{5}\right)\right] \\
&= \frac{1}{5} \ln 5 \approx 0.322
\end{aligned}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Integrate both side

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\ln(1-x)$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$$

(b)

$$\begin{aligned}
P(X = k \mid Y) &= \frac{P(X = k \cap Y)}{P(Y)} \\
&= \frac{P(Y \mid X = k)\, P(X = k)}{P(Y)} \\
&= \frac{\frac{1}{5}\left(\frac{4}{5}\right)^k \frac{1}{k}}{\frac{1}{5}\ln 5} \\
&= \frac{\left(\frac{4}{5}\right)^k}{k \ln 5}
\end{aligned}$$