Fluid Analysis

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Outline

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Introduction

- Treating customer arrival and departure processes as *fluid flows*, and represents teh backlog as a continuous-valued function of time.
- One can obtain *transient* as well as *equilibrium* solutions.
- Since arrival and departure occur with discrete jumps, fluid analysis replaces with *continuous change*. It is a good approximation in heavy-traffic condition:
  - when the queue sizes are large compared to unity, and
  - when the waiting times are large compared to average service times.

  Or the magnitude of the original discontinuities is small relative to the average value of these functions.
- Fluid approximation is a *first-order approximation* since we deal with average values of arrivals, departures and queueing process.

Notations

- $A(t)$ is the *accumulated number of arrivals* up to time t with average denoted as $\overline{A}(t)$. The ratio of deviation $A(t) - \overline{A}(t)$ to the average $\overline{A}(t)$ is negligibly small:

$$\lim_{t \to \infty} \frac{A(t) - \overline{A}(t)}{\overline{A}(t)} \xrightarrow{a.s.} 0. \tag{1}$$

- Fluid approximation replaces the random process $A(t)$ by the continuous deterministic process $\overline{A}(t)$.
- Similary, we replace the departure counting process $D(t)$ by its average $\overline{D}(t)$.
- The amount of backlog (e.g., number of customers in teh system) is $N(t)$. With $N(0) = 0$, can be approximated:

$$\overline{N}(t) = \overline{A}(t) - \overline{D}(t). \tag{2}$$

Notations (continue)

- We define the flow rates of the arrival process as

$$\lambda(t) = \frac{d\overline{A}(t)}{dt} \quad (3)$$

$$\mu(t) = \frac{d\overline{D}(t)}{dt} \quad (4)$$

- Thus, we have

$$\overline{A}(t) = \overline{A}(0) + \int_0^\infty \lambda(u)du \quad (5)$$

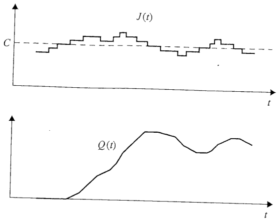
$$\overline{D}(t) = \overline{D}(0) + \int_0^\infty \mu(u)du. \quad (6)$$

Application: Statistical Multiplexer

- There are K statistically independent and identical sources and each source alternates between "on" (or burst) state and the "off" (or silence) state.
- Duration of on (off) state is exponentially distributed with mean β^{-1} (α^{-1}). If the source is "on", it generates cells at teh rate of one cell per unit time.
- Let C be teh multiplexer's service rate, normalized by the data rate per source.
- Let $J(t)$ bethe number of sources that are "on" at time t. If $J(t) < C$, all arriving packets are transmitted immediately to the output link and no queueing. If $J(t) > C$, a queue will develop at the rate of $J(t) - C$.
- For non-trivial analysis, we assume $C < K$.

- Let $Q(t)$ be the number of packets in the buffer.
- The random process $Q(t)$ and $J(t)$ are related via:

$$\frac{dQ(t)}{dt} = \begin{cases} J(t) - C & \text{if } Q(t) > 0 \text{ or } J(t) > C, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$



- While $Q(t) > 0$, it is the integration of the process $J(t)$:

$$Q(t) = \int_{t_0}^{t} J(u)du - C(t - t_0), \tag{8}$$

where t_0 is the msot recent instant such that $Q(t_0) = 0$.

- To derive $Q(t)$, we need to first consider the *pair process* $(J(t), Q(t))$. We define the joint probability function

$$F_j(t,x) = P[J(t) = j, Q(t) \leq x], \quad 0 \leq j \leq K, t > 0, x \geq 0. \tag{9}$$

- Note that $J(t)$ is a birth-death process. Consider a short interval $(t-h, t)$, each "on" source generates r pkts/sec, the server sends rC pkts/sec. We have the following (which is independent of r):

$$\begin{aligned} F_j(t,x) &= \lambda(j-1)hF_{j-1}(t-h, x-(j-C)h) \\ &\quad +\mu(j+1)hF_{j+1}(t-h, x-(j-C)h) \\ &\quad +[1-\lambda(j)h-\mu(j)h]F_j(t-h, x-(j-C)h) + o(h) \end{aligned} \tag{10}$$

where $\lambda(j) = (K-j)\alpha$, and $\mu(j) = j\beta$, for $0 \leq j \leq K$.

(continue)

- First, we have the following Taylor expansion:

$$F_j(t-h, x-(j-C)h) = F_j(t-h, x) - \frac{\partial}{\partial x} F_j(t-h, x)(j-C)h + o(h) \tag{11}$$

- Use the above Taylor expansion on the 3^{rd} term of Eq.(10)

$$\begin{aligned} F_j(t,x) - F_j(t-h,x) + (j-C)\frac{\partial}{\partial x}F_j(t-h,x)h \\ = -[\lambda(j)+\mu(j)]F_j(t-h, x-(j-C)h)h \\ +\lambda(j-1)F_{j-1}(t-h, x-(j-C)h)h \\ +\mu(j+1)F_{j+1}(t-h, x-(j-C)h)h + o(h) \end{aligned}$$

(continue)

- Dividing the above equation by h and let $\lim_{h \to 0}$:

$$\frac{\partial F_j(t,x)}{\partial t} + (j-C)\frac{\partial F_j(t,x)}{\partial x} = -[\lambda(j)+\mu(j)]\,F_j(t,x)+\lambda(j-1)F_{j-1}(t,x) + \mu(j+1)F_{j+1}(t,x) \tag{12}$$

for $0 \leq j \leq K$ and $x \geq 0$. With boundary conditions

$$F_{-1}(t,x) = F_{K+1}(t,x) = 0 \quad \text{for all } t \text{ and } x \geq 0. \tag{13}$$

- We are interested in the equilibrium solution $F_j(x) = \lim_{t\to\infty} F_j(t,x)$ for $0 \leq j \leq K, x \geq 0$. Taking $t \to \infty$, Eq (12) becomes:

$$(j-C)\frac{dF_j(x)}{dx} = -[\lambda(j)+\mu(j)]\,F_j(x)+\lambda(j-1)F_{j-1}(x) + \mu(j+1)F_{j+1}(x) \tag{14}$$

with $F_{-1}(x) = F_{K+1}(x) = 0$ for $x \geq 0$.

(continue)

- The above equilibrium solutions can be solved:
  - Numerically.
  - Laplace Transform method.
  - Spectral Matrix Expansion
- For the last two approaches, please refer to the textbook by Hisashi Kobayashi, "System Modeling and Analysis".
- In the textbook (Chapter 13), it also covers:
  - Rare event for buffer overflow, or $P[Q(t) > B]$
  - Infinite Source Model